

---

## Preface

### Background

By awarding Harry Markowitz, William Sharpe and Merton Miller the 1990 Nobel Prize in Economics, the Nobel Prize Committee brought to worldwide attention the fact that the previous forty years had seen the emergence of a new scientific discipline, the “theory of finance.” This theory attempts to understand how financial markets work, how to make them more efficient, and how they should be regulated. It explains and enhances the important role these markets play in capital allocation and risk reduction to facilitate economic activity. Without losing its application to practical aspects of trading and regulation, the theory of finance has become increasingly mathematical, to the point that problems in finance are now driving research in mathematics.

Harry Markowitz’s 1952 Ph.D. thesis *Portfolio Selection* laid the groundwork for the mathematical theory of finance. Markowitz developed a notion of mean return and covariances for common stocks which allowed him to quantify the concept of “diversification” in a market. He showed how to compute the mean return and variance for a given portfolio and argued that investors should hold only those portfolios whose variance is minimal among all portfolios with a given mean return. Although the language of finance now involves stochastic (Itô) calculus, management of risk in a quantifiable manner is the underlying theme of the modern theory and practice of quantitative finance.

In 1969 Robert Merton introduced stochastic calculus into the study of finance. Merton was motivated by the desire to understand how prices are set in financial markets, which is the classical economics question of “equilibrium,” and in later papers he used the machinery of stochastic calculus to begin investigation of this issue.

At the same time as Merton’s work and with Merton’s assistance, Fischer Black and Myron Scholes were developing their celebrated option pricing formula. This work won the 1973 Nobel Prize in Economics. It provided a satisfying solution to an important practical problem, that of finding a fair price for a European call option, i.e., the right to buy one share of a given stock at a specified price and time. In the period 1979–1983, Harrison, Kreps and

Pliska used the general theory of continuous-time stochastic processes to put the Black-Scholes-Merton option pricing formula on a solid theoretical basis, and as a result, showed how to price numerous other “derivative” securities.

Many of the theoretical developments in finance have found immediate application in financial markets. To understand how they are applied, we digress for a moment on the role of financial institutions. A principal function of a nation’s financial institutions is to act as a risk-reducing intermediary among customers engaged in production. For example, the insurance industry pools premiums of many customers and must pay off only the few who actually incur losses. But risk arises in situations for which pooled-premium insurance is unavailable. For instance, as a hedge against higher fuel costs, an airline may want to buy a security whose value will rise if oil prices rise. But who wants to sell such a security? The role of a financial institution is to design such a security, determine a “fair” price for it, and sell it to airlines. The security thus sold is usually “derivative” (i.e., its value is “derived” from the value of other “primary” securities). “Fair” in this context means that the financial institution earns just enough from selling the security to enable it to trade in other securities whose relation with oil prices is such that, if oil prices do indeed rise, the firm can pay off its increased obligation to the airlines. An “efficient” market is one in which risk-hedging securities are widely available at “fair” prices.

The Black-Scholes-Merton option pricing formula provided, for the first time, a theoretical method of fairly pricing a risk-hedging security. If an investment bank offers a derivative security at a price which is higher than “fair,” it may be underbid. If it offers the security at less than the “fair” price, it runs the risk of substantial loss. This makes the bank reluctant to offer many of the derivative securities which would contribute to market efficiency. In particular, the bank only wants to offer derivative securities whose “fair” price can be determined in advance. Furthermore, if the bank sells such a security, it must then address the hedging problem: how should it manage the risk associated with its new position? The mathematical theory growing out of the Black-Scholes-Merton option pricing formula provides solutions for both the pricing and hedging problems. It thus has enabled the creation of a host of specialized derivative securities. This theory is the subject of this text.

### **Origin of this text**

This text has evolved from mathematics courses in the Master of Science in Computational Finance (MSCF) program at Carnegie Mellon University.

Since its inception in 1994, MSCF program has graduated hundreds of students. These people, who have come from a variety of educational and professional backgrounds, have been a joy to teach. They have been eager to learn, asking questions which stimulated thinking, working hard to understand the material both theoretically and practically, and often requesting the inclusion of additional topics. Many came from the finance industry, and were gracious

in sharing their knowledge in ways which enhanced the classroom experience for all.

This text and my own store of knowledge have benefited greatly from interactions with the MSCF students, and I continue to learn from the MSCF alumni. I take this opportunity to express gratitude to these students and former students by dedicating this work to them.

## Relationship between Volumes I and II

Volume II treats the continuous-time theory of stochastic calculus within the context of finance applications. The presentation of this theory is the *raison d'être* of this work. Volume I presents many of the same finance applications, but within the simpler context of the discrete-time binomial model. It prepares the reader for Volume II by treating several fundamental concepts, including martingales, Markov processes, change of measure and risk-neutral pricing in this less technical setting. However, Volume II has a self-contained treatment of these topics, and strictly speaking, it is not necessary to read Volume I before reading Volume II. It is helpful in that the difficult concepts of Volume II are first seen in a simpler context in Volume I.

In the Carnegie Mellon Master's program in Computational Finance, the course based on Volume I is a prerequisite for the courses based on Volume II. However, Ph.D. students in computer science, finance, mathematics, physics and statistics frequently take the courses based on Volume II without first taking the course based on Volume I.

The reader who begins with Volume II may use Volume I as a reference. As several concepts are presented in Volume II, reference is made to the analogous concepts in Volume I. The reader can at that point choose to read only Volume II or to refer to Volume I for a discussion of the concept at hand in a more transparent setting.

## Dependence among Sections of Volume II

Chapter 1, *General Probability Theory* and Chapter 2, *Information and Conditioning* of Volume II lay the measure-theoretic foundation for probability theory required for a treatment of continuous-time models. Chapter 1 presents probability spaces, Lebesgue integrals, and change of measure. Independence, conditional expectations, and properties of conditional expectations are introduced in Chapter 2. These chapters are used extensively throughout the text, but some readers, especially those with exposure to probability theory, may choose skip this material at the outset, referring to it as needed.

Chapter 3, *Brownian Motion*, introduces Brownian motion and its properties. The most important of these for stochastic calculus is quadratic variation, presented in Section 3.4. All of this material is needed in order to proceed, except Sections 3.6 and 3.7, which are used only in Chapter 7, *Exotic Options* and Chapter 8, *Early Exercise*.

The core of the text is Chapter 4, *Stochastic Calculus*. The only material which the reader may omit is Section 4.7, *Brownian Bridge*. This topic is

included because of its importance in Monte Carlo simulation, but it is not used elsewhere in the text.

Chapter 5, *Risk-Neutral Pricing*, states and proves Girsanov's Theorem, which underlies change of measure. This permits a systematic treatment of risk-neutral pricing and the Fundamental Theorems of Asset Pricing (Section 5.4). Section 5.5, *Dividend-Paying Stocks*, is not used elsewhere in the text. Section 5.6, *Forwards and Futures*, appears later in Section 9.4 and in some exercises

Chapter 6, *Connections with Partial Differential Equations*, develops the connection between stochastic calculus and partial differential equations. This is used frequently in later chapters.

With the exceptions noted above, the material in Chapters 1–6 is essential for reading the later chapters. After Chapter 6, the reader has choices.

Chapter 7, *Exotic Options*, is not used in subsequent chapters, nor is Chapter 8, *Early Exercise*. Chapter 9, *Change of Numéraire*, plays an important role in Section 10.4, *Forward LIBOR model*, but is not otherwise used. Chapter 10, *Term Structure Models*, and Chapter 11, *Introduction to Jump Processes*, are not used elsewhere in the text.

## Acknowledgments

Conversations with several people, including my colleagues David Heath and Dmitry Kramkov, have influenced this text. Lukasz Kruk read much of the manuscript and provided numerous comments and corrections. Other students and faculty have pointed out errors in and suggested improvements of earlier drafts of this work. Some of these are Bogdan Doytchinov, Steven Gillispie, Sean Jones, Anatoli Karolik, Andrzej Krause, Sergey Myagchilov, Nicki Rasmussen, Isaac Sonin, Massimo Tassan-Solet, David Whitaker and Uwe Wystup. In some cases, users of these earlier drafts have suggested exercises or examples, and their contributions are acknowledged at appropriate points in the text. To all those who aided in the development of this text, I am most grateful.

During the creation of this text, the author was partially supported by the National Science Foundation under grants DMS-9802464, DMS-0103814, and DMS-0139911. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.

*Steven E. Shreve  
December 2003  
Pittsburgh, PA*



<http://www.springer.com/978-0-387-40100-3>

Stochastic Calculus for Finance I  
The Binomial Asset Pricing Model

Shreve, S.

2004, XV, 187 p., Hardcover

ISBN: 978-0-387-40100-3