Chapter 2

Varieties in Projective Spaces and Their Gauss Maps

In this chapter, after introducing in Sections 2.1 and 2.2 the basic notions (such as the tangent, osculating and normal subspaces, the second fundamental tensor and the second fundamental form, and the asymptotic lines and asymptotic cone) associated with a variety in a projective space $\mathbb{P}^N$, in Section 2.3, we define the rank of a variety and varieties with degenerate Gauss maps. In Section 2.4, we consider the main examples of varieties with degenerate Gauss maps (cones, torses, hypersurfaces, joins, etc.). In Section 2.5, we study the duality principle and its applications, consider another example of varieties with degenerate Gauss maps (the cubic symmetroid) and correlative transformations, and in Section 2.6, we investigate a hypersurface with a degenerate Gauss map associated with a Veronese variety and find its singular points.

2.1 Varieties in a Projective Space

2.1.1 Equations of a Variety. Let M be an n-dimensional connected differentiable manifold, and let f be a nondegenerate almost everywhere differentiable mapping of M into a projective space $\mathbb{P}^N$:

$$f : M \to \mathbb{P}^N,$$

where $n < N$. The image $X = f(M)$ of the manifold M under this mapping is also differentiable almost everywhere. We shall call X an *n-dimensional variety* (or sometimes *subvariety*). Note that the manifold M is differentiable while the variety $X = f(M)$ is almost everywhere differentiable.

For a point $x \in X$ of a variety $X \subset \mathbb{P}^N$, we have $\dim T_x X \geq \dim X = n$. If $\dim T_x X = \dim X = n$, then a point x is called *regular* (or *smooth*), and if $\dim T_x X > \dim X = n$, a point x is called *singular* (see Shafarevich [Sha 88], Chapter 2, §1).

We denote the *locus of smooth points* of X by X_{sm} and the *locus of singular points* of X by $\operatorname{Sing} X$, so

$$\begin{aligned} X_{sm} &= \{x \in X : \dim T_x X = \dim X\}, \\ \operatorname{Sing} X &= \{x \in X : \dim T_x X > \dim X\}. \end{aligned}$$

It is obvious that $\operatorname{Sing} X \subset X$, $X_{sm} \subseteq X$, $\dim X_{sm} = n$, dim $\operatorname{Sing} X < n$.

If $t^i,\ i = 1, \ldots, n$, are differentiable coordinates on the manifold M, then the variety X can be given by the equations

$$x^u = x^u(t^i), \quad u = 0, 1, \ldots, N, \tag{2.1}$$

where $x^u(t^i)$ are almost everywhere differentiable functions of the variables t^i, and the rank of the matrix $(\frac{\partial x^u}{\partial t^i})$ does not exceed n. Because x^u are homogeneous coordinates of a point x of the space $\mathbb{P}^N$, the functions x^u admit multiplication by a common factor, which can be not only a number but also a function $f(t^i)$.

The locus of singular points $\operatorname{Sing} X$ is determined by the condition

$$\operatorname{rank} \left(\frac{\partial x^u}{\partial t^i} \right) < n.$$

The variety X can also be given locally by a system consisting of $N - n$ independent equations of the form

$$F^\alpha(x^0, x^1, \ldots, x^N) = 0, \quad \alpha = n+1, \ldots, N, \tag{2.2}$$

where F^α are homogeneous almost everywhere differentiable functions. In a neighborhood of a nonsingular point x, the Jacobi matrix $\left(\frac{\partial F^\alpha}{\partial x^u}\right)$ is of rank $N - n$. Hence without loss of generality, we may assume that if in a neighborhood of a point $x \in X$, $\det \left(\frac{\partial F^\alpha}{\partial x^\beta}\right) \neq 0, \alpha, \beta = n+1, \ldots, N$, then equations (2.2) can be solved for the variables x^α:

$$x^\alpha = x^\alpha(x^0, x^1, \ldots, x^n), \quad \alpha = n+1, \ldots, N. \tag{2.3}$$

Here the right-hand sides are homogeneous functions of first degree. Therefore, these right-hand sides and the right-hand sides of equations (2.1) contain n essential variables that determine the location of a point on the variety X. If we set $x^i/x^0 = t^i$, we reduce equations (2.3) to the form (2.1).

In Section 1.5 we considered some algebraic submanifolds in a projective space. Certainly, those are differentiable manifolds. Moreover, equations (1.163) defining the image $\Omega(m,n)$ of the Grassmannian $\mathbb{G}(m,n)$ in the space $\mathbb{P}^N$, where $N = \binom{n+1}{m+1} - 1$, are of form (2.2), and equations (1.165) and (1.171), defining the Segre and Veronese varieties, respectively, are of form (2.1). However, the parameters in equations (1.165) and (1.171) are homogeneous while the parameters in equations (2.1) are nonhomogeneous. But as we indicated for equation (2.3), in a neighborhood of a nonsingular point, it is easy to change homogeneous parameters for nonhomogeneous ones.

2.1.2 The Bundle of First-Order Frames Associated with a Variety. Let X be an almost everywhere differentiable variety of dimension n in the projective space $\mathbb{P}^N$, and let x be its nonsingular point. In what follows, we assume that a point $x \in X$ under consideration is nonsingular without also specifying this. Consider all smooth curves passing through a point $x \in X_{sm}$. The tangent lines to these curves at the point x lie in an n-dimensional subspace $T_x(X)$ of the space $\mathbb{P}^N$, called the *tangent subspace to the variety X at the point x*. For brevity, we also use the symbol T_x for the subspace $T_x(X)$.

If x is a regular point of the variety X, then the tangent subspace $T_x(X)$ can be considered in two ways: as a vector space L^{n+1} formed by the vectors $\boldsymbol{v} = \overline{xy}$, where $y \in T_x(X)$ or as a projective subspace $\mathbb{P}^n$ of the projective space $\mathbb{P}^N$ with the fixed point $x \in X$. In what follows, we will adhere to the second point of view. Unless otherwise stated, we will conduct all our considerations in a neighborhood of a regular point $x \in X$.

We associate a family of moving frames $\{A_u\}$, $u = 0, 1, \ldots, N$, with each point $x \in X_{sm}$, and assume that for all these frames the point A_0 coincides with the point x, and the points A_i, $i = 1, \ldots, n$, lie in the tangent subspace T_x. The frames of this family are called *first-order frames*. Because the differential $dx = dA_0$ of the point x belongs to the tangent subspace T_x, its decomposition with respect to the vertices of the frame $\{A_u\}$ can be written as:

$$dA_0 = \omega_0^0 A_0 + \omega_0^i A_i. \tag{2.4}$$

Thus, in the space $\mathbb{P}^N$, the variety X along with the family of first-order frames is defined by the following system of Pfaffian equations:

$$\omega_0^\alpha = 0, \ \alpha = n+1, \ldots, N, \tag{2.5}$$

and the forms ω_0^i in equation (2.4) are linearly independent and form a cobasis in the tangent subspace T_x. For brevity, we denote these forms by ω^i:

$$\omega_0^i = \omega^i.$$

We call equations (2.5) the *basic* equations of the variety X.

By the structure equations (1.73) of a projective space $\mathbb{P}^N$ and by equations (2.5), the exterior differentials of the forms ω^i can be written as

$$d\omega^i = \omega^j \wedge (\omega_j^i - \delta_j^i \omega_0^0). \tag{2.6}$$

This implies that the 1-forms

$$\theta_j^i = \omega_j^i - \delta_j^i \omega_0^0 \tag{2.7}$$

are the base forms of the frame bundle $\mathcal{R}^1(M)$ of first-order frames on the manifold M of parameters of the variety X. The forms ω^i are the basis forms of the manifold M as well as of the variety X. By relation (1.64), if the point x is held fixed, the forms ω^i satisfy the differential equations

$$\delta\omega^i + \omega^j(\pi_j^i - \delta_j^i \pi_0^0) = 0, \tag{2.8}$$

where, as in Chapter 1, the symbol δ denotes the restriction of the differential d to the fiber $\mathcal{R}_x^1$ of the frame bundle $\mathcal{R}^1(M)$, and $\pi_v^u = \omega_v^u(\delta)$.

If the point x is held fixed on the variety X, then the forms ω^i vanish, $\omega^i = 0$. In this case, the tangent subspace T_x is also fixed. Hence the forms ω_i^α also vanish. Thus, if the point x is held fixed, then the admissible transformations of the moving frames are determined by the following derivational equations:

$$\begin{cases} \delta A_0 = \pi_0^0 A_0, \\ \delta A_i = \pi_i^0 A_0 + \pi_i^j A_j, \\ \delta A_\alpha = \pi_\alpha^0 A_0 + \pi_\alpha^i A_i + \pi_\alpha^\beta A_\beta. \end{cases} \tag{2.9}$$

The 1-forms $\pi_0^0, \pi_i^0, \pi_i^j, \pi_\alpha^0, \pi_\alpha^i$ and π_α^β in (2.9) define the group of transformations of first-order frames associated with the point $x = A_0$. This group is called the *stationary subgroup* of the plane element (x, T_x) of X.

Because the family of first-order frames is associated with each point x of the variety X, the *bundle* $\mathcal{R}^1(X)$ *of first-order frames* is defined on the whole variety X. The base of this bundle is the variety X itself, its base forms are the forms ω^i, its typical fiber is a set of first-order frames associated with a point $x = A_0$, and its fiber forms are the forms $\omega_0^0, \omega_i^0, \omega_i^j, \omega_\alpha^0, \omega_\alpha^i$, and ω_α^β.

Consider the projectivization $\widetilde{T}_x = T_x/A_0$ of the tangent subspace T_x with the center $A_0 = x$ (see Section 1.3.3). This projectivization is a projective space $\widetilde{\mathbb{P}}^{n-1}$ whose elements are the straight lines of the space T_x passing through the point x.

As indicated in Section 1.3, this projectivization defines an equivalence relation in the set of points of the space T_x. This explains why it is natural to denote this projectivization by T_x/A_0:

$$\widetilde{T}_x = T_x/A_0.$$

A frame in the space $\widetilde{T}_x = \widetilde{\mathbb{P}}^{n-1}$ is formed by the points $\widetilde{A}_i = A_i/A_0$, and the forms ω^i become homogeneous coordinates of the point $\widetilde{Y} \in \widetilde{\mathbb{P}}^{n-1}$, i.e.,

$$\widetilde{Y} = \omega^i \widetilde{A}_i.$$

Consider also the projectivization of the space $\mathbb{P}^N$ with the tangent subspace T_x as the center of projectivization. The elements of this projectivization are $(n+1)$-dimensional subspaces of the space $\mathbb{P}^N$ containing the n-dimensional subspace T_x. We denote this projectivization by $\widetilde{\mathbb{P}}^{N-n-1} = \mathbb{P}^N/T_x$. The basis points of the space $\widetilde{\mathbb{P}}^{N-n-1}$ are the points $\widetilde{A}_\alpha = A_\alpha/T_x$, determined by $(n+1)$-dimensional subspaces passing through the points A_α and the center T_x of projectivization. The space $\widetilde{\mathbb{P}}^{N-n-1} = \mathbb{P}^N/T_x$ is called the *first normal subspace* of the variety X at its point x and is denoted by $N_x(X) = \mathbb{P}^N/T_x$.

2.1.3 The Prolongation of Basic Equations. The further investigation of a variety X in a projective space $\mathbb{P}^N$ is concerned with differential prolongations of equations (2.5) defining this variety along with the family of first-order moving frames associated with it. Exterior differentiation of these equations gives the exterior quadratic equations

$$\omega^i \wedge \omega_i^\alpha = 0. \tag{2.10}$$

Applying the Cartan lemma to these exterior equations, we obtain the expressions of the forms ω_i^α in terms of the basis forms ω^i of the variety X:

$$\omega_i^\alpha = b_{ij}^\alpha \omega^j, \quad b_{ij}^\alpha = b_{ji}^\alpha. \tag{2.11}$$

The 1-forms $\{\omega_0^\alpha, \omega_i^\alpha\}$ are the basis forms of the Grassmannian $\mathbb{G}(n, N)$ whose elements are the subspaces $p = A_0 \wedge A_1 \wedge \ldots \wedge A_n$. But on the variety X, we have $\omega^\alpha = 0$ (see (2.5)). Thus, equation (2.11) defines a mapping of the variety X into the Grassmannian $\mathbb{G}(n, N)$. This mapping is called the *Gauss map*. We denote it by γ:

$$\gamma : X \to \mathbb{G}(n, N).$$

Its name is related to the fact that this map is a projective generalization of the spherical map, introduced by Gauss, of a surface V^2 of a three-dimensional Euclidean space R^3 into a sphere S^2 by means of unit normal vectors.

To establish the nature of the geometric object with the components b^α_{ij}, we evaluate the exterior differentials of equations (2.11) by means of structure equations (1.73) of the space $\mathbb{P}^N$. This results in the following exterior equations:

$$\nabla b^\alpha_{ij} \wedge \omega^j = 0, \tag{2.12}$$

where

$$\nabla b^\alpha_{ij} = db^\alpha_{ij} - b^\alpha_{kj}\theta^k_i - b^\alpha_{ik}\theta^k_j + b^\beta_{ij}\theta^\alpha_\beta, \tag{2.13}$$

and the forms θ^j_i are determined by formulas (2.7). As we noted earlier, these forms are connected with transformations of the first-order frames in the subspace $T_x(M)$ tangent to the manifold M of parameters of the variety X. Similarly, the forms

$$\theta^\alpha_\beta = \omega^\alpha_\beta - \delta^\alpha_\beta \omega^0_0 \tag{2.14}$$

determine admissible transformations of moving frames in the space $N_x(X)$.

Applying the Cartan lemma to exterior quadratic equation (2.12), we obtain the equations

$$\nabla b^\alpha_{ij} = b^\alpha_{ijk}\omega^k, \tag{2.15}$$

where the coefficients b^α_{ijk} are symmetric with respect to all lower indices. It follows from these equations that if $\omega^i = 0$, we have

$$\nabla_\delta b^\alpha_{ij} = \delta b^\alpha_{ij} - b^\alpha_{kj}\sigma^k_i - b^\alpha_{ik}\sigma^k_j + b^\beta_{ij}\sigma^\alpha_\beta = 0, \tag{2.16}$$

where

$$\sigma^j_i = \pi^j_i - \delta^j_i \pi^0_0, \;\; \sigma^\alpha_\beta = \pi^\alpha_\beta - \delta^\alpha_\beta \pi^0_0.$$

Comparing equations (2.16) with equations (1.13), we see that the quantities b^α_{ij} form a tensor relative to the indices i and j. They also form a tensor relative to the index α under transformations of moving frames in the space $N_x(X)$. Such tensors are called *mixed tensors*.

2.2 The Second Fundamental Tensor and the Second Fundamental Form

2.2.1 The Second Fundamental Tensor, the Second Fundamental Form, and the Osculating Subspace of a Variety. The tensor b^α_{ij} is connected with the second-order differential neighborhood of a point x of the variety X. For this reason, this tensor is called the *second fundamental tensor of the variety* X. Let us clarify the geometric meaning of this tensor. To do this, we compute the second differential of the point $x = A_0$ by differentiating the relation (2.4):

$$d^2 A_0 = (d\omega_0^0 + (\omega_0^0)^2 + \omega_0^i \omega_i^0) A_0 + (\omega_0^0 \omega_0^i + \omega_0^j \omega_j^i) A_i + \omega_0^i \omega_i^\alpha A_\alpha. \qquad (2.17)$$

Factorizing the latter relation by the tangent subspace $T_x = A_0 \wedge A_1 \wedge \ldots \wedge A_n$, we obtain

$$d^2 A_0 / T_x = \omega_0^i \omega_i^\alpha \widetilde{A}_\alpha, \qquad (2.18)$$

where $\widetilde{A}_\alpha$ are basis points of the normal space $N_x = \mathbb{P}^N / T_x$.

Substituting the values of ω_i^α from equations (2.11) into equation (2.18) and denoting the left-hand side by Φ, we find that

$$\Phi = b_{ij}^\alpha \omega^i \omega^j \widetilde{A}_\alpha. \qquad (2.19)$$

This expression is a quadratic form with respect to the coordinates ω^i, having values in the normal subspace N_x. The form Φ is called the *second fundamental form* of the variety X. Thus, the second fundamental form defines a mapping of the tangent subspace $T_x(X)$ into the normal subspace $N_x(X)$:

$$\Phi : \mathrm{Sym}^{(2)} T_x(X) \to N_x(X).$$

This mapping is called the *Meusnier–Euler mapping* (see Griffiths and Harris [GH 79]).

Note that *a variety X is an n-plane or a part of an n-plane if and only if the second fundamental form Φ vanishes on X*. In fact, if $\Phi \equiv 0$, then it follows from formula (2.18) that $\omega_i^\alpha = 0$ on X. This implies that the equations of infinitesimal displacement of a moving frame become:

$$\begin{cases} dA_0 = \omega_0^0 A_0 + \omega^i A_i, \\ dA_i = \omega_i^0 A_0 + \omega_i^j A_j, \end{cases}$$

and as a result, the n-plane $A_0 \wedge A_1 \wedge \ldots \wedge A_n$ is fixed, and the point A_0 moves in this n-plane.

The scalar forms

$$\Phi^\alpha = b_{ij}^\alpha \omega^i \omega^j \qquad (2.20)$$

are the coordinates of the form Φ with respect to the moving frame $\{\widetilde{A}_\alpha\}$ in the space N_x. Let us denote the maximal number of linearly independent forms Φ^α by m. In some instances, it is convenient to consider the bundle of second fundamental forms of the variety X defined by the relation

$$\Phi(\xi) = \xi_\alpha b_{ij}^\alpha \omega^i \omega^j, \qquad (2.21)$$

where $\xi = (\xi_\alpha)$. The number m is the dimension of this bundle. In what follows, we assume that the number m is constant on the variety X.

The quantities ξ_α occurring in (2.21) define a hyperplane $\xi = \xi_\alpha x^\alpha = 0$, which is tangent to the variety X at the point x, and expression (2.21) is called the *second fundamental form of the variety X with respect to the hyperplane ξ*.

In the space N_x, consider the points

$$\widetilde{B}_{ij} = b_{ij}^\alpha \widetilde{A}_\alpha. \tag{2.22}$$

Because $\widetilde{B}_{ij} = \widetilde{B}_{ji}$, the number of these points is equal to $\frac{1}{2}n(n+1)$. However, it is not necessarily the case that all these points are linearly independent. The maximal number of linearly independent points $\widetilde{B}_{ij}$ coincides with the maximal number of linearly independent forms Φ^α, which we denoted by m. Note that according to our general point of view (see the Preface), we suppose that the integer m is the same on the entire variety X in question, and we will make similar assumptions relative to all other integer-valued invariants arising in our further considerations.

It is obvious that the number m satisfies the following inequalities:

$$0 \leq m \leq \frac{n(n+1)}{2} \quad \text{and} \quad m \leq N - n. \tag{2.23}$$

In the space N_x, the points $\widetilde{B}_{ij}$ span the subspace $\widetilde{\mathbb{P}}^{m-1}$.

Next, in the space $\mathbb{P}^N$, we consider the subspace, which is the linear span of the subspace T_x and the points $B_{ij} = b_{ij}^\alpha A_\alpha$. By relation (2.17), this subspace is also the linear span of all two-dimensional osculating planes of all curves of the variety X passing through the point x. For this reason, this subspace is called the *second osculating subspace* of the variety X at its point x, and it is denoted by $T_x^{(2)}$. We consider the tangent subspace T_x as the *first osculating subspace* of the variety X at a point x, $T_x = T_x^{(1)}$.

2.2.2 Further Specialization of Moving Frames and Reduced Normal Subspaces. We will make a further specialization of moving frames $\{A_u\}$ associated with a point $x \in X$. To do this, we place the vertices $A_{n+1}, \ldots, A_{n+m}$ of the frames into the second osculating subspace $T_x^{(2)}$, whose dimension is equal to $n + m$. The frames thus obtained are called the *frames of second order*.

With this specialization, the points B_{ij}, which together with the points A_0 and A_i define the second osculating subspace $T_x^{(2)}$, are expressed in terms of the points A_{i_1} alone: $B_{ij} = b_{ij}^{i_1} A_{i_1}$, $i_1 = n+1, \ldots, n+m$. So, we have

$$b_{ij}^{\alpha_1} = 0, \quad \alpha_1 = n+m+1, \ldots, N, \tag{2.24}$$

and therefore formulas (2.11) break up into two groups:

$$\omega_i^{i_1} = b_{ij}^{i_1}\omega^j, \tag{2.25}$$

$$\omega_i^{\alpha_1} = 0. \tag{2.26}$$

Therefore the second fundamental forms Φ^α of the variety X can be written as follows:

$$\Phi^{i_1} = b_{ij}^{i_1}\omega^i\omega^j, \quad \Phi^{\alpha_1} = 0, \tag{2.27}$$

and formula (2.18) becomes

$$d^2A_0/T_x = \omega^i\omega_i^{i_1}\widetilde{A}_{i_1}. \tag{2.28}$$

The forms Φ^{i_1} are linearly independent, and the matrix $(b_{ij}^{i_1})$ of coefficients of these forms, having m rows and $\frac{1}{2}n(n+1)$ columns, is of rank m.

Consider now the projectivization with the center T_x of the projective space $T_x^{(2)}$. This projectivization is a projective space of dimension $m-1$. We call this space the *reduced first normal subspace* of the variety X and denote it by $\widetilde{N}_x$:

$$\widetilde{N}_x = T_x^{(2)}/T_x. \tag{2.29}$$

If $N > n + m$, then at the point $x \in X$ it is also possible to define the *second normal subspace*

$$N_x^{(2)} = \mathbb{P}^N/T_x^{(2)}, \tag{2.30}$$

whose dimension is equal to $N - n - m - 1$ and whose basis is formed by the points $\widetilde{A}_{\alpha_1} = A_{\alpha_1}/T_x^{(2)}$.

Let us now establish the form of equations (2.15) after the specialization of moving frames indicated earlier. These equations also break into two groups:

$$\nabla b_{ij}^{i_1} = db_{ij}^{i_1} - b_{kj}^{i_1}\theta_i^k - b_{ik}^{i_1}\theta_j^k + b_{ij}^{j_1}\theta_{j_1}^{i_1} = b_{ijk}^{i_1}\omega^k, \tag{2.31}$$

$$\nabla b_{ij}^{\alpha_1} = b_{ij}^{i_1}\omega_{i_1}^{\alpha_1} = b_{ijk}^{\alpha_1}\omega^k. \tag{2.32}$$

Equations (2.31) show that the quantities $b_{ij}^{i_1}$ form a tensor relative to the indices i, j, and i_1. Because the matrix $(b_{ij}^{i_1})$ is of rank m, equations (2.32) can be solved with respect to the forms $\omega_{i_1}^{\alpha_1}$:

$$\omega_{i_1}^{\alpha_1} = c_{i_1k}^{\alpha_1}\omega^k. \tag{2.33}$$

Substituting these expressions of the forms $\omega_{i_1}^{\alpha_1}$ into equations (2.32), we obtain

$$b_{ij}^{i_1}c_{i_1k}^{\alpha_1} = b_{ijk}^{\alpha_1}. \tag{2.34}$$

Because the quantities $b^{\alpha_1}_{ijk}$ are symmetric with respect to the indices j and k, we find from (2.34) that

$$b^{i_1}_{ij} c^{\alpha_1}_{i_1 k} = b^{i_1}_{ik} c^{\alpha_1}_{i_1 j}. \tag{2.35}$$

This equation can also be obtained as a result of exterior differentiation of equations (2.26).

In the same manner as for the tensor b^{α}_{ij}, we can prove that the quantities $b^{\alpha_1}_{ijk}$ form a tensor relative to the indices i, j, k, and α_1. This and the relations (2.34) imply that the quantities $c^{\alpha_1}_{i_1 k}$ also form a tensor relative to the indices k, i_1, and α_1. As to the quantities $b^{i_1}_{ijk}$ in relations (2.31), it is easy to verify that they do not form a tensor, but rather they depend on the choice of the subspace $A_0 \wedge A_{n+1} \wedge \ldots \wedge A_{n+m}$, which is complementary to the subspace T_x in the osculating subspace $T^{(2)}_x$.

2.2.3 Asymptotic Lines and Asymptotic Cone. A curve on a two-dimensional surface V^2 of a Euclidean space E^3 is called *asymptotic* if its osculating planes coincide with the tangent planes to the surface V^2 or are undetermined (see, for example, Blaschke's books [Bl 21], p. 52, or [Bl 50], p. 65). This definition is projectively invariant and can be generalized to the case where we have a variety of any dimension n in a projective space $\mathbb{P}^N$. Namely, a curve l on a variety X is said to be *asymptotic* if its two-dimensional osculating plane at any of its points x belongs to the tangent subspace T_x to the variety X at this point or is undetermined.

If a curve l is given on the variety X by a parametric equation $x = x(t)$, then its osculating plane is determined by the points $x(t), x'(t)$ and $x''(t)$. But because $x = A_0$, this plane can also be defined by the points A_0, dA_0, and $d^2 A_0$. Because for an asymptotic line the second differential of its point belongs to the tangent subspace T_x, it follows from equation (2.17) that on this curve we have

$$\Phi = \omega^i \omega^{\alpha}_i A_{\alpha} = 0, \tag{2.36}$$

i.e., the second fundamental form of the variety X vanishes on l. Thus in coordinate form, the equations of asymptotic lines have the form

$$b^{\alpha}_{ij} \omega^i \omega^j = 0. \tag{2.37}$$

On a curve l passing through the point x, the basis forms ω^i have the form $\omega^i = \xi^i dt$, where ξ^i are coordinates of a tangent vector to the curve. Substituting these expressions into equations (2.37), we obtain

$$b^{\alpha}_{ij} \xi^i \xi^j = 0. \tag{2.38}$$

These equations define a cone C_x of directions with vertex x. This cone belongs to the tangent subspace T_x and is called the *asymptotic cone*.

If we place the points $A_{i_1},\ i_1 = n+1, \dots, n+m$, of our moving frames into the second osculating subspace $T_x^{(2)}$, as we did in Section 2.2.2, then by (2.38), the equations of the asymptotic cone C_x at the point x can be written as

$$b_{ij}^{i_1} \xi^i \xi^j = 0, \quad i_1 = n+1, \dots, n+m. \tag{2.39}$$

The problem of existence of asymptotic directions at the point x of the variety X is reduced to finding nontrivial solutions of the system of equations (2.39). This is an algebraic problem. In general, nontrivial solutions exist if $m \leq n-1$. However, in some special cases, nontrivial solutions of equations (2.39) may exist even if $m > n-1$.

2.2.4 The Osculating Subspace, the Second Fundamental Form, and the Asymptotic Cone of the Grassmannian. As an example, we now consider the second osculating subspace and the second fundamental form for the Grassmannian $\mathbb{G}(m,n)$.

As in Section 1.4, we denote by $\Omega(m,n)$ the image of the Grassmannian $\mathbb{G}(m,n)$ under the Grassmann mapping. This image is a variety of dimension $\rho = (m+1)(n-m)$ in the projective space $\mathbb{P}^N$, where $N = \binom{n+1}{m+1} - 1$.

With each element $p = \mathbb{P}^m$ of $\mathbb{G}(m,n)$ we associate a family of moving frames whose points $A_i,\ i = 0, 1, \dots, m$, span the subspace $\mathbb{P}^m$. Then we have

$$dA_i = \omega_i^j A_j + \omega_i^\alpha A_\alpha, \quad \alpha = m+1, \dots, n, \tag{2.40}$$

where ω_i^α are the basis forms of $\mathbb{G}(m,n)$.

The subspace $\mathbb{P}^m$ can be represented as

$$p = A_0 \wedge A_1 \wedge \ldots \wedge A_m, \tag{2.41}$$

where the symbol $\wedge$ denotes the exterior product. Differentiating (2.41) and using (2.40), we obtain

$$dp = \omega p + \omega_i^\alpha p_\alpha^i, \tag{2.42}$$

where $\omega = \omega_0^0 + \omega_1^1 + \ldots + \omega_m^m$, and

$$p_\alpha^i = A_0 \wedge A_1 \wedge \ldots \wedge A_{i-1} \wedge A_\alpha \wedge A_{i+1} \wedge \ldots \wedge A_m.$$

This implies that the tangent subspace T_p to the variety $\Omega(m,n)$ is the span of the points p and p_α^i.

Formula (2.42) proves that the forms ω_i^α are coordinates of a point in the projective space T_p/p with respect to the moving frame $\widetilde{p}_\alpha^i = p_\alpha^i / p$.

To find the second differential of the point p, we first differentiate the points p_α^i and then apply projectivization with the center T_p. This gives

$$dp_\alpha^i / T_p = \omega_j^\beta \widetilde{p}_{\alpha\beta}^{ij}, \tag{2.43}$$

where

$$\widetilde{p}^{ij}_{\alpha\beta} = p^{ij}_{\alpha\beta}/T_p$$

and

$$p^{ij}_{\alpha\beta} = A_0 \wedge A_1 \wedge \ldots \wedge A_{i-1} \wedge A_\alpha \wedge A_{i+1} \wedge \ldots \wedge A_{j-1} \wedge A_\beta \wedge A_{j+1} \wedge \ldots \wedge A_m.$$

Thus, the points $p^{ij}_{\alpha\beta}$ are skew-symmetric with respect to both the upper and lower indices. By equation (2.43), the projectivization with the center T_p of the second differential of the point p has the form

$$d^2p/T_p = \frac{1}{2}\sum_{\alpha,\beta,i,j}(\omega_i^\alpha\omega_j^\beta - \omega_i^\beta\omega_j^\alpha)\widetilde{p}^{ij}_{\alpha\beta}. \tag{2.44}$$

The right-hand side of this expression is the second fundamental form Φ of the image $\Omega(m,n)$ of the Grassmannian $\mathbb{G}(m,n)$. The coordinates of this form are written as follows:

$$\omega^{\alpha\beta}_{ij} = \omega_i^\alpha\omega_j^\beta - \omega_i^\beta\omega_j^\alpha. \tag{2.45}$$

It follows that the forms $\omega^{\alpha\beta}_{ij}$ are skew-symmetric in both the upper and lower indices. If $i < j$ and $\alpha < \beta$, the points $p^{ij}_{\alpha\beta}$ are linearly independent, and their number is equal to $\rho_1 = \binom{m+1}{2}\binom{n-m}{2}$. The number of linearly independent forms $\omega^{\alpha\beta}_{ij}$ is equal to the same number ρ_1. The points p, p^i_α, and $p^{ij}_{\alpha\beta}$ determine the second osculating subspace $T^{(2)}_p$ of the variety $\Omega(m,n)$ at the point p. Because the dimension of the tangent space T_p of $\Omega(m,n)$ is equal to

$$\dim T_p = (m+1)(n-m) = \binom{m+1}{1}\binom{n-m}{1}, \tag{2.46}$$

the dimension of its second osculating subspace $T^{(2)}_p$ is given by the formula:

$$\dim T^{(2)}_p = \binom{m+1}{1}\binom{n-m}{1} + \binom{m+1}{2}\binom{n-m}{2}. \tag{2.47}$$

The equation of the asymptotic cone C of the variety $\Omega(m,n)$ has the form

$$\omega^{\alpha\beta}_{ij} = \omega_i^\alpha\omega_j^\beta - \omega_i^\beta\omega_j^\alpha = 0. \tag{2.48}$$

Because the forms $\omega^{\alpha\beta}_{ij}$ are the minors of second order of the rectangular matrix

$$M = (\omega_i^\alpha), \tag{2.49}$$

equations (2.48) are equivalent to the conditions

$$\text{rank } M = 1. \tag{2.50}$$

But as we noted in Section 1.4, in the projective space T_p/p this condition defines the Segre variety $S(m-1, n-m-1)$ carrying plane generators of dimensions $m-1$ and $n-m-1$. The Segre variety $S(m-1, n-m-1)$ is the projectivization of the asymptotic cone C, which is the *Segre cone* $C(m, n-m)$. The vertex of this cone is the point p, and its director manifold is the Segre variety $S(m-1, n-m-1)$.

2.2.5 Varieties with One-Dimensional Normal Subspaces. Consider an n-dimensional variety $X = V^n$ belonging to a projective space $\mathbb{P}^{n+1}$. Such a variety is called a *hypersurface*. For a hypersurface X, equations (2.5), (2.11), and (2.20) have the forms

$$\omega_0^{n+1} = 0, \tag{2.51}$$

$$\omega_i^{n+1} = b_{ij}\omega^j, \;\; b_{ij} = b_{ji}, \tag{2.52}$$

$$\Phi = b_{ij}\omega^i\omega^j, \tag{2.53}$$

where $b_{ij} = b_{ij}^{n+1}$ is the second fundamental tensor of the hypersurface X.

If $\Phi \equiv 0$ at any point $x \in X$, then as we showed in Section 2.2.1, the hypersurface X coincides with its first osculating subspace, i.e., it degenerates into a hyperplane.

If the form Φ does not identically vanish, then the osculating subspace $T_x^{(2)}$ coincides with the space $\mathbb{P}^{n+1}$. Moreover, in this case, the normal subspace N_x is of dimension 1 and coincides with the reduced normal subspace $\widetilde{N}_x$. The hypersurface X has a single relatively invariant second fundamental form Φ, which at any point x determines the cone $C_x \subset T_x$ of asymptotic directions with vertex at x. The cone C_x is defined by the equation

$$\Phi = b_{ij}\omega^i\omega^j = 0. \tag{2.54}$$

Consider a variety $X = V^n$ in the space $\mathbb{P}^N$, and suppose that all second fundamental forms Φ^α, $\alpha = n+1, \ldots, N$, of X are proportional. In this case, the points of the variety X are called *axial*, and the reduced normal subspaces $\widetilde{N}_x$ of X are of dimension 1, as was the case for a hypersurface.

Specializing the moving frames in the same way as in Section 2.2.2, we obtain

$$\Phi^{n+1} = b_{ij}\omega^i\omega^j, \tag{2.55}$$

$$\Phi^{\alpha_1} = 0, \; \alpha_1 = n+2, \ldots, N. \tag{2.56}$$

Thus, equations (2.25) and (2.26) have the form

$$\omega_i^{n+1} = b_{ij}\omega^j, \;\; \omega_i^{\alpha_1} = 0. \tag{2.57}$$

Because now the index i_1 takes only one value, formula (2.33) can be written as follows:

$$\omega_{n+1}^{\alpha_1} = c_k^{\alpha_1}\omega^k, \quad \alpha_1 = n+2, \dots, N, \tag{2.58}$$

and formula (2.52) can be written as

$$b_{ij}c_k^{\alpha_1} = b_{ik}c_j^{\alpha_1}. \tag{2.59}$$

We can now prove the following result.

Theorem 2.1. *If all points of a variety $X = V^n$ of a projective space $\mathbb{P}^N$ are axial, then either the variety X belongs to its fixed osculating subspace $T_x^{(2)}$ of dimension $n+1$, or this variety is a torse, i.e., it is an envelope of a one-parameter family of n-dimensional subspaces.*

Proof. Suppose that rank $\Phi = r \geq 2$. Then the matrix of this form can be reduced to a diagonal form, i.e., $b_{ij} = 0,\, i \neq j, b_{aa} \neq 0, b_{uu} = 0, a = 1, \dots r;$ $u = r+1, \dots, n$. As a result, equations (2.59) take the form

$$b_{aa}c_k^{\alpha_1} = 0, \;\; k \neq a.$$

But because the index a takes more than one value, this implies that

$$c_k^{\alpha_1} = 0 \text{ for any } \; k = 1, \dots, n.$$

Thus, we have $\omega_{n+1}^{\alpha_1} = 0$, the subspace $T_x^{(2)} = A_0 \wedge A_1 \wedge \ldots \wedge A_n \wedge A_{n+1}$ remains fixed when the point x moves along the variety X, and $X \subset T_x^{(2)}$.

If rank $\Phi = r = 1$, then the matrix of Φ can be reduced to the form in which

$$b_{11} \neq 0, \, b_{ij} = 0 \text{ if } i \neq 1 \text{ or } j \neq 1.$$

As a result, equations (2.59) take the form

$$b_{11}c_k^{\alpha_1} = 0, \;\; k \neq 1.$$

It follows that $c_k^{\alpha_1} = 0$, and the forms $\omega_{n+1}^{\alpha_1}$ become

$$\omega_{n+1}^{\alpha_1} = c_1^{\alpha_1}\omega^1.$$

Thus, the family of tangent subspaces T_x of the variety X depends on one parameter, and therefore this variety is a torse (see Example 2.5 in Section 2.4). □

In the case when X is a variety of an N-dimensional space of constant curvature, a similar theorem was proved by C. Segre (see [SegC 07], p. 571), and for this reason, it is called the *Segre theorem*. The proof given above implies that the result of Segre's theorem does not depend on a metric but is of pure projective nature. So our theorem is a *generalized Segre theorem*.

2.3 Rank and Defect of Varieties with Degenerate Gauss Maps

To a regular point $x \in X \subset \mathbb{P}^N$, there corresponds the tangent subspace T_x. Because T_x is an element of the Grassmannian $\mathbb{G}(n, N)$, the variety X defines a map

$$\gamma : \ X \to \mathbb{G}(n, N). \tag{2.60}$$

As we said earlier, under this map, we have $\gamma(x) = T_x(X)$. We called the map γ the *Gauss map*.

We denote the image of the variety X under the Gauss map γ by $\gamma(X)$. Denote by r the rank of the Gauss map $\gamma(X)$, rank $\gamma(X) = r$. It is obvious that $0 \leq r \leq n$. The *rank* of the variety X is defined as the rank of the map γ: rank $X =$ rank $\gamma(X)$.

Because $T_x = A_0 \wedge A_1 \wedge \ldots \wedge A_n$, the basis forms of the Grassmannian $\mathbb{G}(n, N)$ are the forms $\{\omega_0^\alpha, \omega_i^\alpha\}$. Thus, the Gauss map $\gamma(X)$ is defined by equations (2.5) and (2.11). It follows from these equations that

$$\operatorname{rank} \gamma(X) = \operatorname{rank} X = \operatorname{rank} (\omega_i^\alpha) = \operatorname{rank} (b_{ij}^\alpha \omega^j). \tag{2.61}$$

Let $x \in X$ be a regular point of a variety $X \subset \mathbb{P}^N$, and Φ_x be its second fundamental form at this point. Consider the subspace

$$T'_x = \{\xi \in T_x | \Phi_x(\xi, \eta) = 0 \ \text{ for any } \ \eta \in T_x\}.$$

By (2.20), in a coordinate form, this subspace is defined by the system of equations

$$b_{ij}^\alpha \xi^i = 0. \tag{2.62}$$

The number $l = \dim T'_x$ is called the *Gauss defect* of a variety X (see the book [FP 01], p. 89, by Fischer and Piontkowski) or the *index of relative nullity* of the second fundamental form Φ of the variety X at the point x (see the paper [CK 52] by Chern and Kuiper).

Comparing equations (2.61) and (2.62), we find that

$$l + r = n,$$

i.e., *the sum of the defect and the rank of a variety X coincides with its dimension.*

In what follows, we assume that at all points $x \in X$, its rank (and therefore its defect) takes a constant value.

If $r = \text{rank } X = n$, then the Gauss map γ is nondegenerate. In this case, the tangent subspace $T_x(X)$ to the variety X depends on n parameters, and the variety X is called *tangentially nondegenerate.* For such a variety, the forms ω_i^α in equations (2.11) cannot be expressed in terms of fewer than n linearly independent forms ω^i.

If $r = \text{rank } X < n$, then the Gauss map γ is degenerate. In this case, its Gauss image $\gamma(X)$ depends on r parameters, where $0 \leq r < n$. Then we say that the variety X is *tangentially degenerate of rank r*, or X is a *variety with a degenerate Gauss map of rank r*. We denote such variety by $X = V_r^n$, rank $X = r < n$. Varieties with a degenerate Gauss map of rank r foliate into their leaves L of dimension $l = n - r$, along which the tangent subspace $T_x(X)$ is fixed. This foliation is called the Monge–Ampère foliation (see Section 3.1.1). We will prove in Theorem 3.1 (see Section 3.1.3) that the leaves of this foliation are l-planes.

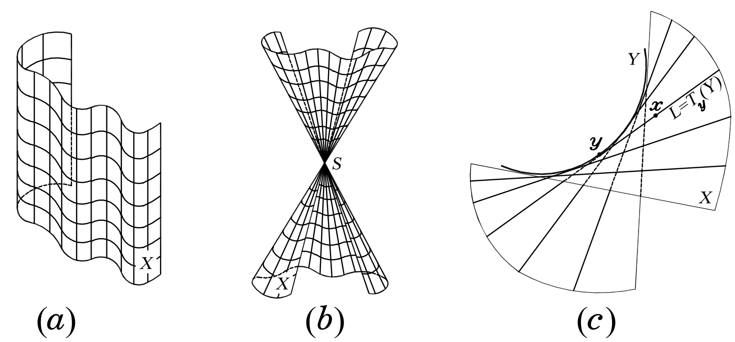

Figure 2.1

In a three-dimensional Euclidean space $\mathbb{E}^3$ $(N = 3, n = 2, r = 1)$ varieties with degenerate Gauss maps are known as *developable surfaces.* There are three classes of developable surfaces in E^3: cylinders, cones, and tangent developables of space curves (see Figure 2.1 $(a), (b), (c)$).

If rank $X = 0$, then the matrix (b_{ij}^α) is the zero matrix, the form Φ is also 0, $\Phi = 0$, and a variety X is a flat variety, i.e., X *is an n-dimensional projective subspace $\mathbb{P}^n$ of the space $\mathbb{P}^N$, or it is an open part of $\mathbb{P}^n$.*

2.4 Examples of Varieties with Degenerate Gauss Maps

Consider a few examples of varieties with degenerate Gauss maps.

Example 2.2. If $\operatorname{rank} X = \dim X = n$, then X is a variety of complete rank. X is also called *tangentially nondegenerate* in the space $\mathbb{P}^N$. Such varieties do not have singular points.

For example, the quadric Q defined in a three-dimensional projective space $\mathbb{P}^3$ by the equation

$$x_0 x_3 - x_1 x_2 = 0$$

is tangentially nondegenerate. For the quadric Q, we have $n = 2, N = 3$, $r = 2$, $l = 0$. Such a quadric bears two families of rectilinear generators. However, the tangent plane $T(Q)$ is not constant along these generators, i.e., none of these families compose the Monge–Ampère foliation.

Example 2.3. As we showed in Section 2.2.1, for $r = 0$, a variety X is an n-dimensional subspace $\mathbb{P}^n$, $n < N$. This variety is the only variety with a degenerate Gauss map without singularities in $\mathbb{P}^N$.

Example 2.4. Suppose that S is a subspace of the space $\mathbb{P}^N$, $\dim S = l - 1$, and T is its complementary subspace, $\dim T = N - l$, $T \cap S = \emptyset$. Let Y be a smooth tangentially nondegenerate variety of the subspace T, $\dim Y = \operatorname{rank} Y = r < N - l$. Consider an r-parameter family of l-dimensional subspaces $L_y = S \wedge y$, $y \in Y$. This variety is a *cone* X with vertex S and the director manifold Y. The subspace $T_x(X)$ tangent to the cone X at a point

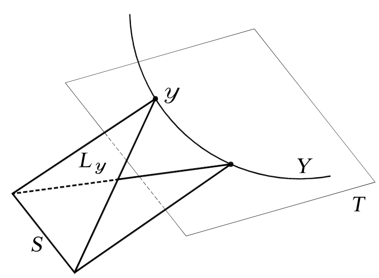

Figure 2.2

$x \in L_y (x \notin S)$ is defined by its vertex S and the subspace $T_y(Y)$, $T_x(X) = S \wedge T_y(Y)$, and $T_x(X)$ remains fixed when a point x moves in the subspace L_y. As a result, the cone X is a variety with a degenerate Gauss map of dimension $n = l + r$ and rank r, with plane generators L_y of dimension l (see

Figure 2.2). The generators L_y of the cone X are leaves of the Monge–Ampère foliation associated with X.

Example 2.5. Consider a smooth curve Y in the space $\mathbb{P}^N$ not belonging to a subspace $\mathbb{P}^{l+1} \subset \mathbb{P}^N$ and the set of its osculating subspaces L_y of order and dimension l. This set forms a variety $X = \cup_{y \in Y} L_y$ of dimension $l+1$ and rank $r = 1$ in $\mathbb{P}^N$. Such a variety is called a *torse* (cf. Section 2.2.5). The subspace $T_y = L_y + \frac{dL_y}{dy}$ is the tangent subspace to X at all points of its generator L_y. Thus, the subspaces L_y are the leaves of the Monge–Ampère foliation associated with the torse X. The subspace $F_y = L_y \cap \frac{dL_y}{dy}$ describes also a torse of dimension l. This process of construction of torses departing from X can be continued in both directions: from one side until we reach a smooth curve Y for which the subspace L_y is the osculating subspace of order $l-1$, and from the other side until we reach an $(N-1)$-dimensional variety (hypersurface) with a degenerate Gauss map. Figure 2.3 shows a torse in $\mathbb{P}^3$.

Conversely, a variety of dimension n and rank 1 is a torse formed by a family of osculating subspaces of order $n-1$ of a curve of class C^p, $p \geq n-1$, in the space $\mathbb{P}^N$.

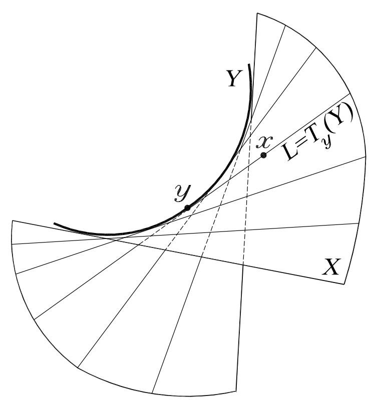

Figure 2.3

In what follows, unless otherwise stated, we always assume that $r > 1$.

In particular, we consider the spatial third-degree curve[1] C defined in the space $\mathbb{P}^3$ by the parametric equations $x(t) = (t^3, t^2, t, 1)$. The tangent line to C is determined by the point $x(t)$ and the point $x'(t) = (3t^2, 2t, 1, 0)$. The parametric equations of this tangent line have the form

$$y(t,s) = x(t) + sx'(t) = (t^3 + 3t^2 s, t^2 + 2ts, t + s, 1).$$

[1] Cayley [Cay 64] called such a curve a *twisted cubic*.

A surface swept by these tangent lines is a torse—a variety with a degenerate Gauss map of rank one and dimension two in the space $\mathbb{P}^3$. In this case we have $n = 2$, $N = 3$, $l = 1$, $r = 1$. The tangents to the line $x(t)$ are the leaves of the Monge–Ampère foliation associated with this third-degree curve.

In order to obtain an equation of form (2.2) of the torse X formed by the tangents to the third-degree curve, we need to exclude parameters t and s from the parametric equations of the third-degree curve and its tangent line. An equation of this torse can also be obtained by a method indicated by Cayley (see [Cay 64]).

Let (y_0, y_1, y_2, y_3) be homogeneous coordinates of the space $\mathbb{P}^3$. Consider the nonhomogeneous polynomial

$$\psi(t) := y_0 t^3 + y_1 t^2 + y_2 t + y_3.$$

An osculating plane of the third-degree curve $x(t) = (t^3, t^2, t, 1)$ is defined by the points $x(t), x'(t) = (3t^2, 2t, 1, 0)$, and $x''(t) = (6t, 2, 0, 0)$. So, the equation of this plane is

$$\begin{vmatrix} y_0 & y_1 & y_2 & y_3 \\ t^3 & t^2 & t & 1 \\ 3t^2 & 2t & 1 & 0 \\ 6t & 2 & 0 & 0 \end{vmatrix} = 0$$

or

$$\psi^*(t) := y_0 - 3y_1 t + 3y_2 t^2 - y_3 t^3 = 0.$$

It follows from this form of $\psi^*(t)$ that the dual curve $x^*(t)$ has the parameterization $x^*(t) = (1, -3t, 3t^2, -t^3)$. Its osculating plane is defined by the points $x^*(t), (x^*)'(t) = (0, -3, 6t, -3t^2)$, and $(x^*)''(t) = (0, 0, 6, -6t)$. Thus, its equation is

$$\begin{vmatrix} y_0 & y_1 & y_2 & y_3 \\ 1 & -3t & 3t^2 & -t^3 \\ 0 & -3 & 6t & -3t^2 \\ 0 & 0 & 6 & -6t \end{vmatrix} = 0.$$

Easy computation shows that this equation is

$$\psi(t) = 0.$$

The torse X is the envelope of the family of osculating planes $\psi^*(t) = 0$ of the third-degree curve $x(t)$, and the torse X^* is the envelope of the family of osculating planes $\psi(t) = 0$ of the dual curve $x^*(t)$.

We find equations of both torses X and X^*.

According to Cayley [Cay 64], an equation of the torse X^* is

$$\text{Disct } \psi(t) = 0,$$

where Disct $\psi(t)$ is the discriminant of the polynomial $\psi(t)$. Computing the discriminant

$$\text{Disct } \psi(t) = \begin{vmatrix} y_0 & y_1 & y_2 & y_3 & 0 \\ 0 & y_0 & y_1 & y_2 & y_3 \\ 3y_0 & 2y_1 & y_2 & 0 & 0 \\ 0 & 3y_0 & 2y_1 & y_2 & 0 \\ 0 & 0 & 3y_0 & 2y_1 & y_2 \end{vmatrix}$$

up to a factor of y_0, we obtain the following equation of the torse X^*:

$$\Psi := 27y_0^2y_3^2 - 18y_0y_1y_2y_3 + 4y_0y_2^3 + 4y_1^3y_3 - y_1^2y_2^2 = 0.$$

We can find an equation of the torse X (which is the envelope of the osculating planes $\psi^*(t) = 0$ of the third-degree curve $x^*(t)$) by computing the discriminant of the polynomial $\psi^*(t)$.

However, it is easier to find this equation by making the substitution

$$y_0 \to y_3, \quad y_1 \to -3y_2, \quad y_2 \to 3y_1, \quad y_3 \to -y_0$$

in the equation of the torse X. The result is

$$\Psi^* := y_0^2y_3^2 - 6y_0y_1y_2y_3 + 4y_0y_2^3 + 4y_1^3y_3 - 3y_1^2y_2^2 = 0.$$

This equation shows that the surface swept by the tangents to the third-degree curve is an algebraic fourth-degree surface.

Note that Cayley [Cay 64] took equations of the family of osculating planes of the torse X in the form

$$y_0t^3 + 3y_1t^2 + 3y_2t^2 + y_3 = 0.$$

Comparing this with $\psi^*(t) = 0$, we see that Cayley used the following parametrization of a third-degree curve: $(1, -t, t^2, -t^3)$. It is easy to check that the equations of the torses X and X^* for Cayley's parameterization are precisely the same as for our parameterization. Namely, the torses X and X^* for Cayley's parameterization are defined by the equations $\Psi^* = 0$ (see [Cay 64]) and $\Psi = 0$, respectively.

In his paper [Ca 64], Cayley found equations of torses formed by the tangents to two special fourth-degree curves (quartics) $u(t)$ and $v(t)$ in the space

$\mathbb{P}^3$. He did not indicate the equations of these fourth-degree curves—he found equations of the torses as envelopes of the families of osculating planes of the dual curves $u^*(t)$ and $v^*(t)$.

In addition, in his paper [Cay 64], Cayley considered in $\mathbb{P}^3$ the fourth-degree curves $u(t) = (81, -27t, 9t^2, t^4)$ and $v(t) = (-2, t, -t^3, 2t^4)$ and found equations of the torses formed by the tangents to these curves. These torses are defined by the algebraic equations

$$y_0^3 y_3^2 + 6y_0^2 y_2^2 y_3 - 24 y_0 y_1^2 y_2 y_3 + 9 y_0 y_2^4 + 16 y_1^4 y_3 - 8 y_1^2 y_2^3 = 0$$

and

$$y_0^3 y_3^3 - 12 y_0^2 y_1 y_2 y_3^2 - 27 y_0^2 y_2^4 - 6 y_0 y_1^2 y_2^2 y_3 - 27 y_1^4 y_3^2 - 64 y_1^3 y_2^3 = 0.$$

These equations can be derived in a way similar to what we used to find the equation $\Psi^*(t) = 0$ of the torse formed by the tangents to the third-degree curve $x(t) = (t^3, t^2, t, 1)$.

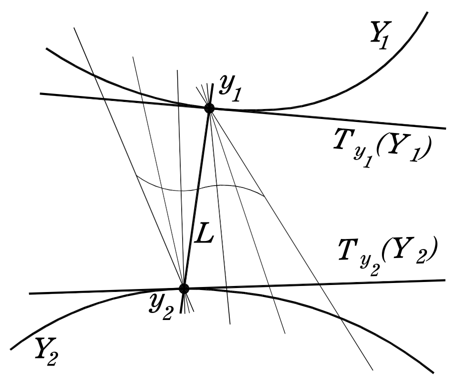

Figure 2.4

Example 2.6. In the space $\mathbb{P}^N$, $N \geq 4$, we take two arbitrary smooth space curves, Y_1 and Y_2, that do not belong to the same three-dimensional space, and the set of all straight lines intersecting these two curves (see Figure 2.4). These straight lines form a three-dimensional variety X. Such a variety is called the *join*. Its dimension is three, dim $X = 3$. It is easy to see that the variety X has a degenerate Gauss map. In fact, the three-dimensional tangent subspace $T_x(X)$ to X at a point x lying on a rectilinear generator L is defined by this generator L and two straight lines tangent to the curves Y_1 and Y_2 at the points y_1 and y_2 of their intersection with the line L. Because this tangent subspace does not depend on the location of the point x on the generator L, the variety under consideration is a variety $X = V_2^3$ with a degenerate Gauss map of rank two.

This example can be generalized by taking k spatial curves in the space $\mathbb{P}^N$, where $N \geq 2k$ and $k > 2$, and considering a k-parameter family of $(k-1)$-planes intersecting all these k curves.

Example 2.7. Let $N = n + 1$, and let Y be an r-parameter family of hyperplanes ξ in general position in $\mathbb{P}^{n+1}$, $r < n$. Such a family has an n-dimensional envelope X that is a variety with a degenerate Gauss map of dimension n and rank r in the space $\mathbb{P}^{n+1}$. It foliates into an r-parameter family of plane generators L of dimension $l = n-r$, along which the tangent subspace $T_x(X)$, $x \in L$, is fixed and coincides with a hyperplane ξ of the family in question. Thus, X is a hypersurface with a degenerate Gauss map of rank r with $(n-r)$-dimensional plane generators L in the space $\mathbb{P}^{n+1}$.

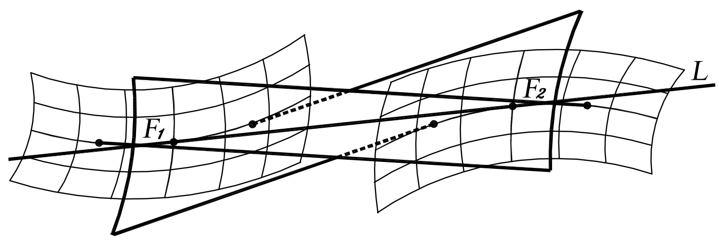

Figure 2.5

Figure 2.5 represents the case $n = 3, r = 2$, i.e., a hypersurface $X = V_2^3 \subset \mathbb{P}^4$.

2.5 Application of the Duality Principle

2.5.1 Dual Variety. For construction of new examples of varieties with degenerate Gauss maps we employ the duality principle in a projective space introduced in Section 1.3.2.

By the duality principle, to a point x of a projective space $\mathbb{P}^N$, there corresponds a hyperplane ξ. A set of hyperplanes of space $\mathbb{P}^N$ forms the dual projective space $(\mathbb{P}^N)^*$ of the same dimension N. Under this correspondence, to a subspace $P \subset \mathbb{P}^N$ of dimension p, there corresponds a subspace $\mathbb{P}^* \subset (\mathbb{P}^N)^*$ of dimension $N - p - 1$. Under the dual map, the incidence of subspaces is reversed, that is, if $\mathbb{P}_1 \subset \mathbb{P}_2$, then $\mathbb{P}_1^* \supset \mathbb{P}_2^*$.

Let X be an irreducible, almost everywhere smooth variety of dimension n, $\dim X = n$, in the space $\mathbb{P}^N$, let x be a smooth point of X, and let $T_x X$ be the tangent subspace to X at the point x. A hyperplane ξ is said to be *tangent to* X *at* x if $T_x \subset \xi$. The bundle of hyperplanes ξ tangent to X at x is of dimension $N - n - 1$.

The set of all hyperplanes ξ tangent to the variety X at its smooth points composes a variety

$$X^\wedge = \{\xi \subset P^N | \exists\, x \in X_{sm} \text{ such that } T_x X \subseteq \xi\}.$$

But this variety can be not closed if X has singular points. The *dual variety* X^* of a variety X is the closure of the variety $X^\wedge$:

$$X^* = \overline{X^\wedge} = \overline{\{\xi \subset P^N | \exists\, x \in X_{sm} \text{ such that } T_x X \subseteq \xi\}}. \tag{2.63}$$

The dual variety X^* can also be described as the envelope of the family of hyperplanes ξ dual to the points $x \in X$. This gives a practical way for finding X^*, which we will use in examples.

If a variety X is tangentially nondegenerate, i.e., if its rank $r = n$, then in the general case, the dimension n^* of its dual variety X^* is equal to

$$n^* = \dim X^* = (N - n - 1) + n = N - 1. \tag{2.64}$$

Equation (2.64) means that the variety X^* is a hypersurface with a degenerate Gauss map in the space $(\mathbb{P}^N)^*$. The rank r of X^* equals the dimension n of the variety X, $r = \operatorname{rank} X^* = n$, and its Gauss defect $\delta_\gamma(X^*) = l^* = n^* - r = N - r - 1$.

However, it may happen that $\dim X^* < N - 1$. Then the number

$$\delta_* = N - 1 - \dim X^*$$

is called the *dual defect* of the variety X, and the variety X itself is said to be *dually degenerate*.

An example of a dually degenerate smooth variety is the Segre variety $X = \operatorname{Seg}\, (\mathbb{P}^m \times \mathbb{P}^n) \subset \mathbb{P}^{mn+m+n}$, whose dual defect equals $|m - n|$ (see Example 2.11).

If a variety X has a degenerate Gauss map (i.e., if its rank $r < n$), then the dual variety X^* is a fibration whose fiber is the bundle $\Xi = \{\xi \subset \mathbb{P}^N | \xi \supseteq T_L X\}$ of hyperplanes ξ containing the tangent subspace $T_L X$ and whose base is the manifold $B = X^*/\Xi$. The dimension of a fiber Ξ of this fibration (as in the case $r = n$) equals $N - n - 1$, $\dim \Xi = N - n - 1$, and the dimension of the base B equals r, $\dim B = r$, i.e., the dimension of B coincides with the rank of the variety X. Therefore, in the general case, the dimension n^* of its dual variety X^* is determined by the formula

$$\dim X^* = (N - n - 1) + r = N - l - 1, \tag{2.65}$$

where $l = \dim L = \delta_\gamma(X) = n - r$, and its Gauss defect is equal to $\delta_\gamma(X^*) = l^* = n^* - r = (N - l - 1) - r = N - n - 1 = \dim \Xi$.

However, it may happen that $\dim X^* < N - l - 1$. Then the number

$$\delta_* = N - l - 1 - \dim X^*$$

is called the *dual defect* of the variety X, and the variety X itself is said to be *dually degenerate*. Note that the dual defect of tangentially nondegenerate varieties (see p. 71) can be obtained from this new definition by taking $l = 0$.

Note also that dually degenerate smooth varieties in the projective space $\mathbb{P}^N$ are few and far between. As to dually degenerate varieties with degenerate Gauss maps, we are aware of only a few examples of dually degenerate varieties X with degenerate Gauss maps: the varieties X with degenerate Gauss maps of ranks three and four in $\mathbb{P}^N$ were considered by Piontkowski [Pio 02b].

This is why *in this book we consider only dually nondegenerate varieties in the space* $\mathbb{P}^N$, i.e., we assume that for the variety $X \subset \mathbb{P}^N$ of dimension n and rank r, the dimension of its dual variety is determined by formula (2.65).

2.5.2 The Main Theorem. The following theorem follows immediately from the preceding considerations.

Theorem 2.8. *Let X be a dually nondegenerate variety with a degenerate Gauss map of dimension n and rank r in the space $\mathbb{P}^N$. Then the leaves L of the Monge–Ampère foliation of X are of dimension $l = n - r$. The dual variety $X^* \subset (\mathbb{P}^N)^*$ is of dimension*

$$n^* = N - l - 1 \tag{2.66}$$

and the same rank r, and the leaves L^* of the Monge–Ampère foliation of X^* are of dimension

$$l^* = N - n - 1. \tag{2.67}$$

Under this map, the plane generator L^* corresponds to a tangent subspace $T_x(X)$ of the variety X, and the tangent subspace $T_\xi(X^*)$ of the variety X^* corresponds to a plane generator L, i.e., on X the tangent bundle $T(X)$ and the Monge–Ampère foliation $L(X)$ are mutually dual.

In particular, if a variety $X \subset \mathbb{P}^N$ is tangentially nondegenerate, then we have $n = r$, $l = 0$ (i.e., $n^* = N - 1$), and the dual map (*) sends X to a hypersurface $X^* \subset (\mathbb{P}^N)^*$ with a degenerate Gauss map of rank n with the leaves L^* of the Monge–Ampère foliation of dimension $l^* = N - n - 1$.

Conversely, if X is a hypersurface with a degenerate Gauss map of rank $r < N-1$ in $\mathbb{P}^N$, then the variety X^* dual to X is a tangentially nondegenerate variety of dimension r and rank r.

In particular, the dual map (*) sends a tangentially nondegenerate variety $X \subset \mathbb{P}^N$ of dimension and rank $r = n = N-2$ to a hypersurface $X^* \subset (\mathbb{P}^{n+2})^*$ with a degenerate Gauss map of rank r, and X^* bears an r-parameter family of rectilinear generators. Each of these rectilinear generators possesses r foci if each is counted as many times as its multiplicity. The hypersurface X^* is torsal and foliates into r families of torses. The original variety X bears a net of conjugate lines corresponding to the torses of the variety X^*. Of course, the correspondence indicated above is mutual.

We consider an irreducible, almost everywhere smooth variety X of dimension n and rank r in the space $\mathbb{P}^N$ in more detail. The tangent bundle $T(X)$ of X is formed by the n-dimensional subspaces T_x tangent to X at points $x \in X$ and depending on r parameters. The subspaces T_x are tangent to X along the plane generators L of dimension $l = n - r$ composing on X the Monge–Ampère foliation $L(X)$. The bundle $T(X)$ and the foliation $L(X)$ have a common r-dimensional base.

Let (*) be the dual map of $\mathbb{P}^N$ onto $(\mathbb{P}^N)^*$. The dual map (*) sends the variety X to a variety X^*, which is the set of all hyperplanes $\xi \subset (\mathbb{P}^N)^*$ tangent to X along the leaves L of its Monge–Ampère foliation. The map (*) sends the tangent bundle $T(X)$ and the Monge–Ampère foliation $L(X)$ of X to the Monge–Ampère foliation $L(X^*)$ and the tangent bundle $T(X^*)$ of X^*, respectively. Thus, under the dual map (*), we have

$$(T(X))^* = L(X^*), \quad (L(X))^* = T(X^*),$$

where $\dim T(X^*) = \dim X^* = n^* = N - l - 1$ and $\dim L(X^*) = \dim L^* = l^* = N - n - 1$.

We now consider a few examples.

Example 2.9. First, we consider a simple example. Let X be a smooth spatial curve X in a three-dimensional projective space $\mathbb{P}^3$. For this curve, we have $N = 3$, $n = r = 1$, $l = 0$, and $T_x(X)$ is the tangent line to X at x. The dual map (*) sends a point $x \in X$ to a plane $\xi \subset X^*$, and the dual variety X^* is the envelope of the one-parameter family of hyperplanes ξ (see Figure 2.6), i.e, X^* is a torse.

Using the formulas for n^* and l^* written earlier we find that $n^* = 2$, $l^* = 1$. The variety X^* bears rectilinear generators L^* along which the tangent planes

$\xi = T(X^*)$ are constant. Hence rank $X^* = 1$. The generators L^* of the torse X^* are dual to the tangent lines $T(X)$ to the curve X.

Next, we determine which varieties correspond to the varieties with degenerate Gauss maps considered in Examples 2.4, 2.5, and 2.7.

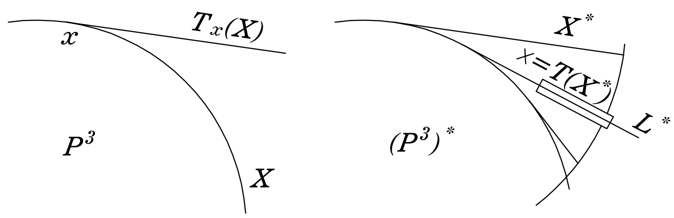

Figure 2.6

Example 2.10. To a cone X of rank r with vertex S of dimension $l-1$ (see Example 2.4), there corresponds a variety X^* lying in the subspace $T = S^*$, $\dim T = N - l$. Because $\dim X^* = n^* = N - l - 1$, the variety X^* is a hypersurface of rank r in the subspace T. Such a hypersurface was considered in Example 2.7.

If a tangentially nondegenerate variety X, $\dim X = \operatorname{rank} X = r$, belongs to a subspace $\mathbb{P}^{n+1} \subset \mathbb{P}^N$, then we can consider two dual maps in the spaces $\mathbb{P}^{n+1}$ and $\mathbb{P}^N$. We denote the first of these maps by $*$ and the second by $\circ$. Then under the first map, the image of X is a hypersurface $X^* \subset \mathbb{P}^{n+1}$, and under the second map, the hypersurface X is transferred into a cone X° of rank r and dimension $n^\circ = N - n + r - 1$ with an $(N-n-2)$-dimensional vertex $S = (\mathbb{P}^{n+1})^\circ$ and $(N-n-1)$-dimensional plane generators $L^\circ = T(X)^\circ$. It follows that Examples 2.4 and 2.7 are mutually dual to each other.

For the torse X (see Example 2.5), we have $n = l + 1$, $r = 1$ and $n^* = N - l - 1$, $l^* = N - l - 2$, i.e., the dual image X^* of a torse X is a torse.

Thus, the varieties considered in Examples 2.4, 2.5, and 2.7 are dual to varieties considered in 2.7, 2.5, and 2.4, respectively.

Example 2.11. *The Segre variety* (see Griffiths and Harris [GH 79] and Tevelev [T 01]) $S(m,n)$ is the embedding of the direct product of the projective spaces $\mathbb{P}^m$ and $\mathbb{P}^n$ in the space $\mathbb{P}^{mn+m+n}$:

$$S : \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^{mn+m+n},$$

defined by the equations

$$z^{ik} = x^i y^k,$$

where $i = 0, 1, \ldots, m$, $k = 0, 1, \ldots, n$, and x^i, y^k, and z^{ik} are the coordinates of points in the spaces $\mathbb{P}^m, \mathbb{P}^n$, and $\mathbb{P}^{mn+m+n}$, respectively. This manifold has the dimension $m + n$, $\dim S(m, n) = m + n$.

Consider in the spaces $\mathbb{P}^m$ and $\mathbb{P}^n$ projective frames $\{A_0, A_1, \ldots, A_m\}$ and $\{B_0, B_1, \ldots, B_n\}$. Then in the space $\mathbb{P}^{mn+m+n}$ we obtain the projective frame

$$\{A_0 \otimes B_0,\ A_0 \otimes B_k,\ A_i \otimes B_0,\ A_i \otimes B_k\}$$

(here and in what follows $i, j = 1, \ldots, m$; $k, l = 1, \ldots, n$) consisting of $(m + 1)(n + 1)$ linearly independent points of the space $\mathbb{P}^{mn+m+n}$. The point $A_0 \otimes B_0$ is the generic point of the variety S.

In the spaces $\mathbb{P}^m$ and $\mathbb{P}^n$, we have the following equations:

$$dA_0 = \omega_0^0 A_0 + \omega_0^i A_i, \quad dB_0 = \sigma_0^0 B_0 + \sigma_0^k B_k$$

(see (1.71)). Hence

$$d(A_0 \otimes B_0) = (\omega_0^0 + \sigma_0^0)(A_0 \otimes B_0) + \omega_0^i (A_i \otimes B_0) + \sigma_0^k (A_0 \otimes B_k),$$

and the subspace in $\mathbb{P}^{mn+m+n}$ spanned by the points $A_0 \otimes B_0, A_i \otimes B_0$, and $A_0 \otimes B_k$ is the tangent subspace to the Segre variety S at the point $A_0 \otimes B_0$:

$$T_{A_0 \otimes B_0} = \text{Span}\ (A_0 \otimes B_0,\ A_i \otimes B_0,\ A_0 \otimes B_k).$$

The second differential of the point $A_0 \otimes B_0$ has the form:

$$d^2(A_0 \otimes B_0) = 2\,\omega_0^i\,\sigma_0^k\,A_i \otimes B_k \pmod{T_{A_0 \otimes B_0}}.$$

Hence the osculating subspace $T^2_{A_0 \otimes B_0}(S)$ to the variety S coincides with the entire space $\mathbb{P}^{mn+m+n}/T_{A_0 \otimes B_0}$, and its second fundamental forms have the form

$$\Phi^{ik} = \omega_0^i \sigma_0^k.$$

The total number of these forms is mn. The equations $\omega_0^i = 0$ determine n-dimensional plane generators on S, and the equations $\sigma_0^k = 0$ determine its m-dimensional plane generators.

Consider a tangent hyperplane to the Segre variety S at the point $A_0 \otimes B_0$. Because such a hyperplane contains the tangent subspace $T_{A_0 \otimes B_0}$, its equation can be written in the form

$$\xi = \xi_{ik} z^{ik} = 0,$$

where $i = 1, \ldots, m$; $k = 1, \ldots, n$, and z^{ik} are coordinates of points in the space $\mathbb{P}^{mn+m+n}/T_{A_0 \otimes B_0}$. As a result, the second fundamental form of the variety S with respect to the hyperplane ξ is

$$\Phi(\xi) = \xi_{ik}\,\omega_0^i\,\sigma_0^k$$

(see (2.21)). The forms $\Phi(\xi)$ constitute the system of the second fundamental forms of the variety S. The mn forms Φ^{ik} are linearly independent forms of this system. The matrix of this system of second fundamental forms has the form

$$\Xi = \begin{pmatrix} 0 & (\xi_{ik}) \\ (\xi_{ki}) & 0 \end{pmatrix}.$$

In this formula (ξ_{ik}) is a rectangular $(m \times n)$-matrix and (ξ_{ki}) is its transpose.

It follows that $\det \Xi = 0$ if $m \neq n$. In this case, the system of the second fundamental forms of the variety S is degenerate, and the dual defect $\delta_*(S)$ of S equals $|n-m| : \delta_*(S) = |n-m|$. The variety S is dually nondegenerate if and only if $m = n$.

2.5.3 Cubic Symmetroid. Now we consider the Veronese variety given as the image of the embedding

$$V^* : \text{Sym}\,(\mathbb{P}^{2*} \times \mathbb{P}^{2*}) \to \mathbb{P}^{5*}$$

into the projective space $\mathbb{P}^{5*}$. This embedding is defined by the equations

$$x_{ij} = u_i u_j, \quad i, j = 0, 1, 2, \tag{2.68}$$

where u_i are projective coordinates in the plane $\mathbb{P}^{2*}$, i.e., tangential coordinates in the plane $\mathbb{P}^2$, and x_{ij} are projective coordinates in the space $\mathbb{P}^{5*}$, $x_{ij} = x_{ji}$.

Let us find an equation of the variety V that is dual to the variety $V^* \subset \mathbb{P}^{5*}$ defined by equations (2.68). This variety V is the envelope of the two-parameter family of hyperplanes defined in the space $\mathbb{P}^5$ by the equation

$$\xi = x^{ij} u_i u_j = 0, \quad i, j = 0, 1, 2. \tag{2.69}$$

Equation (2.69) depends on two affine parameters $u = \frac{u_1}{u_0}$ and $v = \frac{u_2}{u_0}$, and the quantities x^{ij} occurring in (2.69) are projective coordinates in the space $\mathbb{P}^5$. In order to find the equation of the envelope of the family (2.69), we differentiate equation (2.69) with respect to u_i. The result is

$$\frac{\partial \xi}{\partial u_i} = x^{ij} u_j = 0. \tag{2.70}$$

Eliminating the parameters u_j from equations (2.70), we arrive at the equation

$$\det\,(x^{ij}) = 0,$$

or in more detail,

$$F = \det \begin{pmatrix} x^{00} & x^{01} & x^{02} \\ x^{10} & x^{11} & x^{12} \\ x^{20} & x^{21} & x^{22} \end{pmatrix} = 0. \tag{2.71}$$

Equation (2.71) defines in the space $\mathbb{P}^5$ the cubic hypersurface dual to the Veronese variety (2.68) and called the *cubic symmetroid*.

The Veronese variety V^* defined by equation (2.68) is a tangentially nondegenerate variety in the space $\mathbb{P}^{5*}$. Thus, by Theorem 2.8, its dual variety V is a hypersurface with a degenerate Gauss map of rank two in the space $\mathbb{P}^5$ having two-dimensional leaves $L(V)$ of the Monge–Ampère foliation on V. The latter is dual to the tangent bundle $T(V^*)$ of V^*.

Next we find equations of the leaves $L(V)$ of the cubic symmetroid V. Three hyperplanes

$$\alpha_0 x^{0i} + \alpha_1 x^{1i} + \alpha_2 x^{2i} = 0, \quad i = 0, 1, 2, \tag{2.72}$$

of the space $\mathbb{P}^5$ have a common two-dimensional plane. It is easy to see that the coordinates of points of this 2-plane satisfy equation (2.71). In fact, by (2.72), the rows of the determinant on the left-hand side of (2.71) are linearly dependent, and hence the determinant vanishes. Hence equations (2.72) determine two-dimensional plane generators of the symmetroid V. Because equations (2.72) contain two variables $\frac{\alpha_1}{\alpha_0}$ and $\frac{\alpha_2}{\alpha_0}$, the symmetroid V carries a two-parameter family of two-dimensional plane generators.

The equation of the tangent hyperplane ξ at the point $x = (x^{ij})$ to the cubic symmetroid V defined by equations (2.71) has the form

$$\frac{\partial F}{\partial x^{ij}}\, y^{ij} = 0, \tag{2.73}$$

where y^{ij} are coordinates of an arbitrary point $y \in \xi$.

Equation (2.73) can be written in the form

$$F = x^{00}x^{11}x^{22} + 2x^{01}x^{12}x^{20} - x^{00}(x^{12})^2 - x^{11}(x^{02})^2 - x^{22}(x^{01})^2 = 0. \tag{2.74}$$

By (2.74), the coefficients of equation (2.73) are determined by the formulas

$$\begin{aligned}
&\frac{\partial F}{\partial x^{00}} = \begin{vmatrix} x^{11} & x^{12} \\ x^{21} & x^{22} \end{vmatrix}, \quad \frac{\partial F}{\partial x^{01}} = 2 \begin{vmatrix} x^{12} & x^{10} \\ x^{22} & x^{20} \end{vmatrix}, \\
&\frac{\partial F}{\partial x^{11}} = \begin{vmatrix} x^{00} & x^{02} \\ x^{20} & x^{22} \end{vmatrix}, \quad \frac{\partial F}{\partial x^{02}} = 2 \begin{vmatrix} x^{10} & x^{11} \\ x^{20} & x^{21} \end{vmatrix}, \\
&\frac{\partial F}{\partial x^{22}} = \begin{vmatrix} x^{00} & x^{01} \\ x^{10} & x^{11} \end{vmatrix}, \quad \frac{\partial F}{\partial x^{12}} = 2 \begin{vmatrix} x^{01} & x^{00} \\ x^{21} & x^{20} \end{vmatrix}.
\end{aligned} \tag{2.75}$$

Consider the plane generators L^0 of the cubic symmetroid V defined by equations (2.72) with $\alpha_0 = \alpha_1 = 0$, $\alpha_2 \neq 0$. For this generator, equations (2.72) take the form

$$x^{2i} = 0. \tag{2.76}$$

This implies that only one coefficient of equation (2.73), namely $\frac{\partial F}{\partial x^{22}}$, is nonvanishing. Hence, equation (2.73) takes the form

$$y^{22} = 0. \tag{2.77}$$

Equation (2.77) is the equation of the tangent hyperplane to V for all points of the generators L^0. As a result, the tangent hyperplane ξ is constant for all points of the generators L^0.

But all plane generators L of the cubic symmetroid V are projectively equivalent. Thus each of them is a leaf of the Monge–Ampère foliation on V, and the symmetroid V itself is a hypersurface with a degenerate Gauss map of rank $r = 2$ in the space $\mathbb{P}^5$. This corresponds to the contents of Theorem 2.8.

2.5.4 Singular Points of the Cubic Symmetroid. Next we find singular points of the cubic symmetroid V defined by equation (2.71). Such points are determined by the equations

$$\frac{\partial F}{\partial x^{ij}} = 0. \tag{2.78}$$

Because all plane generators of the symmetroid V are projectively equivalent, we will look for singular points on the plane generator L^0 defined by equations

(2.76). On this plane generator, all the determinants (2.75) are identically equal to zero, except the determinant $\frac{\partial F}{\partial x^{22}}$. As a result, singular points on the plane generator (2.76) are determined by the equation

$$\frac{\partial F}{\partial x^{22}} = x^{00}x^{11} - (x^{01})^2 = 0. \tag{2.79}$$

Equation (2.79) defines the locus of singular points in the plane generator L^0. Hence, *the locus of singular points in the plane generator L^0 is a conic.* Similarly, in all other generators L of the cubic symmetroid V, the loci of singular points are the second-degree curves (the focus curves F_L (see Section 3.2, p. 100) of these generators).

From (2.75) and (2.78) it follows that the set of all singular points on the entire cubic symmetroid V is determined by the equation

$$\operatorname{rank} x^{ij} = 1$$

or

$$x^{ij} = x^i x^j, \quad i, j = 0, 1, 2 \tag{2.80}$$

(cf. equations (2.68)). This means that *the set of singular points of the cubic symmetroid $V \subset \mathbb{P}^5$ is a Veronese surface $V^* \subset \mathbb{P}^{5*}$.*

Most likely, all these results are well known in algebraic geometry. However, we obtained them here by the methods of differential geometry.

Now we give one more interpretation of the properties of the cubic symmetroid $V \subset \mathbb{P}^5$. To this end, we denote the entries of the matrix on the left-hand side of (2.71) by a_{ij}, i.e., we write this matrix in the form

$$A = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix}, \text{ where } a_{ij} = a_{ji}.$$

Because the matrix A is defined up to a nonvanishing factor, in the projective plane $\mathbb{P}^2$, it determines a second-degree curve

$$a_{ij} x^i x^j = 0, \;\; i, j = 0, 1, 2$$

(see Figure 2.7 (a)). To the cubic symmetroid V defined in $\mathbb{P}^5$ by the equation

$$\det A = 0, \tag{2.81}$$

there corresponds in $\mathbb{P}^2$ the set of second-degree curves that decompose into two straight lines

$$a_i x^i = 0, \;\; b_i x^i = 0, \quad i = 0, 1, 2, \tag{2.82}$$

(see Figure 2.7 (b)).

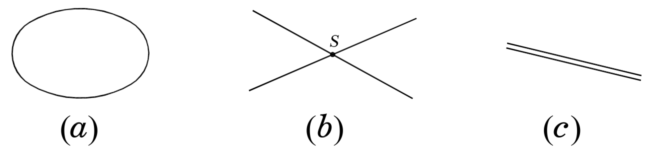

Figure 2.7

To the plane generator $L \subset V$, there corresponds in $\mathbb{P}^2$ the set of second-degree curves of type (2.82) decomposed into two intersecting straight lines with a common point of intersection for all pairs. The family of these plane generators L depends on two parameters because the points of $\mathbb{P}^2$ depend on two parameters.

To a tangent hyperplane of the cubic symmetroid $V \subset \mathbb{P}^5$ at the point $a_{ij} = a_{(i}\, b_{j)}$, there corresponds in $\mathbb{P}^2$ the set of second-degree curves passing through the common point of the straight lines (2.82).

To the set of singular points of the symmetroid $V \subset \mathbb{P}^5$ defined by the equation

$$\operatorname{rank} A = 1,$$

there corresponds in $\mathbb{P}^2$ the set of second-degree curves degenerating into two coinciding straight lines (see Figure 2.7 (c)).

2.5.5 Correlative Transformations. If we have the identification $(\mathbb{P}^N)^* = \mathbb{P}^N$, the duality principle can be realized by a correlative transformation of the space $\mathbb{P}^N$.

Consider a *correlative transformation* $\mathcal{C}$ (a *correlation*) in the space $\mathbb{P}^N$ that maps a point $x \in \mathbb{P}^N$ into a hyperplane $\xi \in \mathbb{P}^N,\ \xi = \mathcal{C}(x)$, and preserves the incidence of points and hyperplanes. A correlation $\mathcal{C}$ maps a k-dimensional subspace $\mathbb{P}^k \subset \mathbb{P}^N$ into an $(N-k-1)$-dimensional subspace $\mathbb{P}^{N-k-1} \subset \mathbb{P}^N$.

We assume that the correlation $\mathcal{C}$ is nondegenerate, i.e., it defines a one-to-one correspondence between points and hyperplanes of the space $\mathbb{P}^N$.

Analytically, a correlation $\mathcal{C}$ can be written in the form

$$\xi_i = c_{ij} x^j, \;\; i, j = 0, 1, \ldots, N,$$

where x^i are point coordinates and ξ_i are tangential coordinates in the space $\mathbb{P}^N$ (cf. formulas (1.76) on p. 23). A correlation $\mathcal{C}$ is nondegenerate if $\det(c_{ij}) \neq 0$.

Consider a smooth curve C in the space $\mathbb{P}^N$ and suppose that this curve does not belong to a hyperplane. A correlation $\mathcal{C}$ maps points of C into hyperplanes forming a one-parameter family. The hyperplanes of this family envelope a hypersurface with a degenerate Gauss map of rank one with $(N-2)$-dimensional generators (see Figure 2.6 on p. 58).

If the curve C lies in a subspace $\mathbb{P}^s \subset \mathbb{P}^N$, then a correlation $\mathcal{C}$ maps points of C into hyperplanes that envelop a hypercone with an $(N-s-1)$-dimensional vertex.

Further, let $X = V^r$ be an arbitrary tangentially nondegenerate r-dimensional variety in the space $\mathbb{P}^N$. A correlation $\mathcal{C}$ maps points of such V^r into hyperplanes forming an r-parameter family. The hyperplanes of this family envelop a hypersurface $Y = V_r^{N-1}$ with a degenerate Gauss map of rank r. The generators of this hypersurface X are of dimension $N-r-1$ and correspond to the tangent subspaces $T_x(V^r)$.

If the tangentially nondegenerate variety V^r belongs to a subspace $\mathbb{P}^s \subset \mathbb{P}^N$, $s > r$, then the hypersurface $Y = V_r^{N-1}$ corresponding to V^r under a correlation $\mathcal{C}$ is a hypercone with an $(N-s-1)$-dimensional vertex.

Now let $X = V_r^n$ be a variety with a degenerate Gauss map of rank r. Then we can prove the following result, which fully corresponds to Theorem 2.8.

Theorem 2.12. *A correlation $\mathcal{C}$ maps an n-dimensional dually nondegenerate variety $X = V_r^n$ with a degenerate Gauss map of rank r with plane generators of dimension $l = n - r$ into a variety $X^* = V_r^{N-l-1}$, with a degenerate Gauss map of the same rank r with $(N-n-1)$-dimensional plane generators.*

Proof. A correlation $\mathcal{C}$ sends an l-dimensional plane generator $L \subset X$ to an $(N-l-1)$-dimensional plane $\mathbb{P}^{N-l-1}$, and a tangent subspace $T_x(X)$ to an $(N-n-1)$-dimensional plane $\mathbb{P}^{N-n-1}$, where $\mathbb{P}^{N-n-1} \subset \mathbb{P}^{N-l-1}$. Because both of these planes depend on r parameters, the planes $\mathbb{P}^{N-n-1}$ are generators of the variety $\mathcal{C}(X)$, and the planes $\mathbb{P}^{N-l-1}$ are its tangent subspaces. Thus, the variety $\mathcal{C}(X)$ is a variety $X^* = V_r^{N-l-1}$ of dimension $N-l-1$ and rank r. □

2.6 Hypersurface with a Degenerate Gauss Map Associated with a Veronese Variety

2.6.1 Veronese Varieties and Varieties with Degenerate Gauss Maps. Consider a real five-dimensional projective space $\mathbb{RP}^5$ with points whose coordinates are defined by symmetric matrices

$$x = \begin{pmatrix} x^{00} & x^{01} & x^{02} \\ x^{10} & x^{11} & x^{12} \\ x^{20} & x^{21} & x^{22} \end{pmatrix},$$

and its dual space $(\mathbb{RP}^5)^*$ with points whose coordinates are defined by the matrices

$$\xi = \begin{pmatrix} x_{00} & x_{01} & x_{02} \\ x_{10} & x_{11} & x_{12} \\ x_{20} & x_{21} & x_{22} \end{pmatrix} = (x_{ij}),$$

where $i, j = 0, 1, 2$; $x_{ij} = x_{ji}$. In the space $(\mathbb{RP}^5)^*$, a frame consists of the points

$$A^{00} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A^{11} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A^{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$A^{01} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A^{02} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad A^{12} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{2.83}$$

and an arbitrary point $\xi \in (\mathbb{RP}^5)^*$ can be represented as a linear combination of the vertices of this frame:

$$\xi = x_{ij} A^{ij}.$$

A Veronese variety V in the space $(\mathbb{RP}^5)^*$ can be given by the following parametric equations:

$$\xi = \begin{pmatrix} u^2 & uv & uw \\ vu & v^2 & vw \\ wu & wv & w^2 \end{pmatrix}, \tag{2.84}$$

where (u, v, w) are projective coordinates in the plane $\mathbb{RP}^2$. Thus, the variety V is the embedding

$$\psi : \operatorname{Sym}(\mathbb{P}^{2*} \times \mathbb{P}^{2*}) \to \mathbb{P}^{5*}.$$

By (2.83), formula (2.84) can also be written in the form

$$\xi = u^2 A^{00} + v^2 A^{11} + w^2 A^{22} + 2uvA^{01} + 2vwA^{12} + 2uwA^{02}. \tag{2.85}$$

Consider now the projection Pr of the space $(\mathbb{RP}^5)^*$ from the point

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

not belonging to the Veronese variety V, onto the subspace $(\mathbb{RP}^4)^*$ not tangent to the variety V and defined in $(\mathbb{RP}^5)^*$ by the equation

$$x_{00} + x_{11} + x_{22} = 0. \tag{2.86}$$

First, we find the projections of the vertices A^{ij} of the frame of the space $(\mathbb{RP}^5)^*$ onto the subspace $(\mathbb{RP}^4)^*$. Because the vertices A^{01}, A^{12}, and A^{02} belong to the subspace $(\mathbb{RP}^4)^*$, the projections coincide with these points:

$$\text{Pr } A^{01} = A^{01}, \ \text{Pr } A^{12} = A^{12}; \ \text{Pr } A^{02} = A^{02}$$

(see Figure 2.8).

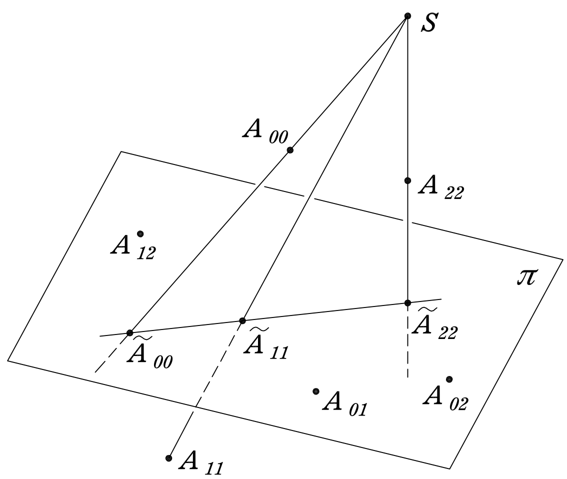

Figure 2.8

The projection of the vertex A_{00} can be found from the condition

$$\text{Pr } A^{00} = A^{00} - \lambda S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in (\mathbb{RP}^4)^*.$$

By (2.86), it follows that

$$1 - 3\lambda = 0, \ \ \lambda = \frac{1}{3},$$

i.e.,

$$\text{Pr } A^{00} = \frac{2}{3}A^{00} - \frac{1}{3}A^{11} - \frac{1}{3}A^{22} = \frac{1}{3}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

In a similar way, we find that

$$\mathrm{Pr}\ A^{11} = \frac{1}{3}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \ \text{and}\ \mathrm{Pr}\ A^{22} = \frac{1}{3}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

The points Pr A^{00}, Pr A^{11}, and Pr A^{22} are linearly dependent because

$$\mathrm{Pr}\ A^{00} + \mathrm{Pr}\ A^{11} + \mathrm{Pr}\ A^{22} = 0. \tag{2.87}$$

Thus, we can take the independent points

$$A^{01}, A^{12}, A^{20} \ \text{and}\ \mathrm{Pr}\ A^{00} = \widetilde{A}^{00}, \mathrm{Pr}\ A^{22} = \widetilde{A}^{22} \tag{2.88}$$

as a basis of the subspace $(\mathbb{RP}^4)^*$. By (2.87), for the point Pr A^{11} we obtain the expression

$$\mathrm{Pr}\ A^{11} = -\widetilde{A}^{00} - \widetilde{A}^{22}. \tag{2.89}$$

Next, we find the projection of the Veronese variety V onto the subspace $(\mathbb{RP}^4)^*$ from the point S. By (2.85), (2.88), and (2.89), we have

$$\mathrm{Pr}\ \xi = (u^2 - v^2)\widetilde{A}^{00} + (w^2 - v^2)\widetilde{A}^{22} + 2uvA^{01} + 2vwA^{12} + 2uwA^{02}.$$

Note that a similar projection of a Veronese variety into a four-dimensional projective space was considered earlier by Sasaki [Sas 91]. In the space $\mathbb{RP}^4$ dual to the subspace $(\mathbb{RP}^4)^*$, the last equation defines a two-parameter family of hyperplanes ξ corresponding to the points x^* of the space $(\mathbb{RP}^4)^*$. The equation of a hyperplane ξ has the form

$$\xi := (u^2 - v^2)x^{00} + (w^2 - v^2)x^{22} + 2uvx^{01} + 2vwx^{12} + 2uwx^{02} = 0, \tag{2.90}$$

where $x^{00}, x^{22}, x^{01}, x^{12}$, and x^{02} are projective coordinates in the space $\mathbb{RP}^4$. The family of hyperplanes ξ depends on two parameters $\frac{u}{w}$ and $\frac{v}{w}$. Hence the envelope of this family is a hypersurface X with a degenerate Gauss map of rank two in the space $\mathbb{RP}^4$. The hypersurface X bears a two-parameter family of rectilinear generators L that are leaves of the Monge–Ampère foliation on X.

In order to find an equation of the envelope of the family of hyperplanes ξ, we differentiate equation (2.90) with respect to the parameters u, v, and w:

$$\left\{\begin{aligned} &\frac{1}{2}\frac{\partial\xi}{\partial u} = ux^{00} + vx^{01} + wx^{02} = 0, \\ &\frac{1}{2}\frac{\partial\xi}{\partial v} = ux^{01} - v(x^{00} + x^{22}) + wx^{12} = 0, \\ &\frac{1}{2}\frac{\partial\xi}{\partial w} = ux^{02} + vx^{12} + wx^{22} = 0. \end{aligned}\right. \tag{2.91}$$

Because by Euler's theorem on homogeneous functions, we have

$$u\frac{\partial \xi}{\partial u} + v\frac{\partial \xi}{\partial v} + w\frac{\partial \xi}{\partial w} = 2\xi;$$

it follows that by (2.91) equation (2.90) is identically satisfied.

Eliminating the parameters u, v, and w from equations (2.91), we find that

$$\Phi = \det \begin{pmatrix} x^{00} & x^{01} & x^{02} \\ x^{01} & -(x^{00}+x^{22}) & x^{12} \\ x^{02} & x^{12} & x^{22} \end{pmatrix} = 0. \tag{2.92}$$

This equation determines the hypersurface X—the envelope of the family of hyperplanes ξ—in the space $\mathbb{RP}^4$.

This implies the following theorem.

Theorem 2.13. *The hypersurface X dual to the projection of a Veronese variety into a four-dimensional subspace is a cubic hypersurface. This hypersurface has a degenerate Gauss map and is of rank two. It bears a two-parameter family of rectilinear generators that are leaves of the Monge–Ampère foliation on X.*

Moreover, equation (2.92) proves that the hypersurface X is equivalent to the projectivization of the set of symmetric matrices of third order with vanishing determinant and trace.

2.6.2 Singular Points. Let us find singular points of the hypersurface X defined by equation (2.92). In order to do this, we write this equation in the form

$$\begin{aligned} \Phi = &-x^{00}x^{22}(x^{00}+x^{22}) + 2x^{01}x^{02}x^{12} \\ &+(x^{02})^2(x^{00}+x^{22}) - x^{00}(x^{12})^2 - x^{22}(x^{01})^2 = 0. \end{aligned} \tag{2.93}$$

Singular points of the hypersurface X are defined by the equations

$$\frac{\partial \Phi}{\partial x^{00}} = -2x^{00}x^{22} - (x^{22})^2 + (x^{02})^2 - (x^{12})^2 = 0, \tag{2.94}$$

$$\frac{\partial \Phi}{\partial x^{01}} = 2x^{02}x^{12} - 2x^{22}x^{01} = 0, \tag{2.95}$$

$$\frac{\partial \Phi}{\partial x^{12}} = 2x^{01}x^{02} - 2x^{00}x^{12} = 0, \tag{2.96}$$

$$\frac{\partial \Phi}{\partial x^{02}} = 2x^{01}x^{12} + 2x^{02}(x^{00}+x^{22}) = 0, \tag{2.97}$$

and

$$\frac{\partial \Phi}{\partial x^{22}} = -(x^{00})^2 - 2x^{00}x^{22} + (x^{02})^2 - (x^{01})^2 = 0. \tag{2.98}$$

Equations (2.95) and (2.96) imply that

$$x^{12} = \lambda x^{01}, \;\; x^{02} = \lambda x^{00}, \;\; x^{22} = \lambda^2 x^{00}, \tag{2.99}$$

where, of course, $\lambda \neq 0$. Substituting these expressions into equations (2.94), (2.97), and (2.98) and dividing by λ or λ^2, we arrive at the same equation

$$(x^{01})^2 + (1 + \lambda^2)(x^{00})^2 = 0,$$

from which it follows that

$$x^{01} = \pm i\sqrt{1 + \lambda^2}\, x^{00}. \tag{2.100}$$

Equations (2.99) and (2.100) determine the desired singular points F and $\overline{F}$ on the hypersurface X. These points are complex conjugate on the straight line $F \wedge \overline{F}$.

It is easy to see that the straight line $F \wedge \overline{F}$ belongs to the hypersurface X defined by equation (2.84). In fact, it follows from (2.99) and (2.100) that the coordinates $(x^{00}, x^{22}, x^{01}, x^{12}, x^{02})$ of an arbitrary point $F + s\overline{F}$ of this line are

$$(1 + s,\, \lambda^2(1 + s),\, i\sqrt{1 + \lambda^2}(1 - s),\, i\lambda\sqrt{1 + \lambda^2}(1 - s),\, \lambda(1 + s))x^{00}.$$

Substituting these coordinates into the left-hand side of equation (2.93), we obtain zero.

NOTES

2.1–2.2. Our presentation of the projectivization of the tangent and osculating subspaces of a submanifold X is close to that in the paper [GH 79] by Griffiths and Harris (see also the book [AG 93] by Akivis and Goldberg).

The differential geometry of the Grassmannian was considered by Akivis in [A 82].

The osculating spaces, fundamental forms, and asymptotic directions and lines of a submanifold X were investigated by É. Cartan in [C 19]. See more on the second fundamental forms of X in Griffiths and Harris [GH 79] and Landsberg [L 94].

Note that the proof of our Theorem 2.1 is different from that of Theorem 2.2 in [AG 93], which has some inaccuracies.

This theorem generalizes a similar theorem of C. Segre (see [SegC 07], p. 571), which was proved for submanifolds X of dimension n of the space $\mathbb{P}^N$ that have at each point $x \in X$ the osculating subspace T^2_x of dimension $n + 1$. By this theorem, a submanifold X either belongs to a subspace $\mathbb{P}^{n+1}$ or is a torse.

Note that C. Segre proved the theorem named after him for a submanifold of a multidimensional space of constant curvature.

Note also that Theorem 2.1 is similar to Theorem 3.10 from the book [AG 93] by Akivis and Goldberg, which was proved there for submanifolds of a space $\mathbb{P}^N$ bearing a net of conjugate lines.

2.3. Zak [Za 87] (see also his book [Za 93] and the paper [Ra 84] by Ran) proved that the Gauss map of a smooth variety is finite (see also the books [FP 01] by Fischer and Piontkowski (subsections 2.3.3 and 3.1.3); [Ha 92] by Harris (p. 189); [L 99] by Landsberg (p. 48); [T 01] by Tevelev (Sections 3.3 and 4.2); and the book [Za 93] by Zak). In terms of differential geometry, Zak's theorem can be formulated as follows: The image of the Gauss map $\gamma(X)$ of a smooth irreducible variety $X \subset \mathbb{P}^N$ of dimension n, which is different from a linear space, is a smooth irreducible variety $\gamma(X) \subset \mathbb{G}(n, N)$ of the same dimension n.

From the point of view of differential geometry, this result is more or less obvious: If a variety X is smooth in $\mathbb{P}^N$, then its Gauss map $\gamma(X)$ has the rank $r = n$ (i.e., X is tangentially nondegenerate).

Fischer [F 88] extends to the complex analytic case a classical result on ruled surfaces in $\mathbb{E}^3$. He shows that the only developable surfaces in $\mathbb{CP}^3$ are planes, cones, and tangent surfaces of curves. He also shows that a developable ruled surface is uniquely determined by its directrix and its Gauss map.

The origins of the theory of varieties with degenerate Gauss maps are in the works of C. Segre [SegC 07, 10] who studied the local differential geometry of linear spaces. In particular, in [SegC 07, 10], he introduced the Segre cone of such families and used the concepts of the second fundamental forms and foci.

Varieties $X = V_r^n$ with degenerate Gauss maps of rank $r < n$ were considered by É. Cartan in [C 16] in connection with his study of metric deformation of hypersurfaces, and in [C 19] in connection with his study of manifolds of constant curvature; by Yanenko in [Ya 53] in connection with his study of metric deformation of submanifolds of arbitrary class; by Akivis in [A 57, 62], Savelyev in [Sa 57, 60], and Ryzhkov in [Ry 60] (see also the survey paper by Akivis and Ryzhkov [AR 64]) in a projective space $\mathbb{P}^N$. Brauner [Br 38], Wu [Wu 95], and Fischer and Wu [FW 95] studied such varieties in a Euclidean N-space $\mathbb{E}^N$.

Note that a relationship of the rank of varieties V^m and their deformation in a Euclidean N-space was indicated by Bianchi as far back as 1905. In [Bi 05] he proved that a necessary condition for V^m to be deformable is the condition rank $V^m \leq 2$. Allendörfer [Al 39] introduced the notion of type t, $t = 0, 1, , \ldots, m$, of V^m and proved that varieties V_{N-p}, $p > 1$, of type $t > 2$ in $\mathbb{E}^N$ are rigid. For definition of type of V^m, see [Al 39] or Yanenko [Ya 53]. Note only that the notion of type (as well as of rank) is projectively and metrically invariant, and that for a hypersurface, the type coincides with the rank.

Griffiths and Harris in [GH 79] (Section 2, pp. 383–393) considered varieties $X = V_r^n$ with degenerate Gauss maps from the point of view of algebraic geometry. Following [GH 79], Landsberg [L 96] considered varieties with degenerate Gauss maps. His recently published book [L 99] is in some sense an update to the paper [GH 79].

Section 5 (pp. 47–50) of these notes is devoted to varieties with degenerate Gauss maps. In the recently published book [FP 01] by Fischer and Piontkowski, the authors studied ruled varieties from the point of view of complex projective algebraic geometry. One section of this book was devoted to varieties with degenerate Gauss maps (they called such varieties developable). Following Griffiths and Harris's paper [GH 79], the authors employed a bilinear second fundamental form for studying developable varieties, gave detailed and more elementary proofs of some results in [GH 79], and reported on some recent progress in this area. In particular, they gave a classification of developable varieties of rank two in codimension one. Rogora in [Rog 97] and Mezzetti and Tommasi in [MT 02a] also considered varieties with degenerate Gauss maps from the point of view of algebraic geometry.

Recently Ishikawa published four papers [I 98, 99a, 99b] and [IM 01] on varieties with degenerate Gauss maps (called "developable" in these papers). In [IM 01], Ishikawa and Morimoto found the connection between such varieties and solutions of Monge–Ampère equations; they named the foliation of plane generators L of X ($\dim L = l$) the Monge–Ampère foliation. In [IM 01], the authors proved that the rank r of a compact C^∞-hypersurface $X \subset \mathbf{R}\,\mathbb{P}^N$ with a degenerate Gauss map is an even integer r satisfying the inequality $\frac{r(r+3)}{2} > N$, $r \neq 0$. In particular, if $r < 2$, then X is necessarily a projective hyperplane of $\mathbf{R}\,\mathbb{P}^N$, and if $N = 3$ or $N = 5$, then a compact C^∞-hypersurface with a degenerate Gauss map is a projective hyperplane.

In [I 98, 99b], Ishikawa found a real algebraic cubic nonsingular hypersurface with a degenerate Gauss map in $\mathbf{R}\mathbb{P}^N$ for $N = 4, 7, 13, 25$, and in [I 99a] he studied singularities of C^∞-hypersurfaces with degenerate Gauss maps.

The notion of the index l of relative nullity was introduced by Chern and Kuiper in their joint paper [CK 52] (see also the book by Kobayashi and Nomizu [KN 63], vol. 2, p. 348) for a variety $X = V^n$ embedded into a Riemannian manifold V^N.

However, the second fundamental forms of a submanifold X are related not so much to the metric structure of X as to its projective structure, because these forms are preserved under projective transformations of the Riemannian submanifold X. This was noticed by Akivis in [A 87b], who also proved the relation $l + r = n$.

Note that if $l > 0$, then the point x is called a *parabolic* point of the variety X. If all points of a variety X are parabolic, then the variety X is called *parabolic* (cf. the papers [Bor 82, 85] by Borisenko). The varieties X, for which the index l is constant and greater than 0 for all points $x \in X$, are called *strongly parabolic*.

In 1997 Borisenko published the survey paper [Bor 97] in which he discussed results on strongly parabolic varieties and related questions in Riemannian and pseudo-Riemannian spaces of constant curvature and, in particular, in a Euclidean space $\mathbb{E}^N$. Among other results, he gives a description of certain classes of varieties of arbitrary codimension that are analogous to the class of parabolic surfaces in a Euclidean space $\mathbb{E}^3$. Borisenko also investigates the local and global metric and topological properties, indicates conditions that imply that a variety of a Euclidean space $\mathbb{E}^N$ is cylindrical, presents results on strongly parabolic varieties in pseudo-Riemannian spaces of constant curvature, and finds the relationship with minimal surfaces.

2.4. The results presented in this section are due to Akivis [A 57] (see also

Section 4.2 in the book [AG 93] by Akivis and Goldberg). In our presentation, we follow the recently published paper [AG 01a] by Akivis and Goldberg. Other examples of varieties with degenerate Gauss maps can be found in the papers [A 87a] by Akivis, [AG 93, 98b, 98c, 01a, 01b, 02b] by Akivis and Goldberg, [AGL 01] by Akivis, Goldberg, and Landsberg, [C 39] by Cartan, [FW 95] by Fischer and Wu, [GH 79] by Griffiths and Harris, [I 98, 99a, 99b, 00a] by Ishikawa, [Pio 01, 02a, 02b] by Piontkowski, [S 60] by Sacksteder, [Wu 95] by Wu, [WZ 02] by Wu and F. Zheng, and in the books [L 99] by Landsberg and [FP 01] by Fischer and Piontkowski. Examples of varieties with degenerate Gauss maps on the sphere S^n were constructed in the recent papers [IKM 01, 02] by Ishikawa, Kimura, and Miyaoka.

2.5. The reader can find more details on the dual varieties and the dual defect of a tangentially nondegenerate variety, for example, in the following books: Fischer and Piontkowski [FP 01] (Sections 2.1.4, 2.1.5, 2.3.4, 2.5.1, 2.5.3, and 2.5.7); Harris [Ha 92] (pp. 196–199); Landsberg [L 99] (pp. 16–17 and 52–57); and Tevelev [T 01] (Chapters 1, 6, and 7). Formula (2.65) for the expected dimension of the dual variety of a variety with degenerate Gauss map appeared also in the paper [Pio 2b] by Piontkowski and implicitly in the books Landsberg [L 99] (see 7.2.1.1 and 7.3i) and Fischer and Piontkowski [FP 01] (Section 2.3.4).

During the last 20 years, the smooth dually degenerate varieties (for which $\dim X^* < N-1$) were considered in many articles (see, for example, the papers [GH 79] by Griffiths and Harris, Zak [Za 87], Ein [E 85, 86] and the books [Ha 92] by Harris, [L 99] by Landsberg, [T 01] by Tevelev, [FP 01] by Fisher and Piontkowski). Note that Harris [Ha 92] (p. 197) uses the term *deficient* for such varieties and the term *deficiency* for their defect.

The classification of dually degenerate smooth varieties of small dimensions n with positive dual defect δ_* was found by Ein [E 85, 86] for $n \leq 6$, by Ein [E 85, 86] and Lanteri and Strupa [LS 87] for $n = 7$, and by Beltrametti, Fania, and Sommese [BFS 92] for $n \leq 10$ (see also Section 9.2.C in the book [T 01] by Tevelev).

For applications of the duality principle see also the book [AG 93] by Akivis and Goldberg. In our presentation of these applications, we follow our recently published papers [AG 01a, 02b] and Section 4.1 of the book [AG 93].

The dual defect of a variety X must be defined as the difference between an expected dimension of the dual variety X^* and its true dimension. Thus, the definition given on p. 71 and used in the literature (see, for example, Fischer and Piontkowski [FP 01] (p. 55); Harris [Ha 92] (p. 199); Landsberg [L 99] (p. 16); and Tevelev [T 01] (p. 3) is correct for smooth varieties because for them an expected dimension of the dual variety X^* equals $N-1$. In the books mentioned above, the definitions of the dual defect and dually degenerate varieties given on p. 71, which are correct for tangentially nondegenerate varieties, are automatically extended to varieties with degenerate Gauss maps. In our opinion, this is incorrect, because for the latter varieties, an expected dimension of X^* is $N-l-1 < N-1$ (see formula (2.65)), and for them the correct definition of the dual defect (and dually degenerate varieties) must be the definitions given on p. 72. Note that the definition on p. 72 includes the definition on p. 71: the latter can be obtained from the former if one takes $l = 0$. Note also that by definition on p. 72, the dual defect δ_* of a dually nondegenerate variety equals 0 (and this is natural), while by the definition on p. 71, $\delta_* = \delta_\gamma = n - r > 0$.

2.6. The constructions we made in Section 2.6 can be generalized for the projective space $\mathbb{KP}$ over the algebras $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, where $\mathbb{C}$ is the algebra of complex numbers, $\mathbb{H}$ is the algebra of quaternions, and $\mathbb{O}$ is the algebra of Cayley's octonions or octaves (see more on octonions and the algebra of Cayley's octonions in Rosenfeld [Ro 97], Section 1.3.1). Then $\dim \mathbb{K} = 2^{i-1}$, $i = 1, 2, 3, 4$. In all these algebras, there is an involutive or antiinvolutive automorphism $z \to \overline{z}$.

This was done by Ishikawa in [I 99a], who constructed examples of real algebraic cubic nonsingular hypersurfaces with degenerate Gauss maps in $\mathbb{RP}^n$ for $n = 4, 7, 13, 25$. These hypersurfaces have the structure of homogeneous spaces of groups $\mathbf{SO}(3), \mathbf{SU}(3), \mathbf{Sp}(3)$, and F_4, respectively, and their projective duals are linear projections of Veronese embeddings of projective planes $\mathbb{KP}^2$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$.

Springer

http://www.springer.com/978-0-387-40463-9

Differential Geometry of Varieties with Degenerate
Gauss Maps
Akivis, M.A.; Goldberg, V.V.
2004, XXI, 255 p. 16 illus., Hardcover
ISBN: 978-0-387-40463-9