Preface

The Origin of This Book

This text grew out of two types of real analysis courses taught by the author at Bard College, one for undergraduate mathematics majors, and the other for students in the mathematics section of Bard's Masters of Arts in Teaching (M.A.T.) Program. Bard's undergraduate real analysis course is a standard introductory course at the junior–senior level, but the M.A.T. real analysis course, as explained below, is somewhat less standard. The author was therefore unable to find an existing real analysis textbook that exactly met the needs of the students in the M.A.T. course, and so this text was written to fill the gap. To make this text more broadly useful, however, it has been written in a way that makes it sufficiently flexible to meet the needs of a standard undergraduate real analysis course as well, though with a few distinguishing features.

One of the principles on which Bard's M.A.T. Program was founded is that secondary school teachers need, in addition to sufficient training in pedagogy, a substantial background in their subject areas. In the Bard M.A.T. Program in Mathematics, not only are all students required to have completed the equivalent of a B.A. in mathematics to enroll in the program, but they are required to take four mathematics courses in the M.A.T. Program, one of which is in real analysis. The M.A.T. mathematics courses are different from standard first-year mathematics graduate courses, in that rather than directing the students toward more advanced mathematical topics, the emphasis is on giving the students an advanced look at the material taught in secondary school mathematics courses. For example, it is important for prospective teachers of calculus to have a good understanding of the properties of the real numbers (including decimal expansion), and a detailed look at logarithmic, exponential and trigonometric functions, none of which is usually treated in detail in standard undergraduate real analysis courses. Of course, a prospective teacher of calculus must also have a good grasp of limits, differentiation and integration, as found in any real analysis course. By contrast, it is not as important for prospective secondary teachers to spend valuable course time on some standard introductory real analysis topics such as sequences and series of functions. Hence, the focus of a real analysis course for M.A.T. students is somewhat different from a standard undergraduate real analysis course.

This text contains all the material needed for both a standard introductory course in real analysis and for variants of such a course aimed at prospective teachers. It is the hope of this author that for each intended audience, this text will offer a clear, accessible and interesting exposition of this beautiful material.

Audience

This text is aimed at three target audiences:

1. Mathematics majors taking a standard introductory real analysis course;
2. Prospective secondary school mathematics teachers taking an introductory real analysis course;
3. Prospective secondary school mathematics teachers taking a second real analysis course.

For undergraduate mathematics majors taking an introductory real analysis course, this text covers all the standard topics that are typically treated in an introductory single-variable real analysis book. The order of the material is slightly different than usual (with sequences being treated after derivatives and integrals), and as a result a few of the proofs are different, but all the standard topics are present, as well as a few extras.

For prospective secondary school mathematics teachers taking an introductory real analysis course, this text has, in addition to the standard topics one would encounter in any undergraduate real analysis course, a thorough treatment of the properties of the real numbers, and an equally thorough treatment of logarithmic, exponential and trigonometric functions. Additionally, the book contains some historical information that a mathematics teacher could use to enliven a calculus course.

For prospective secondary school mathematics teachers taking a second real analysis course (for example, M.A.T. students in mathematics who have already had an undergraduate real analysis course), this text has, in addition to a review of the basic topics of real analysis (limits, derivatives, integrals, sequences), a development of the real numbers starting with the Peano Postulates, a detailed discussion of the decimal expansion of real numbers via least upper bounds, a thorough treatment of logarithmic, exponential and trigonometric functions, and additional topics not usually found in introductory real analysis texts (for example, a discussion of π in terms of the circumference and area of circles, and a proof of the equivalence of various theorems such as the Extreme Value Theorem and the Bolzano–Weierstrass Theorem with the Least Upper Bound Property). It is the belief of this author that for those M.A.T. students who have already had an undergraduate course in real analysis, the proper training for prospective teachers is not to offer a course in more advanced topics in analysis (for example, Lebesgue measure and integration, or metric spaces), but rather to discuss in more detail those aspects of single-variable real analysis that are most directly related to the topics that teachers encounter in secondary schools.

Pedagogical Concerns

Regardless of any particular choices in the selection and order of the material in this text, at heart this text is a detailed and rigorous introduction to real analysis designed for students who have not previously studied the subject.

Some of the pedagogical concerns of this text are as follows.

Slow and Steady

Though it is fun to rush straight to the most exciting results in a subject, and it is tempting to skip over the details of some proofs (either because they seem too routine or because they seem too long), in the author's experience the best way for students to learn the basics of a technical subject such as real analysis is to work through all the details of the subject slowly and steadily. Students need to be challenged, but in an introductory text such as this one it is best to leave the challenges to the exercises, not the proofs of theorems. Most proofs in this text are written out in full detail, and when details are omitted, that is stated explicitly. When previous results in the text are used in a proof, those results are always referenced. Most other real analysis texts of this length cover more material than we do; our aim is not to fit as much material as possible into the book, but to provide sufficient material for a one-semester course (with a few options for the instructor), and to cover that material as thoroughly as possible.

Careful Writing

Every effort has been made to provide clearly and carefully written definitions, theorems and proofs throughout the book. As seen in the author's book *Proofs and Fundamentals: A First Course in Abstract Mathematics* (Birkhäuser, Boston, 2000), the author views the careful writing of proofs as an important part of both teaching and learning rigorous mathematics, and he has attempted to adhere to the advice he gave about writing in that book.

Minimal Technicalities

Though real analysis is technical by nature, this text attempts to keep technical concepts to a minimum. For example, we omit discussion of limit inferior and limit superior, because we can accomplish everything we need without it.

When technicalities are kept to a minimum, the result is that some particularly slick proofs are not available, which is unaesthetic to the experienced mathematician. For the sake of student learning, however, it it better to use a minimum number of technicalities repeatedly than to have the shortest or cleverest proof of each theorem. For example, there are some theorems involving continuity, differentiation and integration (such as the Extreme Value Theorem and the Intermediate Value Theorem) that can be proved very efficiently by using sequences, but which we prove using only the basic properties of the real numbers (and in particular the Least Upper Bound Property).

Another example of keeping technicalities to a minimum is our choice of Dedekind cuts rather than Cauchy sequences for constructing the real numbers from the rational numbers; the method of Dedekind cuts is slightly longer, but it avoids both sequences and equivalence classes. (Of course, there is a discussion of Cauchy sequences in this text, but it is in its natural place in the chapter on sequences, which is well after the chapter on the construction of the real numbers.)

Features

There are many undergraduate books in real analysis, but the author has not found any with the exact same choice of material and pedagogical concerns as this text.

Some of the distinguishing features of this text are as follows.

Thorough Treatment of the Real Numbers

At the heart of real analysis are the properties of the real numbers. Whereas most introductory real analysis texts move as quickly as possible to the core topics of calculus (such as limits, derivatives and integrals) by giving relatively brief treatments of the axioms for the real numbers and the consequences of those axioms, this text emphasizes the importance of the properties of the real numbers as the basis of real analysis. Hence, the real numbers and their properties are developed in more detail than is found in most other introductory real analysis texts. The goal of the text is for students to have a thorough understanding of the fundamentals of real analysis, not to cover as much ground as possible.

Multiple Entryways

A particularly distinctive feature of this text is that it offers three ways to enter into the study of the real numbers.

Entry 1, which yields the most complete treatment of the real numbers, begins with the Peano Postulates for the natural numbers, and then leads to the construction of the integers, the rational numbers and the real numbers, proving the main properties of each set of numbers along the way.

Entry 2, which is more efficient than Entry 1 but more detailed than Entry 3, skips over the axiomatic treatment of the natural numbers, and begins instead with an axiomatic treatment of the integers. It is first shown that inside the integers sits a copy of the natural numbers, and after that the rational numbers and the real numbers are constructed, and their main properties proved.

Entry 3, which is the most efficient approach to the real numbers, starts with an axiomatic treatment of the real numbers. It is shown that inside the real numbers sit the natural numbers, the integers and the rational numbers. This approach is the one taken in most contemporary introductions to real analysis, though we give a bit more details about the natural numbers, integers and rational numbers than is common.

The existence of the three entryways into the real numbers allows for great flexibility in the use of this text. For a first real analysis course, whether for mathematics majors or prospective secondary school mathematics teachers, Entry 3 should be used; for a second real analysis course for prospective secondary school mathematics teachers, or as supplementary reading for a standard introductory real analysis course, Entry 1 or Entry 2 should be used. No matter which entry is used, all students end up knowing the same properties of the real numbers, and hence are equally prepared for the subsequent material.

Follows Order of Material in Calculus Courses

Undergraduate real analysis courses are often organized according to the goal of preparing the students for more advanced mathematics courses. Such a design, however, does not necessarily lead to the best pedagogical approach. Whereas many of the more advanced aspects of real analysis are quite abstract, the motivation for introductory real analysis is the need for a rigorous foundation for calculus. Given that the students in an introductory real analysis course have already had courses in calculus, and given that pedagogically it is best to relate new material to that which is already familiar, this text presents the basic material in real analysis in an order that is closer to that encountered in calculus courses than is found in most real analysis books.

Sequences Later in the Text

In standard calculus courses, sequences usually receive very minimal treatment, and are discussed only as much as they are needed as partial sums of series. In real analysis, by contrast, sequences are a very important tool, and they are treated in great detail in most real analysis texts. Moreover, in most such texts sequences are located right after the preliminary treatment of the real numbers, and prior to the discussion of limits, derivatives and integrals, both because the definition of the convergence of sequences is viewed as slightly easier to learn than the definition of the convergence of functions, and because some of the major theorems about sequences (such as the Bolzano–Weierstrass Theorem) are used in the proofs of some important theorems about continuous functions, derivatives and integrals (such as the Extreme Value Theorem).

In this text, by contrast, sequences are treated after the chapters on limits, derivatives and integrals, similarly to the order of material in a calculus course. Whereas sequences are used in many real analysis books in the proofs of some of the important theorems concerning functions, it turns out that all such theorems can be proved without the use of sequences, where instead of using the Bolzano–Weierstrass Theorem and similar results, a direct appeal is made to the Least Upper Bound Property, or to direct consequences of that property. As such, it is possible to treat continuous functions, derivatives and integrals without the added technicality of sequences, and in the order familiar from calculus courses. Moreover, the use of sequences in proofs where they are not needed, while sometimes making for short and clever proofs, may at times obscure the essential ideas of the theorem being proved.

Of course, wherever they are placed, sequences are a very important topic in real analysis, and they are given a thorough treatment in this text, with all the usual theorems proved.

Integration via Riemann Sums

There are two standard ways of defining the Riemann Integral that are found in introductory real analysis texts: via Riemann sums, and via upper and lower integrals. The latter approach is used by many (if not most) current introductory real analysis books, and it gives a fast route to proving the important theorems about integrals. However, given that the treatment of integrals in calculus courses is via Riemann sums, this text also uses that approach in its definition of integrals, so that students can understand the rigorous treatment of integrals in terms of what they had previously seen in calculus courses.

Equivalence of Various Theorems with the Least Upper Bound Property

Every student in a real analysis course learns that the Least Upper Bound Property is at the heart of what the real numbers consist of, and it is the basis for the proofs of many of the main theorems of real analysis, such as the Extreme Value Theorem and the Bolzano–Weierstrass theorem. Many of these theorems, for example the two just mentioned, are in fact logically equivalent to the Least Upper Bound Property, and in this text we present a proof of this logical equivalence, which is not commonly found in real analysis books.

Thorough Discussion of Transcendental Functions

Logarithmic, exponential and trigonometric functions are familiar to students from precalculus and calculus courses. Whereas most introductory real analysis books either ignore these functions or give them a cursory treatment, in this text these functions are defined rigorously, and their basic properties are proved in detail. Of particular note is our treatment of the sine and cosine functions; these functions are trickier to define rigorously than logarithms and exponentials, but nonetheless deserve a thorough exposition.

Discussion of Area and Arc Length

The main motivation for the development of the definite integral is to compute areas of certain regions in the plane. However, the very important fact that the definite integral of a non-negative function yields the area under the graph of the function, while regularly asserted, is rarely proved in introductory real analysis books (indeed, the concept of area is rarely defined rigorously), which leads not only to a gap in rigor but also to an incomplete understanding of the concept of area. In this text, we give a thorough discussion of area and arc length, starting with geometric definitions of these concepts, and then proofs that in appropriate cases, they can be computed via definite integrals.

More about π

Students are familiar with the number π from a very young age, where it is discussed in the context of the circumference and area of circles. The number π is also introduced into the study of trigonometric functions in precalculus and calculus courses. In real analysis, if the trigonometric functions are to be studied, then the number π cannot be avoided. In this text, a particularly detailed treatment of π is given, in order to clarify the relation between the geometric approach to this number (via the circumference of circles) and analytic approach to it (via the definition of the trigonometric functions using integrals).

Reflections for Every Section

The heart of mathematics is the details, and students in a real analysis course quite naturally get very caught up in the ε's and δ's. However, it is useful at times to step back from the details and ask broader questions, such as: why are things done as they are; why are some aspects of the material straightforward and other aspects not; whether all the hypotheses of the theorems are really needed; and whether there might be an easier way to define or prove things. In real analysis, moreover, it is also helpful to compare the way things are done in that course with the way they were done (or not done) in calculus courses that the student took previously. Hence, every section of this text concludes with some very brief remarks that look back upon the material in the section, often in the context of what the student has seen prior to real analysis. The main purpose of these remarks is not, however, simply to convey the author's thoughts about the material, but is rather to encourage the reader to engage in her own similar reflections upon the material discussed in this text, and upon other mathematical ideas encountered subsequently.

Historical Remarks for Every Chapter

The material in this book is presented in the logical order of development that is now standard for real analysis, but which is quite different from the way the subject developed historically. Though it would be very inefficient to learn the details of real analysis in the order in which it occurred historically, because it took mathematicians a rather circuitous route to reach the understanding we have today, it is nonetheless beneficial for mathematicians to know something about how important topics such as calculus arose. Such historical context is especially valuable for prospective teachers, though it can benefit all students of real analysis, not in understanding the details of rigorous definitions and proofs, but in seeing the bigger picture. Hence, each chapter concludes with a historical discussion of the material in the chapter.

The author is not a historian, and he hopes that the historical material provided is both useful and informative. For a thorough and engaging treatment of the history of mathematics in general, the reader is referred to [Kat98]. Because of the availability of the wonderful website [OR], which has extensive biographical information on every mathematician about whom the reader has heard (and many others as well), the historical material in this text does not include biographical information (other than dates of birth and death) about the mathematicians who developed real analysis.

Errors

In spite of the author's best effort, there will inevitably be some errors in this text. If the reader finds any such errors, it would be very helpful if she would send them to the author at `bloch@bard.edu`. An updated list of errors is available at `http://math.bard.edu/bloch/rnra_errata.pdf`.

Acknowledgments

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It is impossible to acknowledge every source for every idea, theorem or exercise in this text. Most of the theorems, proofs and exercises are either standard or are variants of standard results; some of these the author first encountered as a student, others were learned from a variety of sources. The following are texts that the author consulted regularly, and which notably influenced the writing of this book. Real analysis texts: [Gor02], [Lay01], [Pow94], [Ros68], [Sto01], [TBB01], [Tre03], [Wad00]; calculus text: [Spi67]; history of mathematics texts: [Bar69], [Boy49], [Boy91], [Coo05], [Edw79], [Fer08], [Jah03], [Kat98], [OR].

Some parts of Chapter 1 in this text are a revised version of material in [Blo00], which has been removed from [Blo10], the second edition of that book; this material is important for the construction of the real numbers, a topic that is at the heart of real analysis, but was somewhat inappropriately included in the more elementary, and earlier written, textbook [Blo00] due to the author's perhaps excessive enthusiasm for this material. Additionally, although it is assumed that the reader taking a course in real analysis is familiar with the core material from a book on proofs, sets and functions such as [Blo10], there is nonetheless some overlap in the treatment of

induction and recursion between that book and this text, the material being important for both.

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