CHAPTER II

CRITICAL VALUES OF DIRICHLET L-FUNCTIONS

4. The values of elementary Dirichlet series at integers

4.1. We first introduce the Euler numbers and the Euler polynomials not just in the classical sense, but in more generalized forms by putting

$$\frac{(1+c)e^z}{e^{2z}+c} = \sum_{n=0}^{\infty} \frac{E_{c,n}}{n!} z^n, \tag{4.1}$$

$$\frac{(1+c)e^{tz}}{e^{z}+c} = \sum_{n=0}^{\infty} \frac{E_{c,n}(t)}{n!} z^n. \tag{4.2}$$

Here $c = -\mathbf{e}(\alpha)$ with $\alpha \in \mathbf{R}, \notin \mathbf{Z}$. Thus $c \neq -1$. The standard **Euler numbers** and the **Euler polynomials** are $E_{1,n}$ and $E_{1,n}(t)$, that is, the symbols in the case $\alpha = 1/2$ and $c = 1$. Though they are usually written E_n and $E_n(t)$, we include 1 in the subscript for the purpose of distinguishing them from the Eisenstein series that we will introduce later. Thus $E_{c,n}$ and $E_{c,n}(t)$ may be called the **generalized Euler numbers** and the **generalized Euler polynomials.** To make our formulas short, put $b = -(1+c)^{-1}$. Then we have

$$E_{c,n} = 2^n E_{c,n}(1/2), \tag{4.3a}$$

$$E_{c,n}(t) = t^n + b\sum_{k=0}^{n-1} \binom{n}{k} E_{c,k}(t), \quad b = -(1+c)^{-1}, \tag{4.3b}$$

$$(d/dt)E_{c,n}(t) = nE_{c,n-1}(t) \qquad (n > 0), \tag{4.3c}$$

$$E_{c,n}(t+1) + cE_{c,n}(t) = (1+c)t^n, \tag{4.3d}$$

$$E_{1,2n+1} = 0, \tag{4.3e}$$

$$E_{c,n}(1-t) = (-1)^n E_{c^{-1},n}(t), \tag{4.3f}$$

$$E_{c^{-1},n}(0) = (-1)^{n+1} cE_{c,n}(0). \tag{4.3g}$$

All formulas except (4.3c) are valid for $n \geq 0$. Formula (4.3a) can be obtained by substituting $(1/2,\, 2z)$ for $(t,\, z)$ in (4.2). Since $-b(e^z + c) = 1 - b\sum_{n=1}^{\infty} z^n/n!$, (4.2) can be written

$$\sum_{n=0}^{\infty} \frac{t^n z^n}{n!} = e^{tz} = \left\{1 - b\sum_{n=1}^{\infty} \frac{z^n}{n!}\right\} \sum_{n=0}^{\infty} \frac{E_{c,n}(t)z^n}{n!},$$

which produces (4.3b). Applying d/dt to (4.2), we obtain (4.3c). From (4.2) we obtain $\sum_{n=0}^{\infty}\bigl\{E_{c,n}(t+1)+cE_{c,n}(t)\bigr\}z^n/n! = (1+c)e^{tz}$, which gives (4.3d). We will prove (4.3e) and (4.3f) in the Remark after the following theorem. Combining (4.3d) with (4.3f), we obtain (4.3g). Clearly $E_{c,0}(t) = 1$. Using (4.3b), we can easily verify that $E_{c,n}(t)$ is a polynomial in t of degree n whose coefficients are polynomials in b with coefficients in $\mathbf{Z}$. For example,

$$\begin{aligned}(4.3\mathrm{h})\qquad & E_{c,0}(t) = 1, \quad E_{c,1}(t) = t + b, \quad E_{c,2}(t) = t^2 + 2bt + 2b^2 + b,\\ & E_{c,3}(t) = t^3 + 3bt^2 + (6b^2 + 3b)t + 6b^3 + 6b^2 + b, \ \ b = -(1+c)^{-1}.\end{aligned}$$

The significance of $E_{c,n}(t)$ is that it gives the value of the infinite series $F_{c,n}(t)$ (with $c = -\mathbf{e}(\alpha)$ as above) defined by

$$(4.4)\qquad F_{c,n}(t) = \sum_{h\in\mathbf{Z}}(h+\alpha)^{-n-1}\mathbf{e}\bigl((h+\alpha)t\bigr) \qquad (0 \le n \in \mathbf{Z},\ t \in \mathbf{R}).$$

The infinite sum on the right-hand side depends only on α (mod $\mathbf{Z}$), and so the notation $F_{c,n}$ is meaningful. Also, the sum is clearly convergent for $n > 0$. If $n = 0$, we understand that $\sum_{h\in\mathbf{Z}}$ means $\lim_{m\to\infty}\sum_{|h|\le m}$, which is indeed meaningful, as will be shown below. We have

$$(4.4\mathrm{a})\qquad F_{c,n}(t+m) = \mathbf{e}(m\alpha)F_{c,n}(t) \quad \text{if} \quad m \in \mathbf{Z},$$

$$(4.4\mathrm{b})\qquad F_{c,n}(1-t) = (-1)^n cF_{c^{-1},n}(t).$$

Formula (4.4a) is obvious. Replacing h by $-h$ in (4.4), we obtain (4.4b). Taking $\alpha = q/N$ with a positive integer N and an integer q such that $q \notin N\mathbf{Z}$, we easily see that

$$(4.4\mathrm{c})\qquad F_{c,k-1}(Nt) = N^k\sum_{m\in q+N\mathbf{Z}} m^{-k}\mathbf{e}(mt) \qquad (c = -\mathbf{e}(q/N),\ 0 < k \in \mathbf{Z}).$$

Theorem 4.2. *For $0 \le n \in \mathbf{Z}$ and $0 < t < 1$ we have*

$$(4.5)\qquad E_{c,n}(t) = (1 + c^{-1})n!(2\pi i)^{-n-1}F_{c,n}(t).$$

This is true even for $0 \le t \le 1$ if $n > 0$. Moreover,

$$(4.5\mathrm{a})\qquad (1 + c^{-1})(2\pi i)^{-1}F_{c,0}(0) = (1 - c^{-1})/2.$$

Remark. These formulas combined with (4.4a) determine $F_{n,c}(t)$ for every $t \in \mathbf{R}$. Notice that $E_{c,0}(0) = 1$, which is different from (4.5a). Putting $t = \alpha = 1/2$ in (4.4) and (4.5), we obtain (4.3e). Formula (4.3f) follows from (4.4b) and (4.5).

Proof. We obtain (4.5a) directly from (2.18). We prove (4.5) first in the case $n > 0$ and $0 \le t \le 1$ by taking the contour integral

$$(*)\qquad \int_S f(z)dz, \qquad f(z) = \frac{e^{tz}}{z^{n+1}(e^z + c)},$$

where S is a square, with center at the origin, whose vertices are $A \pm iA$ and $-A \pm iA$, $A = 2N\pi$ with $0 < N \in \mathbf{Z}$. The poles of f in $\mathbf{C}$ are 0 and $2\pi i(h+\alpha)$ with $h \in \mathbf{Z}$. The residue of f at $2\pi i(h+\alpha)$ is easily seen to be $-c^{-1}\mathbf{e}\big(t(h+\alpha)\big)(2\pi i(h+\alpha))^{-n-1}$. From (4.2) we see that the residue of f at 0 is $(1+c)^{-1}E_{c,n}(t)/n!$. Thus by the theorem of residues, the integral of (4.5) equals $2\pi i$ times

$$(1+c)^{-1}E_{c,n}(t)/n! - c^{-1}(2\pi i)^{-n-1}\sum_h (h+\alpha)^{-n-1}\mathbf{e}\big((h+\alpha)t\big),$$

where h runs over the integers such that $|h+\alpha| < N$. To make an estimate of the integral on the sides of S, put $z = x + iy$. If $x = A$, then $|e^{tz}| = e^{tA}$ and $|e^z + c| \geq e^A - 1$, and so $|e^{tz}/(e^z+c)| \leq e^A/(e^A-1)$. If $x = -A$, then $|e^{tz}| = e^{-tA} \leq 1$ and $|e^z + c| \geq 1 - e^{-A}$, and so $|e^{tz}/(e^z+c)| \leq e^A/(e^A-1)$. Next suppose $y = \pm iA$. If $e^x > 2$, then $|e^{tz}| = e^{tx} \leq e^x$ and $|e^z + c| \geq e^x - 1$, and so $|e^{tz}/(e^z+c)| \leq e^x/(e^x-1) = 1/(1-e^{-x}) \leq 2$. If $e^x \leq 2$, then $|e^{tz}| = e^{tx} \leq 2$ and $|e^z + c| = |e^x - \mathbf{e}(\alpha)| \geq B$ with a positive constant B depending only on α, and so $|e^{tz}/(e^z+c)| \leq 2/B$. Therefore, because of the factor z^{-n-1}, we see that the integral of $(*)$ tends to 0 as $N \to \infty$. This proves (4.5) for $n > 0$ with $0 \leq t \leq 1$. To prove the case $n = 0$, we need the following lemma.

Lemma 4.3. (i) *Let $\{a_n\}_{n=1}^{\infty}$ be an increasing sequence of positive numbers such that $\lim_{n\to\infty} a_n = \infty$ and let T be a compact subset of $\mathbf{R}$ such that $T \cap \mathbf{Z} = \emptyset$. Then the series $\sum_{n=1}^{\infty} a_n^{-s}\mathbf{e}((\alpha+n)t)$ and $\sum_{n=1}^{\infty} a_n^{-s}\mathbf{e}((\alpha-n)t)$ are uniformly convergent for a fixed $\alpha \in \mathbf{C}$, $t \in T$, and $s > \sigma$ with any positive constant σ.*

(ii) *Let χ be a $\mathbf{C}$-valued function on $\mathbf{Z}/N\mathbf{Z}$ such that $\sum_{a=1}^{N}\chi(a) = 0$, and let $\{a_n\}_{n=1}^{\infty}$ be as in* (i). *Then $\sum_{n=1}^{\infty} a_n^{-s}\chi(n)$ is uniformly convergent for $s > \sigma$ with any positive constant σ.*

Proof. To prove (i), put $M = \mathrm{Max}_{t\in T} 2/|1 - \mathbf{e}(t)|$, $\omega = \mathbf{e}(t)$, and $\gamma_n = \sum_{h=1}^{n}\omega^h$. Then $|\gamma_n| = |(\omega - \omega^{n+1})/(1-\omega)| \leq M$ if $t \in T$, and so for $1 < m \leq n$ we have

$$\begin{aligned}\sum_{h=m}^{n} a_h^{-s}\mathbf{e}(ht) &= \sum_{h=m}^{n}(\gamma_h - \gamma_{h-1})a_h^{-s} \\ &= \sum_{h=m}^{n-1}\gamma_h\{a_h^{-s} - a_{h+1}^{-s}\} + \gamma_n a_n^{-s} - \gamma_{m-1}a_m^{-s}.\end{aligned}$$

Thus $\big|M^{-1}\sum_{h=m}^{n} a_h^{-s}\mathbf{e}(ht)\big| \leq a_m^{-s} + a_n^{-s} + \sum_{h=m}^{n-1}(a_h^{-s} - a_{h+1}^{-s}) \leq 2a_m^{-s}$, which proves the desired uniform convergence of $\sum_{n=1}^{\infty} a_n^{-s}\mathbf{e}((\alpha+n)t)$, as $|\mathbf{e}(\alpha t)|$ is bounded for $t \in T$. The case with $\alpha - n$ in place of $\alpha + n$ can be handled in the same way. To prove (ii), we replace γ_n in the above proof by $\gamma_n' = \sum_{h=1}^{n}\chi(h)$. We easily see that the $|\gamma_n'|$ are bounded for all n, and we obtain the desired fact by the same technique.

To complete the proof of Theorem 4.2, we consider the series

$$\sum_{n=1}^{\infty}(n+\alpha)^{-2}\mathbf{e}((n+\alpha)t)+\alpha^{-2}\mathbf{e}(\alpha t)+\sum_{n=1}^{\infty}(n-\alpha)^{-2}\mathbf{e}((\alpha-n)t),$$
$$\sum_{n=1}^{\infty}(n+\alpha)^{-1}\mathbf{e}((n+\alpha)t)+\alpha^{-1}\mathbf{e}(\alpha t)-\sum_{n=1}^{\infty}(n-\alpha)^{-1}\mathbf{e}((\alpha-n)t).$$

Termwise application of d/dt to the first series produces $2\pi i$ times the second series. Since $(d/dt)E_{c,1}(t)=E_{c,0}(t)$, we thus obtain (4.5) for $E_{c,0}(t)$, provided such termwise differentiation is valid, which is indeed the case, as Lemma 4.3 shows that these series are uniformly convergent for $b \le t \le b'$ with constants b and b' such that $0<b<b'<1$. As for s, we simply take it to be 1 or 2. (Actually, the standard theorem on termwise differentiation requires the uniform convergence only for the latter series.)

4.4. Taking $t=0$ in (4.5) with $n>0$ and comparing the result with (2.16), we find that

$$P_{n+1}(x)=(x-1)^nE_{-x,n}(0) \qquad (n>0), \tag{4.6}$$

where x is an indeterminate. We can make it valid even when $n=0$ by putting $P_1(x)=1$. This combined with (4.3g) resp. (4.3b) proves (2.19) resp. (2.20).

4.5. The right-hand side of (4.4) has $\alpha^{-n-1}\mathbf{e}(\alpha t)$ as the term for $h=0$, which makes the sum meaningless if $\alpha=0$. However, removing that term, for $\alpha=0$ we have $\sum_{0\ne h\in\mathbf{Z}}h^{-n-1}\mathbf{e}(ht)$. We will now show that this sum can be handled by introducing the **Bernoulli numbers** B_n and **Bernoulli polynomials** $B_n(t)$ for $0\le n\in\mathbf{Z}$ as follows:

$$\frac{z}{e^z-1}=\sum_{n=0}^{\infty}\frac{B_n}{n!}z^n, \tag{4.7a}$$

$$\frac{ze^{tz}}{e^z-1}=\sum_{n=0}^{\infty}\frac{B_n(t)}{n!}z^n. \tag{4.7b}$$

We have, for every $n\ge 0$ unless stated otherwise,

$$B_n=B_n(0), \tag{4.8a}$$

$$\sum_{k=0}^{n-1}\binom{n}{k}B_k(t)=nt^{n-1} \qquad (n>0), \tag{4.8b}$$

$$(d/dt)B_n(t)=nB_{n-1}(t) \qquad (n>0), \tag{4.8c}$$

$$B_n(t+1)-B_n(t)=nt^{n-1}, \tag{4.8d}$$

$$B_{2m+1}=0 \qquad (0<m\in\mathbf{Z}), \tag{4.8e}$$

$$B_n(1-t)=(-1)^nB_n(t). \tag{4.8f}$$

These are well known, and can be proved in the same way as for (4.3a–f). Also, $B_n(t)$ is a polynomival in t of degree n with rational coefficients. For example, (4.8f) $B_0(t) = 1,\ B_1(t) = t - \frac{1}{2},\ B_2(t) = t^2 - t + \frac{1}{6},\ B_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t.$

Theorem 4.6. *For $1 \le n \in \mathbf{Z}$ and $0 < t < 1$ we have*

$$B_n(t) = -n!(2\pi i)^{-n} \sum_{0 \ne h \in \mathbf{Z}} h^{-n}\mathbf{e}(ht), \tag{4.9}$$

where $\sum_{0\ne h\in\mathbf{Z}}$ means $\lim_{m\to\infty}\sum_{0<|h|\le m}$ when $n = 1$. This formula is valid even for $0 \le t \le 1$ if $n > 1$.

Proof. This can be proved in the same manner as for Theorem 4.2. We consider

$$\int_S f(z)dz, \qquad f(z) = \frac{e^{tz}}{z^n(e^z - 1)},$$

where S is a square, with center at the origin, whose vertices are $A \pm iA$ and $-A \pm iA$, $A = (2N+1)\pi$ with $0 < N \in \mathbf{Z}$. The poles of f in $\mathbf{C}$ are $2\pi ih$ with $h \in \mathbf{Z}$. If $h \ne 0$, the residue of f at $2\pi ih$ is easily seen to be $(2\pi ih)^{-n}\mathbf{e}(th)$. From (4.7b) we see that the residue of f at 0 is $B_n(t)/n!$. Using the theorem of residues and making N large, we obtain (4.9) for $n > 1$ and $0 \le t \le 1$. Since $B_2'(t) = 2B_1(t)$, we obtain (4.9) for $B_1(t)$ by applying d/dt to the formula for $B_2(t)$, as we can justify termwise differentiation when $0 < t < 1$, by virtue of Lemma 4.3.

Now the main theorem on the values of $D^r_{a,N}(s)$ at integers can be stated as follows:

Theorem 4.7. *Given $0 < N \in \mathbf{Z}$, $a \in \mathbf{Z}$, and $r = 0$ or 1, define $D^r_{a,N}(s)$ by (3.11)* and put $\xi = \mathbf{e}(a/N)$. Then for $0 < m \in \mathbf{Z}$ the following assertions hold:

(i) *The quantities $(2\pi i)^{-2m}D^0_{a,N}(2m)$ and $(2\pi i)^{1-2m}D^1_{a,N}(2m-1)$ for $0 < a < N$ are numbers of $\mathbf{Q}(\xi)$ given as follows:*

$$(2m-1)!N^{2m}(2\pi i)^{-2m}D^0_{a,N}(2m) = \frac{\xi}{\xi-1}E_{-\xi,2m-1}(0) = \frac{\xi P_{2m}(\xi)}{(\xi-1)^{2m}}, \tag{4.10a}$$

$$\begin{aligned}&(2m-2)!N^{2m-1}(2\pi i)^{1-2m}D^1_{a,N}(2m-1)\\ &\quad = \begin{cases} \dfrac{\xi}{\xi-1}E_{-\xi,2m-2}(0) = \dfrac{\xi P_{2m-1}(\xi)}{(\xi-1)^{2m-1}} & (m > 1), \\ \dfrac{\xi+1}{2(\xi-1)} & (m = 1). \end{cases}\end{aligned} \tag{4.10b}$$

(ii) *If $\sigma \in \mathrm{Gal}\big(\mathbf{Q}(\zeta)/\mathbf{Q}\big)$ and $\mathbf{e}(1/N)^\sigma = \mathbf{e}(q/N)$ with $q \in \mathbf{Z}$, then*

$$\big\{(2\pi i)^{r-2m}D^r_{a,N}(2m-r)\big\}^\sigma = (2\pi i)^{r-2m}D^r_{qa,N}(2m-r). \tag{4.11}$$

(iii) *In the case $a = 0$ we have*

$$-(2m)!(2\pi i)^{-2m}N^{2m}D^0_{0,N}(2m) = B_{2m}. \tag{4.12}$$

(iv) *The quantity* $D^r_{a,N}(r+1-2m)$ *for* $0<a<N$ *is a rational number given by*

$$D^r_{a,N}(r+1-2m) = \frac{2}{r-2m}N^{2m-r-1}B_{2m-r}(a/N). \tag{4.13}$$

This is true even for $a=0$ *if* $2m-r>1$.

Remark. From (4.12) we see that $(-1)^{m+1}B_{2m}>0$.

PROOF. The first two formulas are already given in (3.9), as $D^r_{a,N}(2m-r) = M^{2m-r}_{a,N}(0)$ and $P_{n+1}(x) = (x-1)^n E_{-x,n}(0)$; see (3.12) and (4.6). We can also prove them by taking $\alpha = a/N$ and $t=0$ in (4.5), excluding the case involving $D^1_{a,N}(1)$. Formula (4.12) follows from (4.9) with $t=0$. Assertion (ii) can easily be seen from (4.10a, b). Putting $t=a/N$ in (4.9) and comparing the result with (3.16), we obtain (4.13).

4.8. For $1<N\in\mathbf{Z}$ let χ be a primitive or an imprimitive Dirichlet character modulo N. This means that χ is a homomorphism $(\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times$ which can be trivial. We put $\chi(a)=0$ for a not prime to N. We define, as usual, the Dirichlet L-function $L(s,\chi)$ by

$$L(s,\chi) = \sum_{n=1}^{\infty}\chi(n)n^{-s}. \tag{4.14}$$

The right-hand side is absolutely convergent for $\mathrm{Re}(s)>1$, and convergent even for $\mathrm{Re}(s)>0$ provided χ is nontrivial, by virtue of Lemma 4.3 (ii). Suppose $\chi(-1)=(-1)^r$ with $r=0$ or 1; then

$$2L(s,\chi) = \sum_{a=1}^{N}\chi(a)D^r_{a,N}(s) = \sum_{a=1}^{N}\chi(a)M^{-r}_{a,N}\big((s+r)/2\big), \tag{4.15}$$

and so $\Gamma\big((s+r)/2\big)L(s,\chi)$ can be continued as a meromorphic function to the whole s-plane with possible poles at $s=0$ and $s=1$, which occur only when χ is trivial.

Theorem 4.9. *Let* χ *and* N *be as in §4.8; let* k *be a positive integer such that* $\chi(-1)=(-1)^k$, $m=[(N-1)/2]$, *and* $\zeta=\mathbf{e}(1/N)$. *Then*

$$\begin{aligned} N^k(k-1)!(2\pi i)^{-k}L(k,\chi) &= \sum_{a=1}^{m}\chi(a)\frac{\zeta^a E_{-\zeta^a,k-1}(0)}{\zeta^a-1} \\ &= \sum_{a=1}^{m}\chi(a)\frac{\zeta^a P_k(\zeta^a)}{(\zeta^a-1)^k} \qquad (1<k\in\mathbf{Z}), \end{aligned} \tag{4.16a}$$

$$L(1,\chi) = \frac{\pi i}{N}\sum_{a=1}^{m}\chi(a)\frac{\zeta^a+1}{\zeta^a-1} \quad \text{if} \quad \chi(-1)=-1. \tag{4.16b}$$

Moreover, let μ_d denote the primitive character modulo d, where d is 3 or 4. Then we have, with $\zeta=\mathbf{e}(1/d)$,

$$\text{(4.17a)}\quad d^k(k-1)!(2\pi i)^{-k}L(k,\,\mu_d) = \frac{\zeta P_k(\zeta)}{(\zeta-1)^k} = \frac{\zeta E_{-\zeta,k-1}(0)}{\zeta-1} \quad (0<k-1\in 2\mathbf{Z}),$$

$$\text{(4.17b)}\qquad dL(1,\,\mu_d) = \pi i(\zeta+1)/(\zeta-1).$$

PROOF. Since $2L(k+2s,\,\chi) = \sum_{a=1}^{N-1}\chi(a)M^k_{a,N}(s)$ if $\chi(-1) = (-1)^k$, from (3.9) and (4.6) we obtain, for $k > 1$,

$$2N^k(k-1)!(2\pi i)^{-k}L(k,\,\chi) = \sum_{a=1}^{N-1}\chi(a)\frac{\zeta^a E_{-\zeta^a,k-1}(0)}{\zeta^a-1}.$$

Now $N = 2m+1$ or $2m+2$, and so the last sum can be written

$$\sum_{a=1}^{m}\left\{\chi(a)\frac{E_{-\zeta^a,k-1}(0)}{1-\zeta^{-a}} + \chi(N-a)\frac{E_{-\zeta^{-a},k-1}(0)}{1-\zeta^a}\right\},$$

as $\chi(m+1) = 0$ if N is even. Using (4.3g), we obtain the part of (4.16a) involving $E_{*,*}$, which combined with (4.6) gives the remaining part. We can prove (4.16b) in the same manner by means of (3.9) with $\nu = 1$. Formulas (4.17a, b) are mere special cases of (4.16a, b).

The formulas of the above theorem are different from the classical formulas for $L(k,\,\chi)$, which we will present in Theorem 4.12 below.

4.10. Take $N = 2, 3, 4$, or 6 and $a = 0$ or 1. Then we easily see that $D^0_{0,N}(s) = 2N^{-s}\zeta(s)$ and

$$\text{(4.18a)}\qquad D^0_{1,2}(s) = 2(1-2^{-s})\zeta(s),$$

$$\text{(4.18b)}\qquad D^0_{1,3}(s) = (1-3^{-s})\zeta(s),$$

$$\text{(4.18c)}\qquad D^0_{1,4}(s) = (1-2^{-s})\zeta(s),$$

$$\text{(4.18d)}\qquad D^0_{1,6}(s) = (1-2^{-s})(1-3^{-s})\zeta(s).$$

Thus $\zeta(2m)$ and $\zeta(1-2m)$ can be obtained from (4.10a) and (4.13). For instance, for $0 < k \in 2\mathbf{Z}$ we have

$$\begin{aligned}\text{(4.19a)}\quad (k/2)\zeta(1-k) &= \frac{2^{k-1}}{2^k-2}B_k(1/2) = \frac{3^{k-1}}{3^{k-1}-1}B_k(1/3)\\ &= \frac{4^{k-1}}{2^{k-1}-1}B_k(1/4) = \frac{-6^{k-1}}{(2^{k-1}-1)(3^{k-1}-1)}B_k(1/6),\end{aligned}$$

$$\begin{aligned}\text{(4.19b)}\qquad \frac{(k-1)!\zeta(k)}{(2\pi i)^k} &= \frac{-P_k(-1)}{2^{k+1}(2^k-1)} = \frac{\omega^2 P_k(\omega^2)}{(3^k-1)(\omega^2-1)^k}\\ &= \frac{iP_k(i)}{(4^k-2^k)(i-1)^k} = \frac{\omega P_k(\omega)}{(2^k-1)(3^k-1)(\omega-1)^k},\end{aligned}$$

where $\omega = \mathbf{e}(1/6)$. Comparing (4.19b) with (4.12), we obtain the part of the following formula concerning even k.

$$\text{(4.19c)}\qquad \frac{2P_k(i)}{(i-1)^k} = \begin{cases} k^{-1}i(4^k-2^k)B_k & (0<k\in 2\mathbf{Z}),\\ -E_{1,\,k-1} & (0<k-1\in 2\mathbf{Z}).\end{cases}$$

The part for odd k follows from (4.17a) and (4.30b) below.

4.11. Let us now treat the values of $L(k, \chi)$ first in the traditional way and then in several novel ways. We take a primitive Dirichlet character χ modulo d, and define the **Gauss sum** $G(\chi)$ by

$$G(\chi) = \sum_{a=1}^{d} \chi(a)\mathbf{e}(a/d). \tag{4.20}$$

The following properties of $G(\chi)$ are well known:

$$\sum_{a=1}^{d} \chi(a)\mathbf{e}(ab/d) = \overline{\chi}(b)G(\chi) \quad \text{for every } b \in \mathbf{Z}, \tag{4.21}$$

$$G(\chi)G(\overline{\chi}) = \chi(-1)d, \quad \overline{G(\chi)} = \chi(-1)G(\overline{\chi}), \quad |G(\chi)|^2 = d. \tag{4.22}$$

See [S71, Lemma 3.63], for example. Here and throughout the rest of this section d is a positive integer

From (3.7) or (3.15) we can easily derive the functional equation for $L(s, \chi)$. Namely, Let χ be a primitive character modulo d such that $\chi(-1) = (-1)^r$ with $r = 0$ or 1. Put

$$R(s, \chi) = (d/\pi)^{s/2}\Gamma\big((s+r)/2\big)L(s, \chi). \tag{4.23}$$

Then

$$R(s, \chi) = W(\chi)R(1-s, \overline{\chi}) \quad \text{with} \quad W(\chi) = i^{-r}d^{-1/2}G(\chi). \tag{4.24}$$

Indeed, from (3.7) and (4.21) we obtain, with $k = r + 1/2$,

$$\pi^{s-k}\Gamma(k-s)\sum_{a=1}^{d}\overline{\chi}(a)M_{a,d}^{-r}(k-s) = d^{2s-r-1}\pi^{-s}\Gamma(s)i^{-r}G(\overline{\chi})\sum_{b=1}^{d}\chi(b)M_{b,d}^{-r}(s).$$

Substituting $(s+r)/2$ for s and employing (4.22), we obtain (4.24).

Theorem 4.12. (i) *Let $\zeta(s)$ denote the Riemann zeta function. Then for $0 < n \in 2\mathbf{Z}$ we have*

$$2 \cdot n!(2\pi i)^{-n}\zeta(n) = n\zeta(1-n) = -B_n. \tag{4.25}$$

(ii) *Let χ be a nontrivial primitive Dirichlet character modulo d, and let k be a positive integer such that $\chi(-1) = (-1)^k$. Then*

$$2 \cdot k!(2\pi i)^{-k}G(\overline{\chi})L(k, \chi) = kd^{1-k}L(1-k, \overline{\chi}) = -\sum_{a=1}^{d-1}\overline{\chi}(a)B_k(a/d). \tag{4.26}$$

In particular, if $k = 1$ and $\chi(-1) = -1$, then

$$(\pi i)^{-1}G(\chi)L(1, \overline{\chi}) = L(0, \chi) = -\sum_{a=1}^{d-1}\chi(a)a/d. \tag{4.27}$$

Proof. Formula (4.25) can be obtained from (4.8a), (4.12), and (4.13) by taking $N = 1$ and $r = a = 0$, as $D_{0,1}^{0}(s) = 2\zeta(s)$. As for (4.26), by (4.21) and (4.9) we have

$$\begin{aligned}2G(\overline{\chi})L(k,\,\chi) &= \sum_{0\neq h\in\mathbf{Z}} h^{-k}G(\overline{\chi})\chi(h)\\ &= \sum_{0\neq h\in\mathbf{Z}}\sum_{a=1}^{d} h^{-k}\overline{\chi}(a)\mathbf{e}(ha/d) = -\frac{(2\pi i)^k}{k!}\sum_{a=1}^{d}\overline{\chi}(a)B_k(a/d).\end{aligned}$$

If $k=1$, we need to show that $2L(1,\,\chi)=\lim_{m\to\infty}\sum_{0\neq|h|\leq m}\chi(h)h^{-1}$, which follows from the uniform convergence of $\sum_{h=1}^{\infty}\chi(h)h^{-s}$ for $s>1/2$, as guaranteed by Lemma 4.3 (ii). The value $L(1-k,\,\overline{\chi})$ can be obtained by considering $\sum_{a=1}^{d}\overline{\chi}(a)D^{r}_{a,d}(r+1-2m)$ and employing (4.13) and (4.15) with $N=d$ and $k=2m-r$. It also follows from (4.24). Formula (4.27) is merely a special case of (4.26), as $B_1(t)=t-1/2$ and $\sum_{a=1}^{d-1}\overline{\chi}(a)=0$.

Let us insert here a few comments. The first one is historical. Formula (4.25) is due to Euler. Dirichlet proved (4.27) and even found a formula for $L(1,\,\chi)$ when $\chi(-1)=1$. As for (4.26), Hecke stated and proved it in [He40], and he is perhaps the first person who did so, though the fact was possibly known to other number-theorists, as (4.9) had been known since much earlier periods.

In any case, (4.26) is the well known standard formula for $L(k,\,\chi)$, but there is no reason for accepting it as the best or most important result. Indeed, we have a clear-cut formula (4.13), which we can view as more basic than (4.26). If the values of Dirichlet series are our main interest, the series $D^{r}_{a,N}(s)$ and the infinite sum of (4.4c) are most natural objects of study. In the same spirit we will present various new formulas for $L(k,\,\chi)$ which are different from (4.26) and are derived by the idea expressed by (4.4c).

We first prove

Lemma 4.13. (i) *Let χ be a nontrivial primitive Dirichlet character modulo d, and let k be a positive integer such that $\chi(-1)=(-1)^k$. Further let $q=[(d-1)/2]$. Then*

$$\sum_{a=1}^{d-1}\chi(a)B_n(a/d)=\begin{cases}2\sum_{a=1}^{q}\chi(a)B_n(a/d) & \text{if}\quad n=k,\\ 0 & \text{if}\quad n=k-1.\end{cases}\tag{4.28}$$

(ii) *In the setting of (i), suppose d is odd. Then*

$$\sum_{a=1}^{d-1}(-1)^a\chi(a)E_{1,n}(a/d)=\begin{cases}2\sum_{a=1}^{q}(-1)^a\chi(a)E_{1,n}(a/d) & \text{if}\quad n=k-1,\\ 0 & \text{if}\quad n=k.\end{cases}\tag{4.29}$$

(iii) *In the setting of (i), suppose $d=4d_0$ with $1<d_0\in\mathbf{Z}$. Then*

$$\sum_{a=1}^{q}\chi(a)E_{1,n}(2a/d)=\begin{cases}2\sum_{a=1}^{d_0-1}\chi(a)E_{1,n}(2a/d) & \text{if}\quad n=k-1,\\ 0 & \text{if}\quad n=k.\end{cases}\tag{4.30}$$

(iv) *For* χ, d, *and* q *as in (i), suppose* $\chi(-1) = 1$. *Then*

$$\sum_{a=1}^{q} \chi(a) = 0, \tag{4.31a}$$

$$\sum_{a=1}^{d-1} \chi(a)a = 0, \tag{4.31b}$$

$$\sum_{a=1}^{d-1} (-1)^a \chi(a) = 0 \quad \text{if} \quad d \notin 2\mathbf{Z}, \tag{4.31c}$$

$$\sum_{a=1}^{d-1} \chi(a)a^3 = (3d/2) \sum_{a=1}^{d-1} \chi(a)a^2. \tag{4.31d}$$

PROOF. We have $d = 2q + 1$ or $d = 2q + 2$ according as d is odd or even. If d is even, then $4|d$, and so $\chi(q + 1) = 0$. By (4.8f) we have $B_n\big((d - a)/d\big) = (-1)^n B_n(a/d)$, and so

$$\begin{aligned}\sum_{a=1}^{d-1} \chi(a)B_n(a/d) &= \sum_{a=1}^{q} \chi(a)B_n(a/d) + \sum_{a=1}^{q} \chi(d - a)B_n\big((d - a)/d\big) \\ &= \sum_{a=1}^{q} \chi(a)\big\{B_n(a/d) + (-1)^{k+n}B_n(a/d)\big\}.\end{aligned}$$

This proves (4.28). We can similarly prove (4.29); the only difference is that we have $(-1)^{d+k+n}$ instead of $(-1)^{k+n}$. If $d = 4d_0 > 4$ as in (iii), we have $q = 2d_0 - 1$, and so

$$\sum_{a=1}^{q} \chi(a)E_{1,n}(2a/d) = \sum_{a=1}^{d_0-1} \Big\{\chi(a)E_{1,n}(2a/d) + \chi(2d_0 - a)E_{1,n}\big(2(2d_0 - a)/d\big)\Big\},$$

as $\chi(d_0) = 0$. Since $2d_0a - 2d_0 \in d\mathbf{Z}$ for odd a, we have $\chi(2d_0 - a) = \chi(2d_0a - a) = \chi(-a)\chi(1 - 2d_0)$. Thus by (4.3f) the last sum equals

$$\big\{1 + (-1)^{n+k}\chi(1 - 2d_0)\big\} \sum_{a=1}^{d_0-1} \chi(a)E_{1,n}(2a/d).$$

By Lemma 1.12 we have $\chi(1 - 2d_0) = -1$. Thus we obtain (4.30). Finally suppose $\chi(-1) = 1$. Then $\chi(d - a) = \chi(a)$, and so

$$0 = \sum_{a=1}^{d-1} \chi(a) = \sum_{a=1}^{q} \chi(a) + \sum_{a=1}^{q} \chi(d - a) = 2\sum_{a=1}^{q} \chi(a),$$

which proves (4.31a). (Notice that $\chi(q + 1) = 0$ if d is even.) Therefore

$$\sum_{a=1}^{d-1} \chi(a)a = \sum_{a=1}^{q} \chi(a)a + \sum_{a=1}^{q} \chi(d - a)(d - a) = d\sum_{a=1}^{q} \chi(a) = 0,$$

which is (4.31b). Similarly, employing (4.31a), we find that

$$\sum_{a=1}^{d-1}\chi(a)a^2 = \sum_{a=1}^{q}\chi(a)\bigl\{a^2+(d-a)^2\bigr\} = 2\sum_{a=1}^{q}\chi(a)(a^2-da),$$
$$\sum_{a=1}^{d-1}\chi(a)a^3 = \sum_{a=1}^{q}\chi(a)\bigl\{a^3+(d-a)^3\bigr\} = 3d\sum_{a=1}^{q}\chi(a)(a^2-da).$$

Formula (4.31d) follows from these two equalities. If d is odd,

$$\sum_{a=1}^{d-1}(-1)^a\chi(a) = \sum_{b=1}^{d-1}(-1)^{d-b}\chi(d-b) = -\sum_{b=1}^{d-1}(-1)^b\chi(b),$$

from which we obtain (4.31c).

We now state the principal results of this section, among which (4.32), (4.34), and (4.35) are most essential.

Theorem 4.14. (i) *Let χ be a nontrivial primitive Dirichlet character modulo d, and let k be a positive integer such that $\chi(-1)=(-1)^k$. Further let $q=[(d-1)/2]$. Then*

$$(4.32)\qquad (k-1)!(2\pi i)^{-k}G(\overline{\chi})L(k,\,\chi) = \frac{1}{2\bigl(2^k-\chi(2)\bigr)}\sum_{a=1}^{q}\overline{\chi}(a)E_{1,k-1}(2a/d),$$

$$(4.33)\qquad k!(2\pi i)^{-k}G(\overline{\chi})L(k,\,\chi) = -\sum_{a=1}^{q}\overline{\chi}(a)B_k(a/d).$$

(ii) *Suppose in particular $d=4d_0$ with $1<d_0\in\mathbf{Z}$. Then*

$$(4.34)\qquad (k-1)!(\pi i)^{-k}G(\overline{\chi})L(k,\,\chi) = \sum_{a=1}^{d_0-1}\overline{\chi}(a)E_{1,k-1}(2a/d).$$

(iii) *If d is odd, then*

$$(4.35)\qquad (k-1)!(2\pi i)^{-k}G(\overline{\chi})L(k,\,\chi) = \frac{\chi(2)}{2\bigl(2^k-\chi(2)\bigr)}\sum_{b=1}^{q}(-1)^b\overline{\chi}(b)E_{1,k-1}(b/d).$$

In particular, if $k=1$, $m=[q/2]$, and $n=[(q-1)/2]$, then we have

$$(4.36)\qquad (2\pi i)^{-1}G(\overline{\chi})L(1,\,\chi) = \begin{cases} \dfrac{1}{\{1+\chi(2)\}\{2-\chi(2)\}}\displaystyle\sum_{a=1}^{m}\overline{\chi}(a) & \text{if } \chi(2)\neq -1,\\[2ex] \dfrac{1}{\{1-\overline{\chi}(2)\}\{2-\chi(2)\}}\displaystyle\sum_{c=0}^{n}\overline{\chi}(2c+1) & \text{if } \chi(2)\neq 1.\end{cases}$$

(iv) *If μ_d denotes the primitive character modulo d with $d=3$ or 4, then for $k=2m+1$ with $0\le m\in\mathbf{Z}$ we have*

$$(4.37)\qquad k!(-1)^m(2\pi)^{-k}\sqrt{3}\,L(k,\,\mu_3) = -B_k(\tfrac{1}{3}) = \frac{k}{2(2^k+1)}E_{1,\,2m}(\tfrac{2}{3}),$$

$$k!(-4)^{-m}\pi^{-k}L(k,\,\mu_4) = -B_k(\tfrac{1}{4}) = 2^{-2k}kE_{1,\,2m}. \tag{4.38}$$

PROOF. In the proof of Lemma 4.13 we noted that $\chi(q+1)=0$ if $d\in 2\mathbf{Z}$. Therefore by (4.21) we have, for every $m\in\mathbf{Z}$,

$$\begin{aligned}G(\overline{\chi})\chi(m) &= \sum_{a=1}^{q}\overline{\chi}(a)\mathbf{e}(am/d) + \sum_{a=1}^{q}\overline{\chi}(d-a)\mathbf{e}((d-a)m/d)\\ &= \sum_{a=1}^{q}\overline{\chi}(a)\bigl\{\mathbf{e}(am/d)+(-1)^k\mathbf{e}(-am/d)\bigr\}.\end{aligned}$$

Take $n=k-1$, $\alpha=1/2$, and $t=2a/d$ with $1\le a\le q$ in (4.4) and (4.5). Then

$$E_{1,k-1}(2a/d) = 2(k-1)!(2\pi i)^{-k}2^k\sum_{m\ \mathrm{odd}} m^{-k}\mathbf{e}(ma/d). \tag{$*$}$$

On the other hand,

$$\begin{aligned}&2G(\overline{\chi})\bigl(1-\chi(2)2^{-k}\bigr)L(k,\,\chi) = G(\overline{\chi})\sum_{m\ \mathrm{odd}}\chi(m)m^{-k}\\ &= \sum_{a=1}^{q}\overline{\chi}(a)\biggl\{\sum_{m\ \mathrm{odd}} m^{-k}\mathbf{e}(am/d) + \sum_{m\ \mathrm{odd}}(-m)^{-k}\mathbf{e}(-am/d)\biggr\}\\ &= 2\sum_{a=1}^{q}\overline{\chi}(a)\sum_{m\ \mathrm{odd}} m^{-k}\mathbf{e}(am/d).\end{aligned}$$

Combining this with $(*)$, we obtain (4.32). We have to be careful about the case $k=1$, but Lemma 4.3 settles the technical point as explained in the proof of Theorem 4.12. Formula (4.33) follows immediately from (4.26) combined with (4.28).

If $d=4d_0>4$, then $q=2d_0-1$, and we have

$$\sum_{a=1}^{q}\overline{\chi}(a)E_{1,k-1}(2a/d) = \bigl\{1-\overline{\chi}(1-2d_0)\bigr\}\sum_{a=1}^{d_0-1}\overline{\chi}(a)E_{1,k-1}(2a/d),$$

as shown in the proof of Lemma 4.13. (Take $n=k-1$.) Since $(1-2d_0)^2-1\in d\mathbf{Z}$, we have $\chi(1-2d_0)=\pm1$. If $\chi(1-2d_0)=1$, then (4.32) shows that $L(k,\,\chi)=0$, a contradiction. Thus $\chi(1-2d_0)=-1$, and we obtain (4.34) from (4.32), as $\chi(2)=0$. (We already stated the equality $\chi(1-2d_0)=-1$ in Lemma 1.12 and gave an elementary proof.)

As for (iii), putting $m=[q/2]$ and $n=[(q-1)/2]$, we have clearly

$$\overline{\chi}(2)\sum_{a=1}^{q}\overline{\chi}(a)E_{1,k-1}(2a/d) = \sum_{a=1}^{m}\overline{\chi}(2a)E_{1,k-1}(2a/d) + \sum_{a=m+1}^{q}\overline{\chi}(2a)E_{1,k-1}(2a/d).$$

For $m<a\le q$ put $c=q-a$. Then $2a=d-2c-1$, and so $\chi(2a)=\chi(-2c-1)$ and $E_{1,k-1}(2a/d)=(-1)^{k-1}E_{1,k-1}((2c+1)/d)$ by (4.3f). Thus the last sum $\sum_{a=m+1}^{q}$ can be written $-\sum_{c=0}^{n}\overline{\chi}(2c+1)E_{1,k-1}((2c+1)/d)$. Therefore we obtain (4.35). If $k=1$, we recall that $E_{1,0}(t)=1$. We consider (4.32) and $\overline{\chi}(2)$ times (4.35). Taking the sum and difference of these two equalities, we obtain (4.36).

Finally, taking $d = 3$ and 4 in (4.32) and (4.33), we obtain (4.37) and (4.38), as $E_{1,n} = 2^n E_{1,n}(1/2)$. This completes the proof.

We note here an easy fact. Taking $c = 1$ and $t = 0$ in (4.5), we obtain

$$4(2^k - 1)(k-1)!(2\pi i)^{-k}\zeta(k) = E_{1,\,k-1}(0) \qquad (0 < k \in 2\mathbf{Z}). \tag{4.39}$$

Since $P_k(-1) = (-2)^{k-1}E_{1,\,k-1}(0)$, (4.39) is essentially the same as the first equality of (4.19b). Comparing this with (4.25), we obtain

$$2(1 - 2^k)B_k = kE_{1,\,k-1}(0) = (-2)^{1-k}kP_k(-1) \qquad (0 < k \in 2\mathbf{Z}). \tag{4.40}$$

The relation between $L(2m+1,\,\mu_4)$ and the Euler number $E_{1,2m}$ as stated in (4.38) is classical and perhaps due to Euler. Otherwise, the equalities in the above theorem seem to be new, except for (4.33) and (4.37), which may possibly be known. In particular, taking χ to be real and $k = 1$ in (4.32), we obtain a well known class number formula for an imaginary quadratic field, which we will state in (5.9) below. Thus (4.32) includes such a classical result as a special case, but apparently it has never been stated in that general form even when $k = 1$. Notice that for $d = 8$ or 12, the right-hand side of (4.34) contains only one nonvanishing term, and so we can state formulas similar to (4.37) and (4.38) in such cases. We will not give their explicit forms here, since they are included in (6.3) and (6.5) below as special cases. It should also be noted that $E_{c,\,k-1}(t)$ is a polynomial in t of degree $k-1$, while $B_k(t)$ is of degree k.

Since $E_{1,0}(t) = 1$, the formulas of Theorem 4.14 involving $E_{1,k-1}$ have simple forms if $k = 1$. Another simple formula is (4.27). In general, $E_{1,k-1}(t)$ and $B_k(t)$ have more terms, and the matter is not so simple. However, a similar simplification is feasible at least to the extent described in the following theorem, which gives recurrence formulas for $L(k,\,\chi)$ modified by elementary factors.

Theorem 4.15. *Let χ be a nontrivial primitive Dirichlet character of conductor d and k a positive integer such that $\chi(-1) = (-1)^k$; let $q = [(d-1)/2]$ and $m = [(k-1)/2]$.*

(i) *Define $A(k,\,\chi)$ by*

$$A(k,\,\chi) = 2\cdot k!(2\pi i)^{-k}G(\overline{\chi})L(k,\,\chi). \tag{4.41}$$

Then

$$A(k,\,\chi) = \frac{-1}{d^k}\sum_{a=1}^{d-1}\overline{\chi}(a)a^k - \frac{1}{k+1}\sum_{\nu=1}^{m}\binom{k+1}{2\nu+1}A(k-2\nu,\,\chi). \tag{4.42}$$

(ii) *Suppose d is odd or $d > 4$. Define $\Lambda(k,\,\chi)$ by*

$$\Lambda(k,\,\chi) = \begin{cases} 4\{\overline{\chi}(2) - 2^{-k}\}(k-1)!(\pi i)^{-k}G(\overline{\chi})L(k,\,\chi) & \text{if} \quad d \notin 2\mathbf{Z}, \\ 2(k-1)!(\pi i)^{-k}G(\overline{\chi})L(k,\,\chi) & \text{if} \quad d \in 2\mathbf{Z}. \end{cases} \tag{4.43}$$

Then

$$\Lambda(k,\,\chi) = -\frac{1}{2}\sum_{\nu=1}^{m}\binom{k-1}{2\nu}\Lambda(k-2\nu,\,\chi) + \begin{cases} \dfrac{1}{d^{k-1}}\displaystyle\sum_{a=1}^{d-1}(-1)^a\overline{\chi}(a)a^{k-1} & \text{if} \quad d \notin 2\mathbf{Z}, \\ \dfrac{2^{k-1}}{d^{k-1}}\displaystyle\sum_{a=1}^{q}\overline{\chi}(a)a^{k-1} & \text{if} \quad d \in 2\mathbf{Z}. \end{cases} \tag{4.44}$$

Proof. By (4.26), $A(k,\,\chi) = -\sum_{a=1}^{d-1}\overline{\chi}(a)B_k(a/d)$, and by (4.8b) we have

$$(k+1)B_k(t) = (k+1)t^k - \sum_{\mu=0}^{k-1}\binom{k+1}{\mu}B_\mu(t).$$

By (4.28), $\sum_{a=1}^{d-1}\overline{\chi}(a)B_\mu(a/d) = 0$ if $\mu - k \notin 2\mathbf{Z}$. The sum is 0 also for $\mu = 0$, as $B_0(t) = 1$. Thus we obtain (4.42). Similarly we have, by (4.3b),

$$E_{1,k-1}(t) = t^{k-1} - \frac{1}{2}\sum_{\lambda=1}^{k-1}\binom{k-1}{\lambda-1}E_{1,\lambda-1}(t).$$

Therefore we obtain (4.44) by combining (4.34) and (4.35) with (4.30) and (4.29).

If $k = 2$, we can state clear-cut formulas as follows.

Corollary 4.16. *Let χ be a nontrivial Dirichlet character of conductor d such that $\chi(-1) = 1$ and let $q = [(d-1)/2]$. Then*

$$\pi^{-2}G(\overline{\chi})L(2,\,\chi) = \frac{4}{d\big(\chi(2)-4\big)}\sum_{a=1}^{q}\overline{\chi}(a)a, \tag{4.45}$$

$$\pi^{-2}G(\overline{\chi})L(2,\,\chi) = \frac{1}{d^2}\sum_{a=1}^{d-1}\overline{\chi}(a)a^2. \tag{4.46}$$

Proof. Take $k = 2$ in (4.32). Since $E_{1,1}(t) = t - 1/2$, the sum of (4.32) becomes $2\sum_{a=1}^{q}\overline{\chi}(a)a/d - \sum_{a=1}^{q}\overline{\chi}(a)/2$, and so we obtain (4.45) in view of (4.31a). Similarly take $k = 2$ in (4.26). Since $B_2(t) = t^2 - t + 1/6$, we obtain (4.46) in view of (4.31b). We can also take $k = 2$ in (4.44), but obtain nothing better than (4.45).

4.17. We end this section by mentioning a classical formula and its analogues. First, as an immediate consequence of (4.8d) we obtain

$$B_{n+1}(m+1) - B_{n+1} = (n+1)\sum_{k=0}^{m}k^n \qquad (0 \le n \in \mathbf{Z}), \tag{4.47}$$

which is often cited in connection with the Euler-Maclaurin formula. (We understand that $0^0 = 1$.) Likewise, from (4.3d) we can easily derive

$$E_{c,n}(0) - \mathbf{e}(-m\alpha)E_{c,n}(m) = \bigl\{1 - \mathbf{e}(-\alpha)\bigr\} \sum_{k=0}^{m-1} \mathbf{e}(-k\alpha)k^n, \tag{4.48}$$

where $c = -\mathbf{e}(\alpha)$, $\alpha \in \mathbf{R}$, $\notin \mathbf{Z}$. In particular, for $\alpha = 1/2$ we have

$$E_{1,n}(0) - (-1)^m E_{1,n}(m) = 2 \sum_{k=0}^{m-1} (-1)^k k^n \tag{4.49}$$

with the classical Euler polynomial $E_{1,n}(t)$. Apparently no previous researchers took the trouble of stating this rather nice-looking formula. Notice that (4.48) and (4.49) are of degree n in the parameter m, whereas (4.47) is of degree $n+1$.

We will present some more formulas for $L(k,\, \chi)$ in Section 6, but before doing so, we first consider applications of those we already have to certain class number formulas.

5. The class number of a cyclotomic field

5.1. The value $L(1,\, \chi)$ is closely related to the class number of a cyclotomic field, an imaginary quadratic field in particular. Let us now discuss this topic in the easiest cases and present some new class number formulas, assuming that the reader is familiar with some basic facts on cyclotomic fields. For an algebraic number field M of finite degree we denote by $\mathfrak{r}_M$, D_M, R_M, ζ_M, h_M, and w_M the maximal order, discriminant, regulator, Dedekind zeta function, class number of M and the number of roots of unity in M. It is a well known result of Dedekind that

$$\lim_{s\to 1}(s-1)\zeta_M(s) = \frac{2^{r_1}(2\pi)^{r_2} R_M h_M}{w_M |D_M|^{1/2}}, \tag{5.1}$$

where r_1 resp. r_2 is the number of real resp. imaginary archimedean primes of M. We apply this to a subfield of $\mathbf{Q}_{\mathrm{ab}}$. Any subfield M of $\mathbf{Q}_{\mathrm{ab}}$ is either totally real or totally imaginary. In the former case, $r_1 = [M : \mathbf{Q}]$ and $r_2 = 0$; in the latter case $r_1 = 0$ and $r_2 = [M : \mathbf{Q}]/2$.

We now fix a totally imaginary finite extension K of $\mathbf{Q}$ contained in $\mathbf{Q}_{\mathrm{ab}}$ and put $F = \bigl\{x \in K \,\big|\, x^\rho = x\bigr\}$, where ρ is the restriction of complex conjugation to K. Then F is totally real, $w_F = 2$, and $[K : F] = 2$. Let $[K : \mathbf{Q}] = 2t$. From (5.1) we obtain

$$(\zeta_K/\zeta_F)(1) = \frac{2\pi^t}{w_K} \cdot \frac{h_K}{h_F} \cdot \frac{R_K}{R_F} \cdot \left|\frac{D_F}{D_K}\right|^{1/2}. \tag{5.2}$$

Suppose $K \subset \mathbf{Q}(\zeta)$ with $\zeta = \mathbf{e}(1/m)$, where m is a positive integer that is either odd or divisible by 4. Then $\mathrm{Gal}\bigl(\mathbf{Q}(\zeta)/\mathbf{Q}\bigr)$ is isomorphic to $(\mathbf{Z}/m\mathbf{Z})^\times$, and $\mathrm{Gal}\bigl(\mathbf{Q}(\zeta)/K\bigr)$ to a subgroup H of $(\mathbf{Z}/m\mathbf{Z})^\times$. We can show that

$$\zeta_K(s) = \zeta_F(s) \prod_{\chi\in X} L(s,\, \chi), \tag{5.3}$$

where X is the set of all primitive Dirichlet characters χ such that $\chi(-1) = -1$, the conductor of χ divides m, and $\chi(H) = 1$. Thus we obtain the following formula:

$$\frac{h_K}{h_F} = \frac{w_K}{2\pi^t} \cdot \frac{R_F}{R_K} \cdot \left|\frac{D_K}{D_F}\right|^{1/2} \prod_{\chi \in X} L(1, \chi). \tag{5.4}$$

In the simplest case we take K to be an imaginary quadratic field of discriminant $-d$. Then $\zeta_K(s) = L(s, \chi)\zeta(s)$, where χ is a unique real Dirichlet character of conductor d such that $\chi(-1) = -1$, that is, $\chi(a) = \left(\dfrac{-d}{a}\right)$, and (5.4) takes the form

$$h_K = \frac{w_K\sqrt{d}}{2\pi} L(1, \chi). \tag{5.5}$$

This formula was first proved by Dirichlet in a somewhat different context. The functional equations of $\zeta(s)$ and $\zeta_K(s)$ show that $R(s, \chi) = R(1-s, \chi)$, which together with (4.24) proves that

$$G(\chi) = i\sqrt{d} \quad \text{if} \quad \chi(a) = \left(\frac{-d}{a}\right). \tag{5.6}$$

This argument, due to Hecke, is applicable also to a quadratic character χ such that $\chi(-1) = 1$, and even to any Hecke character of a number field corresponding to a quadratic extension; see [S97, (A6.3.4), (A6.4.1)]. Combining (4.27), (5.5), and (5.6), we obtain a classical formula

$$h_K = -\frac{w_K}{2d} \sum_{a=1}^{d} \chi(a)a. \tag{5.7}$$

5.2. Let χ be a primitive (not necessarily real) Dirichlet character of conductor d such that $\chi(-1) = -1$. Since $E_{1,0}(t) = 1$, from (4.32) with $k = 1$ we obtain

$$G(\overline{\chi})L(1, \chi) = \frac{\pi i}{2 - \chi(2)} \sum_{a=1}^{q} \overline{\chi}(a), \quad q = [(d-1)/2]. \tag{5.8}$$

Comparing this with (4.27), we obtain

$$\sum_{a=1}^{d} \chi(a)a = \frac{d}{\overline{\chi}(2) - 2} \sum_{a=1}^{q} \chi(a). \tag{5.8a}$$

In particular, if χ is real and $K = \mathbf{Q}(\sqrt{-d}\,)$, then (5.5) combined with (5.6) and (5.8) shows that

$$h_K = \frac{w_K}{2\big(2 - \chi(2)\big)} \sum_{a=1}^{q} \chi(a), \quad q = [(d-1)/2], \tag{5.9}$$

where the symbols are the same as in (5.5) and (5.7). This formula is also classical. Thus (4.32) or its special case (5.8) can be viewed as a generalization of (5.9). We will present several new formulas for h_K in Corollary 6.4 below.

5.3. We now consider the field $K = \mathbf{Q}(\zeta)$ with $\zeta = \mathbf{e}(1/m)$, $0 < m \in \mathbf{Z}$. We will soon specialize this to the case where m is a prime power, but first recall some basic facts on K under the assumption that m is either odd or divisible by 4. Let $F = \{x \in K \mid x^\rho = x\}$ as in §5.1. We put

$$t = [K : \mathbf{Q}]/2. \tag{5.10}$$

Here are some basic facts that can be found in most textbooks on algebraic number theory:

(5.11a) *$w_K = 2m$ if m is odd, and $w_K = m$ if $4|m$.*

(5.11b) *A prime number p is ramified in K if and only if $p|m$.*

(5.11c) *For a prime number p not dividing m, let f be the smallest positive integer such that $p^f - 1 \in m\mathbf{Z}$ and let $g = 2t/f$. Then p splits into exactly g prime ideals in K each of which has degree f.*

(5.11d) $\mathfrak{r}_K = \mathbf{Z}[\zeta]$.

(5.11e) $h_K/h_F \in \mathbf{Z}$.

To prove the last statement, take the maximal unramified abelian extension J of F. Since K is ramified over F at every archimedean prime, we see that $K \not\subset J$. Thus $h_F = [J : F] = [JK : K]$, which divides h_K, as JK is an unramified abelian extension of K. This proves (5.11e).

We now assume that $m = \ell^r$ with a prime ℓ and $0 < r \in \mathbf{Z}$; $r > 1$ if $\ell = 2$. Then $2t = \ell^{r-1}(\ell - 1)$ and the following statements hold:

(5.12a) $(1-\zeta)\mathfrak{r}_K$ *is a prime ideal in K and* $\ell\mathfrak{r}_K = (1-\zeta)^{2t}\mathfrak{r}_K$.

(5.12b) $|D_K| = \ell^e$, *where* $e = r\ell^r - (r+1)\ell^{r-1})$ *if* $\ell \neq 2$, *and* $e = (r-1)2^{r-1}$ *if* $\ell = 2$; $|D_F| = \ell^{(e-1)/2}$ *if* $\ell \neq 2$, *and* $|D_F| = 2^{(e-2)/2}$ *if* $\ell = 2$.

Lemma 5.4. *If m is a prime power, then $\mathfrak{r}_K^\times = W\mathfrak{r}_F^\times$, where W is the set of all roots of unity in K. Consequently, $R_K = 2^{t-1}R_F$.*

Proof. For $\alpha \in \mathfrak{r}_K^\times$ let $\beta = \alpha^\rho/\alpha$. Then $|\beta^\sigma| = 1$ for every $\sigma \in \mathrm{Gal}(K/\mathbf{Q})$. By Kronecker's theorem, $\beta \in W$. Suppose m is odd. Then $\beta = \varepsilon\gamma^2$ with $\gamma \in W$ and $\varepsilon = \pm 1$. If $\varepsilon = -1$, we have $(\alpha\gamma)^\rho = -\alpha\gamma$, and so $2\alpha\gamma = \alpha\gamma - (\alpha\gamma)^\rho \in d(K/F)$, where $d(K/F)$ is the different of K relative to F. This is a contradiction, since $d(K/F)$ is nontrivial (as can be seen from (5.12a)) and prime to 2. Thus $\varepsilon = 1$, and so $(\alpha\gamma)^\rho = \alpha\gamma$, which shows that $\alpha\gamma \in F$. Therefore $\alpha \in W\mathfrak{r}_F$ as expected. Suppose $m = 2^r$; let $n = 2^{r-1}$ and $\beta = \zeta^{-a}$ with $a \in \mathbf{Z}$. Then $\zeta^n = -1$. If a is even, we can put $\zeta = \gamma^2$ with $\gamma \in W$, which together with the above argument leads to the desired conclusion; so assume a to be odd. Since $\mathfrak{r}_K = \mathbf{Z}[\zeta]$, we have $\alpha = \sum_{s=0}^{n-1} c_s\zeta^s$ with $c_s \in \mathbf{Z}$. Then $\sum c_s\zeta^s = \sum c_s\zeta^{a-s}$. Hence $c_s = \pm c_{s'}$ if $s + s' \equiv a \pmod{n}$. Now the elements ζ^s and ζ^{a-s} for positive odd $s < n$ form a basis of K over $\mathbf{Q}$. In other words, $\zeta^{s'}$ with even s' can be replaced by ζ^{a-s} with some odd $s < n$. Therefore $\alpha = \sum_{s \text{ odd}} c_s\left(\zeta^s \pm \zeta^{a-s}\right)$. Now $\zeta^s \pm \zeta^{a-s} = \zeta^s(1-\zeta^q)$ with some q, and this is divisible by $1-\zeta$, which is a contradiction, since α is a unit. Once we know that $\mathfrak{r}_K^\times = W\mathfrak{r}_F^\times$, the assertion concerning R_K and R_F easily follows from the definition of the regulator.

Lemma 5.5. *Let $m = \ell^r$ with a prime number ℓ and $0 < r \in \mathbf{Z}$; assume $r > 3$ if $\ell = 2$. Let X be the set of all primitive Dirichlet characters χ such that $\chi(-1) = -1$ and the conductor of χ divides m. Then*

$$\ell \neq 2: \quad \prod_{\chi\in X} G(\chi) = i^t\ell^{(e+1)/4}, \quad e = r\ell^r - (r+1)\ell^{r-1},$$

$$\ell = 2: \quad \prod_{\chi\in X} G(\chi) = \sqrt{2}\cdot 2^c, \quad c = (r-1)2^{r-3}.$$

Proof. We can put $X = \bigsqcup_{s=1}^r X_s$, where X_s is the set of all $\chi \in X$ with conductor ℓ^s. Put $e_s = \#(X_s)$.

(I) We first consider the case of odd ℓ. We easily see that $e_1 = (\ell-1)/2$ and $e_s = \ell^{s-2}(\ell-1)^2/2$ if $s > 1$. For $\chi \in X$ we have $\chi = \overline{\chi}$ only if $\ell + 1 \in 4\mathbf{Z}$, in which case there is exactly one such χ, which belongs to X_1. Thus $\chi \neq \overline{\chi}$ for $\chi \in X_s$ if $s > 1$ or $\ell - 1 \in 4\mathbf{Z}$. Therefore by (4.22) we have

$$\prod_{\chi\in X_s} G(\chi) = (-\ell^s)^{e_s/2} \quad \text{if} \quad s > 1 \quad \text{or} \quad \ell - 1 \in 4\mathbf{Z}. \tag{5.13}$$

If $\ell + 1 \in 4\mathbf{Z}$, we have, by (5.6), $\prod_{\chi\in X_1} G(\chi) = i\ell^{1/2}(-\ell)^{(\ell-3)/4}$, which can be written

$$\prod_{\chi\in X_1} G(\chi) = (i\ell^{1/2})^{(\ell-1)/2}. \tag{5.14}$$

This is true also when $\ell - 1 \in 4\mathbf{Z}$. To simplify our notation, put $q = (\ell-1)/2$. Then from (5.13) and (5.14) we obtain $\prod_{\chi\in X} G(\chi) = i^a\ell^{b/2}$ with

$$a = q + \sum_{s=2}^r e_s \quad \text{and} \quad b = q + \sum_{s=2}^r se_s.$$

We easily find that $a = q\ell^{r-1} = t$. To calculate b, we note an elementary equality

$$\sum_{n=1}^k nx^n = \frac{kx^{k+2} - (k+1)x^{k+1} + x}{(x-1)^2}, \tag{5.15}$$

where x is an indeterminate. Now $\sum_{s=2}^r s\ell^{s-2} = 2\sum_{s=2}^r \ell^{s-2} + \sum_{s=2}^r (s-2)\ell^{s-2}$. Applying (5.15) to the last sum, we eventually find that $2b = r\ell^r - (r+1)\ell^{r-1} + 1$.

(II) Next we take $\ell = 2$. Then $e_1 = 0$, $e_2 = 1$, and $e_s = 2^{s-3}$ if $s > 2$. The set X_2 resp. X_3 consists of a real character of conductor 4 resp. 8; otherwise $\chi \neq \overline{\chi}$. Then by the same type of reasoning as for odd ℓ, we find the result as stated in our lemma.

Lemma 5.6. *Let $m = \ell^r$ as in Lemma 5.5, and let p be a prime number other than ℓ. Let Z be the subgroup of $(\mathbf{Z}/m\mathbf{Z})^\times$ generated by p and let $f = [Z:1]$. (In other words, f is the smallest positive integer such that $p^f - 1 \in m\mathbf{Z}$.) Then*

$$\prod_{\chi \in X} \left[1 - \chi(p)p^{-s}\right] = \begin{cases} (1 + p^{-fs/2})^{2t/f} & \text{if } \ -1 \in Z, \\ (1 - p^{-fs})^{t/f} & \text{if } \ -1 \notin Z. \end{cases} \tag{5.16}$$

Moreover, if ℓ is odd, then f is even if and only if $-1 \in Z$.

PROOF. The canonical isomorphism of $(\mathbf{Z}/m\mathbf{Z})^\times$ onto $\mathrm{Gal}(K/\mathbf{Q})$ sends -1 to ρ and Z to the decomposition group of p. Therefore the Euler p-factor of ζ_K is $(1-p^{-fs})^{2t/f}$. Let $\mathfrak{p}$ be the prime factor of p in K. Then $\mathfrak{p}^\rho = \mathfrak{p}$ if and only if $-1 \in Z$. Thus the Euler p-factor of ζ_F equals $(1-p^{-fs/2})^{2t/f}$ if $-1 \in Z$, and $(1-p^{-fs})^{t/f}$ if $-1 \notin Z$. Taking the quotient ζ_K/ζ_F, we obtain (5.16). The last assertion follows from the fact that $(\mathbf{Z}/m\mathbf{Z})^\times$ is cyclic if ℓ is odd.

We now present formulas for h_k/h_F different from classical ones.

Theorem 5.7. *Let K and F be as in §5.3 with $m = \ell^r$, $0 < r \in \mathbf{Z}$, where ℓ is a prime number and $r > 2$ if $\ell = 2$. Let X be the set of all primitive Dirichlet characters χ such that $\chi(-1) = -1$ and the conductor of χ divides m. Then the following assertions hold:*

(i) *Suppose $\ell \neq 2$; let f be the smallest positive integer such that $2^f - 1 \in m\mathbf{Z}$. Put $A = (2^{f/2}+1)^{2t/f}$ or $(2^f - 1)^{t/f}$ according as f is even or odd. For each $\chi \in X$ of conductor ℓ^s put $q_\chi = (\ell^s - 1)/2$. Then we have*

$$\frac{h_K}{h_F} = 2^{1-t}\ell^r A^{-1} \prod_{\chi \in X} \left\{ \sum_{a=1}^{q_\chi} \chi(a) \right\}. \tag{5.17}$$

Moreover, the sum $\sum_{a=1}^{q_\chi} \chi(a)$ can be replaced by

$$\frac{2}{1+\overline{\chi}(2)} \sum_{a=1}^{k_\chi} \chi(a) \quad \text{with} \quad k_\chi = [q_\chi/2] \quad \text{if} \quad \chi(2) \neq -1, \tag{5.17a}$$

$$\frac{2}{1-\overline{\chi}(2)} \sum_{c=0}^{n_\chi} \chi(2c+1) \quad \text{with} \quad n_\chi = [(q_\chi - 1)/2] \quad \text{if} \quad \chi(2) \neq 1. \tag{5.17b}$$

(ii) *Suppose $\ell = 2$; let Y be the set of all $\chi \in X$ of conductor > 4. For $\chi \in Y$ of conductor 2^s, put $b_\chi = 2^{s-2}$. Then*

$$\frac{h_K}{h_F} = 2^{\gamma} \prod_{\chi\in Y} \left\{ \sum_{a=1}^{b_\chi - 1} \chi(a) \right\} \quad \textit{with} \quad \gamma = r - 1 - 2^{r-2}. \tag{5.18}$$

PROOF. We employ (5.4). We have $w_K = 2m$ if $\ell \neq 2$ and $w_K = m$ if $\ell = 2$; $R_F/R_K = 2^{1-t}$ as shown in Lemma 5.4; $|D_K/D_F|$ is given by (5.12b). We have to determine $\prod_{\chi\in X} L(1,\,\chi)$. Suppose $\ell \neq 2$. Then we use (5.8) which involves $2 - \chi(2)$ and the Gauss sum. As for the latter, we employ Lemma 5.5. As for the former, from Lemma 5.6 we obtain

$$\prod_{\chi\in X} \bigl[2 - \chi(2)\bigr] = \begin{cases} (2^{f/2}+1)^{2t/f} & \text{if} \quad f \in 2\mathbf{Z}, \\ (2^f - 1)^{t/f} & \text{if} \quad f \notin 2\mathbf{Z}. \end{cases} \tag{5.19}$$

Combining all these factors, we obtain (5.17). The sum $\sum_{a=1}^{q_\chi} \chi(a)$ can be replaced by (5.17a, b) by virtue of (4.36). If $\ell = 2$, we use (4.34) instead of (5.8). Special care must be given to the character of conductor 4, which is included in X but not in Y. Observing that c of Lemma 5.5 equals $e/4$ with e of (5.12b), we obtain (5.18) for $r > 3$. Actually we see that (5.18) is true even for $r = 2$ and 3.

Remark. For odd ℓ, it often happens that 2 is a primitive root modulo m, in which case we have $f = 2t$, and so $A = 2^t + 1$. Thus we can put $A = \ell^r I$ with a positive integer I, and the factor $\ell^r A^{-1}$ in (5.17) equals I^{-1}.

5.8. Let us now state the classical formulas for h_K/h_F that can be found in the standard literature on this topic:

$$\frac{h_K}{h_F} = B \prod_{\chi\in X} \left\{ -\sum_{a=1}^{c_\chi} \chi(a)a \right\} \quad \text{with} \tag{5.20}$$

$$B = \begin{cases} 2^{1-t}\ell^{\alpha}, & \alpha = r - 2^{-1}(r\ell^r - (r+1)\ell^{r-1} + 1) \quad \text{if} \quad \ell \neq 2, \\ 2^{\beta}, & \beta = r - 1 - 2^{r-2}r \quad \text{if} \quad \ell = 2. \end{cases}$$

Here c_χ is the conductor of χ. This can be obtained from (5.4) by applying (4.27) to $L(1,\,\chi)$ and employing Lemmas 5.4 and 5.5.

Clearly formulas (5.17) and (5.18) are of "smaller sizes" than (5.20). That is especially so for (5.18). As for (5.17), we derived it from (5.20) combined with (5.8a). Equality (5.8a) is an old well known fact at least for real χ. Even for non-real χ, it must have been known at least to some experts, but apparently nobody tried to state (5.17).

In stating formulas for h_K/h_F, we confined ourselves to the case where $K = \mathbf{Q}(\zeta)$, $\zeta = \mathbf{e}(1/m)$, with a prime power m. A formula of type (5.20) for more general cyclotomic fields is known; see [Ha]. We can of course state analogues of (5.17) and (5.18) for such fields, whose precise statements may be left to the reader.

6. Some more formulas for $L(k,\,\chi)$

6.1. Let us now state several sum expressions for $L(k,\,\chi)$ different from those of Section 4. The first type has fewer terms than the sum of (4.32), and are technically more complex than the second type, which follows from (4.5) rather easily. We present the first type as eight formulas depending on the nature of the Dirichlet character and the parameter c of the generalized Euler polynomial $E_{c,n}$. One formula, (6.2), is somewhat different from the other seven. Though it may be possible to state those seven as a single formula, we do not do so, as such would make it cumbersome and less easy to understand. In the following theorem, the product $\chi\lambda$ for two characters χ and λ means the character defined by $(\chi\lambda)(m)=\chi(m)\lambda(m)$, which can be imprimitive if the conductors of χ and λ are not relatively prime. We first prove

Lemma 6.2. *Let ε denote ± 1, and given two positive integers d and s, let $g=[d/s]+1$ or $g=-[d/s]$ according as $\varepsilon=1$ or -1. Then $0\le(\varepsilon/s)+(j/d)\le 1$ for every $j\in\mathbf{Z}$ such that $1-g\le j\le d-g$. Moreover, $0<(\varepsilon/s)+(j/d)<1$ for all such j's if $d/s\notin\mathbf{Z}$.*

PROOF. If $g=[d/s]+1$, then $g-1\le d/s<g$, and so $(g-1)/d\le 1/s<g/d$. For $1-g\le j\le d-g$ we have $(1-g)/d\le j/d\le 1-g/d$, and therefore $0\le(1/s)+(j/d)<1$. If $g=-[d/s]$, then $-g\le d/s<1-g$, and so $(g-1)/d<-1/s\le g/d$, and we obtain $0<(-1/s)+(j/d)\le 1$. If $d/s\notin\mathbf{Z}$, we easily see that $0<(\varepsilon/s)+(j/d)<1$ in both cases.

Theorem 6.3. *Let χ and λ be primitive Dirichlet characters and μ_3, μ_4 be as in Theorem 4.14 (iv); let d be the conductor of χ and k a positive integer; further let ε denote ± 1. Then the following assertions hold:*

(i) *Suppose d is odd, >3, and $\chi(-1)=(-1)^{k+1}$; suppose also $k>1$ if $3|d$; let $g=[d/3]+1$. Then*

$$\begin{aligned}(6.1)\qquad &\sum_{j=1-g}^{d-g}(-1)^j\overline{\chi}(j)E_{1,k-1}\big(\tfrac{1}{3}+\tfrac{j}{d}\big)\\ &\qquad=(k-1)!2\sqrt{-3}(\pi i)^{-k}\overline{\chi}(2)G(\overline{\chi})\big\{1+\chi(2)2^{-k}\big\}L(k,\,\chi\mu_3).\end{aligned}$$

(ii) *Suppose $d=2m+1$ with $0<m\in\mathbf{Z}$ and $\chi(-1)=(-1)^{k+1}$. Then*

$$(6.2)\qquad \sum_{j=1}^{m}(-1)^j\overline{\chi}(j)E_{1,k-1}\big(\tfrac{1}{2}+\tfrac{j}{d}\big)=(k-1)!2i(\pi i)^{-k}\overline{\chi}(2)G(\overline{\chi})L(k,\,\chi\mu_4).$$

(iii) *Suppose d is odd, $\lambda(m)=\big(\frac{\delta}{m}\big)$ with $\delta=\pm 8$, and $(\chi\lambda)(-1)=(-1)^k$. Then*

$$(6.3)\qquad \sum_{j=1-g}^{d-g}(-1)^j\overline{\chi}(j)E_{1,k-1}\big(\tfrac{1}{4}+\tfrac{j}{d}\big)=(k-1)!\sqrt{\delta}(\pi i)^{-k}\overline{\chi}(2)G(\overline{\chi})L(k,\,\chi\lambda),$$

where $g = [d/4] + 1$. *The formula is valid even for* $d = 1$ *and trivial* χ, *in which case the sum on the left-hand side is* $E_{1,k-1}(1/4)$ *and* $G(\overline{\chi}) = 1$.

(iv) *For* d, λ, χ, *and* k *of* (iii) *we have also*

$$\sum_{j=1-g}^{d-g} \mathbf{e}(-dj/4)\overline{\chi}(j)E_{-i,k-1}\left(\tfrac{1}{2} + \tfrac{j}{d}\right) = (k-1)!\sqrt{-2} \cdot 2^k(\pi i)^{-k}\overline{\chi}(4)G(\overline{\chi})L(k, \chi\lambda), \tag{6.4}$$

where $g = [d/2] + 1$. *The formula is valid even for* $d = 1$ *and trivial* χ, *in which case the sum on the left-hand side is* $E_{-i,k-1}(1/2)$ *and* $G(\overline{\chi}) = 1$.

(v) *Suppose* d *is odd,* > 3, $\lambda(m) = \left(\frac{3}{m}\right)$, *and* $\chi(-1) = (-1)^k$. *Then*

$$\sum_{j=1-g}^{d-g} (-1)^j\overline{\chi}(j)E_{1,k-1}\left(\tfrac{1}{6} + \tfrac{j}{d}\right) = (k-1)!2\sqrt{3}(\pi i)^{-k}\overline{\chi}(2)G(\overline{\chi})L(k, \chi\lambda), \tag{6.5}$$

where $g = [d/6] + 1$. *The formula is valid even for* $d = 1$ *and trivial* χ, *in which case* $G(\overline{\chi}) = 1$ *and the sum on the left-hand side is* $E_{1,k-1}(1/6)$.

(vi) *For* d *prime to 6 and* λ, χ, k *of* (v) *we have*

$$\sum_{j=1-g}^{d-g} \mathbf{e}(-dj/6)\overline{\chi}(j)E_{c,k-1}\left(\tfrac{1}{2} + \tfrac{j}{d}\right) = (k-1)! \cdot 3^k i(\pi i)^{-k}\overline{\chi}(6)G(\overline{\chi})L(k, \chi\lambda), \tag{6.6}$$

where $g = [d/2] + 1$ *and* $c = \mathbf{e}(2/3)$. *The formula is valid even for* $d = 1$ *and trivial* χ, *in which case* $G(\overline{\chi}) = 1$ *and the sum on the left-hand side is* $E_{c,k-1}(1/2)$.

(vii) *Suppose* d *is prime to 3; let* λ *be the Dirichlet character modulo 9 such that* $\lambda(2) = \mathbf{e}(2/3)$. *Then*

$$\sum_{j=1-g}^{d-g} \mathbf{e}(-dj/3)\overline{\chi}(j)E_{c,k-1}\left(\tfrac{\varepsilon}{3} + \tfrac{j}{d}\right) = (k-1)!\mathbf{e}(\varepsilon/9)\{1 + \mathbf{e}(1/6)\} \cdot (2\pi i)^{-k}3^k\overline{\chi}(3)G(\overline{\chi}) \cdot \begin{cases} L(k, \chi\lambda^{\varepsilon}) & \text{if } \chi(-1) = (-1)^k, \\ L(k, \mu_3\chi\lambda^{\varepsilon}) & \text{if } \chi(-1) = (-1)^{k+1}, \end{cases} \tag{6.7}$$

where $c = -\mathbf{e}(1/3)$, *and* $g = [d/3] + 1$ *or* $g = -[d/3]$ *according as* $\varepsilon = 1$ *or* $\varepsilon = -1$. *The formula is valid even for* $d = 1$ *and trivial* χ, *in which case* $G(\overline{\chi}) = 1$ *and the sum on the left-hand side is* $E_{c,k-1}(1/3)$ *if* $\varepsilon = 1$ *and* $E_{c,k-1}(2/3)$ *if* $\varepsilon = -1$.

(viii) *Suppose* d *is odd; let* λ *be the Dirichlet character modulo 16 such that* $\lambda(5) = i$ *and* $(\chi\lambda)(-1) = (-1)^k$. *Then*

$$\sum_{j=1-g}^{d-g} \mathbf{e}(-dj/4)\overline{\chi}(j)E_{-i,k-1}\left(\tfrac{\varepsilon}{4} + \tfrac{j}{d}\right) = (k-1)!(1+i)\mathbf{e}\left(\tfrac{\varepsilon}{16}\right) \cdot 2^k(\pi i)^{-k}\overline{\chi}(4)G(\overline{\chi})L(k, \chi\lambda^{\varepsilon}), \tag{6.8}$$

where $g = [d/4] + 1$ or $g = -[d/4]$ according as $\varepsilon = 1$ or $\varepsilon = -1$. The formula is valid even for $d = 1$ and trivial χ, in which case $G(\overline{\chi}) = 1$ and the sum on the left-hand side is $E_{-i,k-1}(1/4)$ if $\varepsilon = 1$ and $E_{-i,k-1}(3/4)$ if $\varepsilon = -1$. (Once χ and k are given, λ is uniquely determined by the conditions $\lambda(5) = i$ and $(\chi\lambda)(-1) = (-1)^k$.)

PROOF. We first consider Case (v), which has a feature that the other cases lack. We use formula (4.5) with $\alpha = 1/2$ and $t = (1/6) + (j/d)$ with j in the sum expression of (6.5). By Lemma 6.2 we have $0 < t < 1$, and so (4.5) is applicable to the present setting for $n \geq 0$. Put $j = pd + 2q$ with $p, q \in \mathbf{Z}$. Then for $h \in \mathbf{Z}$ we have

$$\mathbf{e}\big((h+\alpha)t\big) = \mathbf{e}(1/12)\mathbf{e}(h/6)\mathbf{e}(p/2)\mathbf{e}\big((2h+1)q/d\big).$$

Clearly $\mathbf{e}(p/2) = \mathbf{e}(j/2) = (-1)^j$. Therefore, by (4.5), the left-hand side of (6.5) equals

$$2\mathbf{e}(1/12)(k-1)!(\pi i)^{-k}\sum_{h\in\mathbf{Z}}(2h+1)^{-k}\mathbf{e}(h/6)\sum_j \overline{\chi}(j)\mathbf{e}\big((2h+1)q/d\big).$$

Since $\chi(j) = \chi(2q)$ and q runs over $\mathbf{Z}/d\mathbf{Z}$, the sum $\sum_j$ equals $\overline{\chi}(2)G(\overline{\chi})\chi(2h+1)$. Put $\sum_{h\in\mathbf{Z}}\mathbf{e}(h/6)\chi(2h+1)(2h+1)^{-k} = \sum_{n=1}^{\infty} b_n n^{-k}$. Since $\chi(-1) = (-1)^k$, we have $b_n = \chi(n)\big\{\mathbf{e}((n-1)/12) + \mathbf{e}((-n-1)/12)\big\}$. We easily see that this equals $\{1 + \mathbf{e}(5/6)\}\chi(n)\lambda(n)$. (Notice that $b_n = 0$ if $3|n$.) Therefore we obtain (6.5). If $d = 1$ and χ is trivial, we consider $E_{1,k-1}(1/6)$ and put $G(\overline{\chi}) = 1$. Then the above argument is valid in that case too. Also, if $k = 1$, we have to invoke Lemma 4.3 for the same reason as in the proof of Theorems 4.12 and 4.14.

The other cases except Case (ii) can be proven basically in the same fashion. We take $\alpha = 1/3$ in Case (vii), $\alpha = 1/6$ in Case (vi), $\alpha = 1/4$ in Cases (iv) and (viii), and $\alpha = 1/2$ in the remaining cases. We also take $j = pd + qr$ with $p, q \in \mathbf{Z}$, where $r = \alpha^{-1}$. If $3|d$ in Case (i), then $(1/3) + (j/d) = 0$ for $j = 1 - g$, and therefore (4.5) is applicable only to the case $k > 1$.

In Case (ii), by the same technique with $\alpha = r^{-1} = 1/2$ we first obtain

$$\sum_{j=-m}^{m}(-1)^j\overline{\chi}(j)E_{1,k-1}\big(\tfrac{1}{2}+\tfrac{j}{d}\big) = (k-1)!4i(\pi i)^{-k}\overline{\chi}(2)G(\overline{\chi})L(k,\,\chi\mu_4), \tag{6.9}$$

where $m = [d/2]$. The left-hand side can be written

$$\sum_{j=1}^{m}(-1)^j\overline{\chi}(j)\Big\{E_{1,k-1}\big(\tfrac{1}{2}+\tfrac{j}{d}\big) + \chi(-1)E_{1,k-1}\big(\tfrac{1}{2}-\tfrac{j}{d}\big)\Big\}.$$

By (4.3f) this equals twice the left-hand side of (6.2). Dividing by 2, we obtain (6.2).

Now some class number formulas different from (5.7) and (5.9) can be obtained by taking $k = 1$ and χ to be a real character in (4.34), (4.36), and also in the first six cases of the above theorem. Here are their explicit statements.

Corollary 6.4. *Let d_0 denote a positive squarefree integer > 1 and h_K the class number of the field K given in each case below. Except in Case (i) suppose d_0 is odd and let χ_0 be the real primitive Dirichlet character of conductor d_0. Then the following assertions hold.*

(i) *Suppose $2 < d_0 + 1 \notin 4\mathbf{Z}$ and $K = \mathbf{Q}(\sqrt{-d_0}\,)$; let χ be the primitive quadratic Dirichlet character of conductor $4d_0$ that corresponds to K, and $\nu = [d_0/2]$. Then*

$$h_K = \sum_{a=1}^{d_0-1} \chi(a) = \sum_{b=1}^{\nu} \chi(2b-1). \tag{6.10}$$

(ii) *Suppose $d_0 = 4\mu + 1$ with $0 < \mu \in \mathbf{Z}$ and let $K = \mathbf{Q}(\sqrt{-d_0}\,)$. Then*

$$h_K = \chi_0(2) \sum_{a=1}^{2\mu} (-1)^a \chi_0(a). \tag{6.11}$$

(iii) *Suppose $d_0 = 4\mu + 3$ with $0 < \mu \in \mathbf{Z}$ and let $K = \mathbf{Q}(\sqrt{-d_0}\,)$. Then*

$$h_K = \begin{cases} \displaystyle\sum_{a=1}^{\mu} \chi(a) & (\mu \notin 2\mathbf{Z}), \\ \displaystyle\frac{1}{3}\sum_{c=0}^{\mu} \chi(2c+1) & (\mu \in 2\mathbf{Z}). \end{cases} \tag{6.12}$$

(iv) *Let $K = \mathbf{Q}(\sqrt{-2d_0}\,)$ and $\mu = [d_0/4]$. Then*

$$h_K = 2\chi_0(2) \sum_{a=1}^{\mu} (-1)^a \chi_0(a) \quad \textit{if} \quad d_0 = 4\mu + 1, \tag{6.13}$$

$$h_K = 2\sum_{a=0}^{\mu} (-1)^a \chi_0(2a+1) = 2\chi_0(2) \sum_{b=\mu+1}^{2\mu+1} (-1)^b \chi_0(b) \ \ \textit{if} \ \ d_0 = 4\mu + 3. \tag{6.14}$$

(v) *Suppose that d_0 is prime to 3 and $d_0 + 1 \in 4\mathbf{Z}$; let $K = \mathbf{Q}(\sqrt{-3d_0}\,)$ and $m = [d_0/2]$. Then*

$$h_K = 2\chi_0(2) \Bigl\{ \sum_a \chi_0(a) - \sum_b \chi_0(b) \Bigr\}, \tag{6.15}$$

where $1 \le a \le m$, $1 \le b \le m$, $a \equiv 1, 2 \pmod 6$, and $b \equiv 4, 5 \pmod 6$.

(vi) *Let K and d_0 be the same as in (iv)* and let $m = [d_0/6]$. Then

$$h_K = 2\chi_0(2) \sum_{a=m+1}^{(d_0-1)/2} (-1)^a \chi_0(a). \tag{6.16}$$

(vii) *Suppose that d_0 is prime to 3 and $d_0 - 1 \in 4\mathbf{Z}$; let $K = \mathbf{Q}(\sqrt{-3d_0}\,)$ and $m = [d_0/3]$. Then*

$$h_K = \frac{2}{2\chi_0(2) + 1} \sum_{a=1}^{m} (-1)^a \chi_0(a). \tag{6.17}$$

PROOF. We first recall that $E_{c,0}(t) = 1$ as noted in (4.3h). Take $k = 1$ in (4.34). Then by (5.5) we obtain (6.10), as $\chi(a) = 0$ for even a. Formula (6.12) folllows immediately from (4.36) combined with (5.5). Next we take $k = 1$ and $(\chi_0,\, d_0)$ to be $(\chi,\, d)$ in Theorem 6.3. Let us first consider (6.3). Suppose $d_0 = 4\mu + 3$. Then $\chi_0(-1) = -1$, and the left-hand side of (6.3) equals

$$\sum_{a=-\mu}^{3\mu+2}(-1)^a\chi_0(a) = \sum_{a=-\mu}^{\mu}(-1)^a\chi_0(a) + \sum_{b=\mu+1}^{3\mu+2}(-1)^b\chi_0(b)$$
$$= \sum_{a=1}^{\mu}(-1)^a\bigl\{\chi_0(a) + \chi_0(-a)\bigr\} + \sum_{b=\mu+1}^{2\mu+1}\bigl\{(-1)^b\chi_0(b) + (-1)^{d_0-b}\chi_0(d_0-b)\bigr\}.$$

The first sum of the last line is 0; the second sum equals $2\sum_{b=\mu+1}^{2\mu+1}(-1)^b\chi_0(b)$. Putting $b = 2\mu + 1 - c$, we have $2b = d - 2c - 1$, and so the last sum equals $\chi_0(2)\sum_{c=0}^{\mu}(-1)^{c+1}\chi_0(d_0 - 2c - 1) = \chi_0(2)\sum_{c=0}^{\mu}\chi_0(2c+1)$. Applying (6.3) to the factor $L(1,\,\chi\lambda)$ of (6.3), we obtain (6.14). Taking $d_0 = 4\mu + 1$ in (6.3), we similarly obtain (6.13). These formulas (6.13) and (6.14) can be obtained also from (6.4).

Formula (6.11) follows directly from (6.2) combined with (5.5) applied to the present K. To prove (6.15), we use (6.6). Notice that $d_0 \equiv 7$ or $11 \pmod{12}$. Put $\omega = \mathbf{e}(1/6)$, and suppose $d_0 - 7 \in 12\mathbf{Z}$. Then the left-hand side of (6.6) can be written $\sum_{j=-m}^{m}\chi(j)\omega^{-j}$, which equals $\sum_{j=1}^{m}\chi(j)(\omega^{-j} - \omega^{j})$. We easily see that this equals

$$-i\sqrt{3}\Bigl\{\sum_a \chi_0(a) - \sum_b \chi_0(b)\Bigr\} \tag{6.18}$$

with a and b as in (v). Applying (5.5) to $L(1,\,\chi_0\lambda)$, we can verify that the right-hand side of (6.6) equals $i\sqrt{3}\chi_0(6)h_K/2$. If $d_0 - 11 \in 12\mathbf{Z}$, then ω must be replaced by ω^{-1}, and we have $i\sqrt{3}$ instead of $-i\sqrt{3}$ in (6.18). Now $\chi_0(3) = \bigl(\frac{3}{d_0}\bigr)$, which equals -1 or 1 according as $d_0 \equiv 7$ or $11 \pmod{12}$. Therefore we obtain (6.15) in both cases as expected.

We derive (6.16) from (6.5). Put $m = [d_0/6]$ and $d_0 = 2q + 1$. Then the left-hand side of (6.5) becomes $\sum_{a=-m}^{d_0-m-1}(-1)^a\chi_0(a)$. We have $\sum_{a=-m}^{m}(-1)^a\chi_0(a) = 0$ as $\chi_0(-1) = -1$, and so we only have to consider $\sum_{a=m+1}^{d_0-m-1}(-1)^a\chi_0(a)$, which equals

$$\sum_{a=m+1}^{q}\Bigl\{(-1)^a\chi_0(a) + (-1)^{d_0-a}\chi_0(d_0-a)\Bigr\} = 2\sum_{a=m+1}^{q}(-1)^a\chi_0(a).$$

Thus we obtain (6.16).

Finally, in the setting of (vii) we see that the left-hand side of (6.1) equals

$$\sum_{a=-m}^{d_0-m-1}(-1)^a\chi_0(a) = \sum_{a=-m}^{m}(-1)^a\chi_0(a) + \sum_{b=m+1}^{d_0-m-1}(-1)^b\chi_0(b).$$

The same argument as for (6.14) shows that the last sum over b vanishes, while the sum over $-m \le a \le m$ equals $2\sum_{a=1}^{m}(-1)^a\chi_0(a)$. Thus we obtain (6.17) and our proof is complete.

6.5. Though it may be possible to prove the formulas of the above corollary more directly, our point of presenting them is that they follow easily from a more general principle concerning $L(k,\,\chi)$ with $k \ge 1$. It should also be noted that they are quite different from (5.9). Take, for example, $K = \mathbf{Q}(\sqrt{-2d_0}\,)$ with $d_0 = 4\mu + 1$ as in (6.13). Then the discriminant of K is $-8d_0$, and so the sum of (5.9) has $16\mu + 3$ terms. Since $\chi(a) = 0$ for even a in this case, the number of nonvanishing terms is much smaller, but not so small as that for (6.13).

One more remark may be added, First, we can state formulas for $L(1,\,\chi)$ of various types of χ that are not necessarily real. Take χ_0 to be a primitive character whose conductor d_0 is of the form $d_0 = 4\mu+3$ and such that $\chi_0(-1) = 1$. (Such a χ_0 cannot be real.) Let $\lambda(m) = \left(\frac{-2}{m}\right)$. Then from (6.3) we can derive, by the same technique as in the proof of Corollary 6.4,

$$\pi^{-1}\sqrt{2}\,\chi_0(2)G(\chi_0)L(1,\,\overline{\chi}_0\lambda) = \sum_{a=1}^{\mu}(-1)^a\chi_0(a). \tag{6.19}$$

Similarly we can handle $L(1,\,\overline{\chi}_0\lambda)$ when $d_0 = 4\mu+1$, $\chi_0(-1) = -1$, and $\lambda(m) = \left(\frac{2}{m}\right)$, in which case a formula of type (6.14) appears. Some more formulas for $L(1,\,\chi)$ will be given in Corollary 6.7 below.

We now state some analogues of (4.32), which are not so technically involved as Theorem 6.3.

Theorem 6.6. *Let χ be a primitive character modulo d, and k a positive integer such that $\chi(-1) = (-1)^k$. Then the following assertions hold:*

(i) *Suppose d is prime to 3; let $c = -\mathbf{e}(1/3)$. Then*

$$\begin{aligned}(k-1)!(2\pi i)^{-k}G(\overline{\chi})L(k,\,\chi)\\ = \frac{\chi(3)}{(1+c^{-1})\{3^k - \chi(3)\}}\sum_{a=1}^{d-1}\overline{\chi}(a)\mathbf{e}(-da/3)E_{c,k-1}(a/d).\end{aligned} \tag{6.20}$$

(ii) *Suppose d is prime to 2; let $c = -\mathbf{e}(1/4)$. Then*

$$\begin{aligned}(k-1)!(2\pi i)^{-k}G(\overline{\chi})L(k,\,\chi)\\ = \frac{\chi(4)}{(1+i)\{4^k - \chi(2)2^k\}}\sum_{a=1}^{d-1}\overline{\chi}(a)\mathbf{e}(-da/4)E_{c,k-1}(a/d).\end{aligned} \tag{6.21}$$

(iii) *Suppose d is prime to 6; let $c = -\mathbf{e}(1/6)$. Then*

$$(k-1)!(2\pi i)^{-k}G(\overline{\chi})L(k,\,\chi) \tag{6.22}$$

$$= \frac{\chi(6)}{\mathbf{e}(1/6)\{2^k - \chi(2)\}\{3^k - \chi(3)\}} \sum_{a=1}^{d-1} \overline{\chi}(a)\mathbf{e}(-da/6)E_{c,k-1}(a/d).$$

PROOF. For simplicity, let us write F and E for $F_{c,\,k-1}$ and $E_{c,\,k-1}$. We first prove (iii). Thus $c = -\mathbf{e}(\alpha)$ with $\alpha = 1/6$. Taking $N = 6$, $q = 1$, and $t = b/d$ in (4.4c), we obtain

$$(*) \qquad F(6b/d) = 6^k \sum_{m \in 1+6\mathbf{Z}} m^{-k}\mathbf{e}(mb/d) \qquad (b \in \mathbf{Z}).$$

Clearly $F(6b/d)$ depends only on $b \pmod{d}$. Also, we easily see that

$$(**) \qquad \sum_{m \in 1+6\mathbf{Z}} \chi(m)m^{-k} = \big\{1 - \chi(2)2^{-k}\big\}\big\{1 - \chi(3)3^{-k}\big\}L(k,\, \chi).$$

If $k = 1$, the sums of $(*)$ and $(**)$ should be understood in the sense explained in a few lines below (4.4). Take integers μ and ν so that $1 = d\mu + 6\nu$. Then for $0 < a < d$ we have $a = ad\mu + 6a\nu$ and $F(a/d) = \mathbf{e}(a\mu/6)F(6a\nu/d)$ by (4.4a). Since $\mathbf{e}(a\mu/6) = \mathbf{e}(ad/6)$, we have, in view of $(*)$,

$$\begin{aligned}\sum_{a=1}^{d-1} \overline{\chi}(a)\mathbf{e}(-ad/6)F(a/d) &= \sum_{a=1}^{d-1} \overline{\chi}(a)F(6a\nu/d) \\ &= \chi(\nu)\sum_{a=1}^{d-1} \overline{\chi}(a\nu)6^k \sum_{m \in 1+6\mathbf{Z}} m^{-k}\mathbf{e}(ma\nu/d) \\ &= 6^k\chi(\nu)G(\overline{\chi}) \sum_{m \in 1+6\mathbf{Z}} \chi(m)m^{-k}.\end{aligned}$$

Applying (4.5) to $F(a/d)$ and employing $(**)$, we obtain (6.22), as $\chi(6\nu) = 1$. The other two cases with $\alpha = 1/3$ and $\alpha = 1/4$ can be proved in the same manner.

Corollary 6.7. *Let χ be a primitive character modulo d such that $\chi(-1) = -1$. Then the following assertions hold.*

(i) *If d is prime to 3, we have*

$$(6.23) \qquad (2\pi i)^{-1}G(\overline{\chi})L(1,\, \chi) = \frac{\chi(3)}{3 - \chi(3)}\left\{\sum_{a=1}^{[d/6]} \overline{\chi}(3a) - \sum_{b \in B} \overline{\chi}(b)\right\},$$

$$B = \big\{b \in \mathbf{Z}\,\big|\, 0 < b < d/2,\ b - d \in 3\mathbf{Z}\big\}.$$

(ii) *If d is prime to 2, we have*

$$(6.24) \quad (2\pi i)^{-1}G(\overline{\chi})L(1,\, \chi) = \frac{\chi(4)}{2 - \chi(2)} \cdot \begin{cases} \dfrac{1}{\chi(4)+1}\displaystyle\sum_{a \in A} \overline{\chi}(a) & \text{if } \chi(4) \neq -1, \\ \dfrac{1}{\chi(4)-1}\displaystyle\sum_{b \in B} \overline{\chi}(b) & \text{if } \chi(4) \neq 1, \end{cases}$$

$$A = \big\{a \in \mathbf{Z}\,\big|\, 0 < a < d/2,\ a \equiv 0 \text{ or } -d \pmod{4}\big\},$$
$$B = \big\{b \in \mathbf{Z}\,\big|\, 0 < b < d/2,\ b \equiv 2 \text{ or } d \pmod{4}\big\}.$$

(iii) *If d is prime to 6, we have*

$$(6.25)\qquad (2\pi i)^{-1}G(\overline{\chi})L(1,\,\chi) = \frac{\chi(6)}{\{2-\chi(2)\}\{3-\chi(3)\}} \cdot \Bigl\{ \sum_{a\in A_1}\overline{\chi}(a) + 2\sum_{a\in A_2}\overline{\chi}(a) - \sum_{b\in B_1}\overline{\chi}(b) - 2\sum_{b\in B_2}\overline{\chi}(b)\Bigr\},$$

$$A_1 = \bigl\{a \in \mathbf{Z}\,\big|\, 0 < a < d/2,\ a \equiv 0 \text{ or } -2d \pmod 6\bigr\},$$
$$A_2 = \bigl\{a \in \mathbf{Z}\,\big|\, 0 < a < d/2,\ a \equiv -d \pmod 6\bigr\},$$
$$B_1 = \bigl\{b \in \mathbf{Z}\,\big|\, 0 < b < d/2,\ b \equiv d \text{ or } 3 \pmod 6\bigr\},$$
$$B_2 = \bigl\{b \in \mathbf{Z}\,\big|\, 0 < b < d/2,\ b \equiv 2d \pmod 6\bigr\}.$$

PROOF. We take $k = 1$ in Theorem 6.6 and use the fact that $E_{c,0}(t) = 1$ as noted in (4.3h). We first consider the case $c = -\mathbf{e}(1/6)$. Put $q = (d-1)/2$ and $\zeta = \mathbf{e}(1/6)$. Then the sum on the right-hand side of (6.22) can be written

$$\sum_{a=1}^{q}\overline{\chi}(a)\bigl\{\mathbf{e}(-da/6) - \mathbf{e}((da-1)/6)\bigr\},$$

and $\mathbf{e}(-b/6) - \mathbf{e}((b-1)/6)$ is ζ, $-\zeta$, 2ζ, or -2ζ according to $b \pmod 6$. Then we obtain (6.25) with A_ν and B_ν as given there. Case (i) can be proved in the same manner. In Case (ii), by the same technique we first obtain

$$(6.26)\qquad (2\pi i)^{-1}G(\overline{\chi})L(1,\,\chi) = \frac{\chi(4)}{4-2\chi(2)}(X - Y),$$
$$X = \sum_{a\in A}\overline{\chi}(a), \qquad Y = \sum_{b\in B}\overline{\chi}(b)$$

with A and B as given in (6.24). On the other hand, (4.32) with $k = 1$ shows that this equals $(X+Y)/\bigl(4-2\chi(2)\bigr)$. Thus $X + Y = \chi(4)(X - Y)$. Therefore we can state the result as in (6.24).

If $K = \mathbf{Q}(\sqrt{-d}\,)$ and $-d$ is the discriminant of K and $d > 4$, then the formulas of Corollary 6.7 give $h_K/2$. Of course we cannot attach importance to any of such class number formulas, but we mention them simply because they follow from more general results on $L(k,\,\chi)$, which are well worthy of notice.

http://www.springer.com/978-0-387-72473-7

Elementary Dirichlet Series and Modular Forms
Shimura, G.
2007, VIII, 152 p., Hardcover
ISBN: 978-0-387-72473-7