“We are to admit no more causes of natural things” (as we are told by Newton) than “such as are both true and sufficient to explain their appearances.” This central theme is basic to the pursuit of science, and goes back to the principle known as Occam’s razor: “if presented with a choice between indifferent alternatives, then one ought to select the simplest one.” Unconsciously or explicitly, informal applications of this principle in science and mathematics abound.

The conglomerate of different research threads drawing on an objective and absolute form of this approach appears to be part of a single emerging discipline, which will become a major applied science like information theory or probability theory. We aim at providing a unified and comprehensive introduction to the central ideas and applications of this discipline.

Intuitively, the amount of information in a finite string is the size (number of binary digits, or bits) of the shortest program that without additional data, computes the string and terminates. A similar definition can be given for infinite strings, but in this case the program produces element after element forever. Thus, a long sequence of 1’s such as

\[
\begin{align*}
\text{11111...1} \\
\text{10,000 times}
\end{align*}
\]

contains little information because a program of size about \(\log 10,000\) bits outputs it:

\[
\text{for } i := 1 \text{ to } 10,000 \\
\text{print 1}
\]

Likewise, the transcendental number \(\pi = 3.1415\ldots\), an infinite sequence of seemingly random decimal digits, contains but a few bits of information. (There is a short program that produces the consecutive digits of \(\pi\) forever.) Such a definition would appear to make the amount of information in a string (or other object) depend on the particular programming language used.

Fortunately, it can be shown that all reasonable choices of programming languages lead to quantification of the amount of absolute information in individual objects that is invariant up to an additive constant. We call this quantity the ‘Kolmogorov complexity’ of the object. If an object contains regularities, then it has a shorter description than itself. We call such an object ‘compressible.’

The application of Kolmogorov complexity takes a variety of forms, for example, using the fact that some strings are extremely compressible; using the compressibility of strings as a selection criterion; using the fact that many strings are not compressible at all; and using the fact that
some strings may be compressed in principle, but that it takes a lot of

The theory dealing with the quantity of information in individual objects
goes by names such as ‘algorithmic information theory,’ ‘Kolmogorov
complexity,’ ‘K-complexity,’ ‘Kolmogorov–Chaitin randomness,’ ‘algo-
rithmic complexity,’ ‘stochastic complexity,’ ‘descriptional complexity,’
‘minimum description length,’ ‘program-size complexity,’ and others.
Each such name may represent a variation of the basic underlying idea
or a different point of departure. The mathematical formulation in each
case tends to reflect the particular traditions of the field that gave birth
to it, be it probability theory, information theory, theory of computing,
statistics, or artificial intelligence.

This raises the question about the proper name for the area. Although
there is a good case to be made for each of the alternatives listed above,
and a name like ‘Solomonoff–Kolmogorov–Chaitin complexity’ would
give proper credit to the inventors, we regard ‘Kolmogorov complex-
ity’ as well entrenched and commonly understood, and we shall use it
hereafter.

The mathematical theory of Kolmogorov complexity contains deep and
sophisticated mathematics. Yet one needs to know only a small amount
of this mathematics to apply the notions fruitfully in widely divergent
areas, from sorting algorithms to combinatorial theory, and from induc-
tive reasoning and machine learning to dissipationless computing.

Formal knowledge of basic principles does not necessarily imply the
wherewithal to apply it, perhaps especially so in the case of Kolmogorov
complexity. It is our purpose to develop the theory in detail and outline
a wide range of illustrative applications. In fact, while the pure theory of
the subject will have its appeal to the select few, the surprisingly large
field of its applications will, we hope, delight the multitude.

The mathematical theory of Kolmogorov complexity is treated in Chap-
ters 2, 3, and 4; the applications are treated in Chapters 5 through 8.
Chapter 1 can be skipped by the reader who wants to proceed immedi-
ately to the technicalities. Section 1.1 is meant as a leisurely, informal
introduction and peek at the contents of the book. The remainder of
Chapter 1 is a compilation of material on diverse notations and disci-
plines drawn upon.

We define mathematical notions and establish uniform notation to be
used throughout. In some cases we choose nonstandard notation since
the standard one is homonymous. For instance, the notions ‘absolute
value,’ ‘cardinality of a set,’ and ‘length of a string’ are commonly de-
noted in the same way as $|\cdot|$. We choose distinguishing notations $|\cdot|$, $d(\cdot)$, and $l(\cdot)$, respectively.
Briefly, we review the basic elements of computability theory and probability theory that are required. Finally, in order to place the subject in the appropriate historical and conceptual context we trace the main roots of Kolmogorov complexity.

This way the stage is set for Chapters 2 and 3, where we introduce the notion of optimal effective descriptions of objects. The length of such a description (or the number of bits of information in it) is its Kolmogorov complexity. We treat all aspects of the elementary mathematical theory of Kolmogorov complexity. This body of knowledge may be called algorithmic complexity theory. The theory of Martin-Löf tests for randomness of finite objects and infinite sequences is inextricably intertwined with the theory of Kolmogorov complexity and is completely treated. We also investigate the statistical properties of finite strings with high Kolmogorov complexity. Both of these topics are eminently useful in the applications part of the book. We also investigate the recursion-theoretic properties of Kolmogorov complexity (relations with Gödel’s incompleteness result), and the Kolmogorov complexity version of information theory, which we may call ‘algorithmic information theory’ or ‘absolute information theory.’

The treatment of algorithmic probability theory in Chapter 4 presupposes Sections 1.6, 1.11.2, and Chapter 3 (at least Sections 3.1 through 3.4). Just as Chapters 2 and 3 deal with the optimal effective description length of objects, we now turn to optimal (greatest) effective probability of objects. We treat the elementary mathematical theory in detail. Subsequently, we develop the theory of effective randomness tests under arbitrary recursive distributions for both finite and infinite sequences. This leads to several classes of randomness tests, each of which has a universal randomness test. This is the basis for the treatment of a mathematical theory of inductive reasoning in Chapter 5 and the theory of algorithmic entropy in Chapter 8.

Chapter 5 develops a general theory of inductive reasoning and applies the developed notions to particular problems of inductive inference, prediction, mistake bounds, computational learning theory, and minimum description length induction in statistics. This development can be viewed both as a resolution of certain problems in philosophy about the concept and feasibility of induction (and the ambiguous notion of ‘Occam’s razor’), as well as a mathematical theory underlying computational machine learning and statistical reasoning.

Chapter 6 introduces the incompressibility method. Its utility is demonstrated in a plethora of examples of proving mathematical and computational results. Examples include combinatorial properties, the time complexity of computations, the average-case analysis of algorithms such as Heapsort, language recognition, string matching, pumping lemmas in
formal language theory, lower bounds in parallel computation, and Turing machine complexity. Chapter 6 assumes only the most basic notions and facts of Sections 2.1, 2.2, 3.1, 3.3.

Some parts of the treatment of resource-bounded Kolmogorov complexity and its many applications in computational complexity theory in Chapter 7 presuppose familiarity with a first-year graduate theory course in computer science or basic understanding of the material in Section 1.7.4. Sections 7.5 and 7.7 on universal optimal search and logical depth only require material covered in this book. The section on logical depth is technical and can be viewed as a mathematical basis with which to study the emergence of life-like phenomena—thus forming a bridge to Chapter 8, which deals with applications of Kolmogorov complexity to relations between physics and computation.

Chapter 8 presupposes parts of Chapters 2, 3, 4, the basics of information theory as given in Section 1.11, and some familiarity with college physics. It treats physical theories like dissipationless reversible computing, information distance and picture similarity, thermodynamics of computation, statistical thermodynamics, entropy, and chaos from a Kolmogorov complexity point of view. At the end of the book there is a comprehensive listing of the literature on theory and applications of Kolmogorov complexity and a detailed index.

Acknowledgments

We thank Greg Chaitin, Péter Gács, Leonid Levin, and Ray Solomonoff for taking the time to tell us about the early history of our subject and for introducing us to many of its applications. Juris Hartmanis and Joel Seiferas initiated us into Kolmogorov complexity in various ways.

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tested in one-semester courses and seminars at the University of Amsterdam in 1988 and 1989, the University of Waterloo in 1989, Dartmouth College in 1990, the Universitat Polytècnica de Catalunya in Barcelona in 1991/1992, the University of California at Santa Barbara, Johns Hopkins University, and Boston University in 1992/1993.

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The London Mathematical Society kindly gave permission to reproduce a long extract by A.M. Turing. The Indian Statistical Institute, through the editor of Sankhyā, kindly gave permission to quote A.N. Kolmogorov.

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One of us [PV] gives very special thanks to his lovely wife Pauline for insisting from the outset on the significance of this enterprise. The Aiken Computation Laboratory of Harvard University, Cambridge, Massachusetts, USA; the Computer Science Department of York University, Ontario, Canada; the Computer Science Department of the University of Waterloo, Ontario, Canada; and CWI, Amsterdam, the Netherlands provided the working environments in which this book could be written.

When this book was conceived ten years ago, few scientists realized the width of scope and the power for applicability of the central ideas. Partially because of the enthusiastic reception of the first edition, open problems have been solved and new applications have been developed. We have added new material on the relation between data compression and minimum description length induction, computational learning, and universal prediction; circuit theory; distributed algorithmics; instance complexity; CD compression; computational complexity; Kolmogorov random graphs; shortest encoding of routing tables in communication networks; resource-bounded computable universal distributions; average case properties; the equality of statistical entropy and expected Kolmogorov complexity; and so on. Apart from being used by researchers and as a reference work, the book is now commonly used for graduate courses and seminars. In recognition of this fact, the second edition has...
been produced in textbook style. We have preserved as much as possible
the ordering of the material as it was in the first edition. The many ex-
cercises bunched together at the ends of some chapters have been moved
to the appropriate sections. The comprehensive bibliography on Kol-
mogorov complexity at the end of the book has been updated, as have
the ‘History and References’ sections of the chapters. Many readers were
kind enough to express their appreciation for the first edition and to send
notification of typos, errors, and comments. Their number is too large
to thank them individually, so we thank them all collectively.

Preface to the
Third Edition

The general area of reasoning based on shortest description length con-
tinues to coalesce. Simultaneously, the emphasis in handling of informa-
tion in computers and communication networks continues to move from
being random-variable based to being individual-outcome based. Prac-
tically speaking, this has resulted in a number of spectacular real-life
applications of Kolmogorov complexity, where the latter is replaced by
compression programs. The general area has branched out into subareas,
each with its own specialized books or treatments. This work, through
its subsequent editions, has been both a catalyst and an outcome of
these trends. The third edition endeavors to capture the essence of the
state of the art at the end of the first decade of the new millennium. It
is a corrected and greatly expanded version of the earlier editions. Many
people contributed, and we thank them all collectively.

How to Use
This Book

The technical content of this book consists of four layers. The main
text is the first layer. The second layer consists of examples in the main
text. These elaborate the theory developed from the main theorems. The
third layer consists of nonindented, smaller-font paragraphs interspersed
with the main text. The purpose of such paragraphs is to have an ex-
planatory aside, to raise some technical issues that are important but
would distract attention from the main narrative, or to point to alter-
native or related technical issues. Much of the technical content of the
literature on Kolmogorov complexity and related issues appears in the
fourth layer, the exercises. When the idea behind a nontrivial exercise is
not our own, we have tried to give credit to the person who originated
the idea. Corresponding references to the literature are usually given in
comments to an exercise or in the historical section of that chapter.

Starred sections are not really required for the understanding of the se-
quel and can be omitted at first reading. The application sections are not
starred. The exercises are grouped together at the end of main sections.
Each group relates to the material in between it and the previous group.
Each chapter is concluded by an extensive historical section with full
references. For convenience, all references in the text to the Kolmogorov complexity literature and other relevant literature are given in full where they occur. The book concludes with a References section intended as a separate exhaustive listing of the literature restricted to the theory and the direct applications of Kolmogorov complexity. There are reference items that do not occur in the text and text references that do not occur in the References. We added a very detailed Index combining the index to notation, the name index, and the concept index. The page number where a notion is defined first is printed in boldface. The initial part of the Index is an index to notation. Names such as ‘J. von Neumann’ are indexed European style ‘Neumann, J. von.’

The exercises are sometimes trivial, sometimes genuine exercises, but more often compilations of entire research papers or even well-known open problems. There are good arguments to include both: the easy and real exercises to let the student exercise his comprehension of the material in the main text; the contents of research papers to have a comprehensive coverage of the field in a small number of pages; and research problems to show where the field is (or could be) heading. To save the reader the problem of having to determine which is which: “I found this simple exercise in number theory that looked like Pythagoras’s Theorem. Seems difficult.” “Oh, that is Fermat’s Last Theorem; it took three hundred and fifty years to solve it . . .,” we have adopted the system of rating numbers used by D.E. Knuth [The Art of Computer Programming, Volume 1: Fundamental Algorithms, Addison-Wesley, 1973. Second Edition, pp. xvii–xix]. The interpretation is as follows:

00 A very easy exercise that can be answered immediately, from the top of your head, if the material in the text is understood.

10 A simple problem to exercise understanding of the text. Use fifteen minutes to think, and possibly pencil and paper.

20 An average problem to test basic understanding of the text and may take one or two hours to answer completely.

30 A moderately difficult or complex problem taking perhaps several hours to a day to solve satisfactorily.

40 A quite difficult or lengthy problem, suitable for a term project, often a significant result in the research literature. We would expect a very bright student or researcher to be able to solve the problem in a reasonable amount of time, but the solution is not trivial.

50 A research problem that, to the authors’ knowledge, is open at the time of writing. If the reader has found a solution, he is urged to write it up for publication; furthermore, the authors of this book would appreciate hearing about the solution as soon as possible.
This scale is logarithmic: a problem of rating 17 is a bit simpler than average. Problems with rating 50, subsequently solved, will appear in a next edition of this book with rating about 45. Rates are sometimes based on the use of solutions to earlier problems. The rating of an exercise is based on that of its most difficult item, but not on the number of items. Assigning accurate rating numbers is impossible—one man’s meat is another man’s poison—and our rating will differ from ratings by others.

An orthogonal rating M implies that the problem involves more mathematical concepts and motivation than is necessary for someone who is primarily interested in Kolmogorov complexity and applications. Exercises marked HM require the use of calculus or other higher mathematics not developed in this book. Some exercises are marked with a ●; and these are especially instructive or useful. Exercises marked O are problems that are, to our knowledge, unsolved at the time of writing. The rating of such exercises is based on our estimate of the difficulty of solving them. Obviously, such an estimate may be totally wrong.

Solutions to exercises, or references to the literature where such solutions can be found, appear in the Comments paragraph at the end of each exercise. Nobody is expected to be able to solve all exercises.

The material presented in this book draws on work that until now was available only in the form of advanced research publications, possibly not translated into English, or was unpublished. A large portion of the material is new. The book is appropriate for either a one- or a two-semester introductory course in departments of mathematics, computer science, physics, probability theory and statistics, artificial intelligence, cognitive science, and philosophy. Outlines of possible one-semester courses that can be taught using this book are presented below.

Fortunately, the field of descriptional complexity is fairly young and the basics can still be comprehensively covered. We have tried to the best of our abilities to read, digest, and verify the literature on the topics covered in this book. We have taken pains to establish correctly the history of the main ideas involved. We apologize to those who have been unintentionally slighted in the historical sections. Many people have generously and selflessly contributed to verify and correct drafts of the various editions of this book. We thank them below and apologize to those we forgot. In a work of this scope and size there are bound to remain factual errors and incorrect attributions. We greatly appreciate notification of errors or any other comments the reader may have, preferably by email, in order that future editions may be corrected.
Outlines of One-Semester Courses

We have mapped out a number of one-semester courses on a variety of topics. These topics range from basic courses in theory and applications to special-interest courses in learning theory, randomness, or information theory using the Kolmogorov complexity approach.

PREREQUISITES: Sections 1.1, 1.2, 1.7 (except Section 1.7.4).

I. Course on Basic Algorithmic Complexity and Applications

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II. Course on Algorithmic Complexity

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IV. Course on Algorithmic Information Theory and Applications

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V. Course on Algorithmic Probability Theory, Learning, Inference, and Prediction

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VI. Course on the Incompressibility Method

Chapter 2 (Sections 2.1, 2.2, 2.4, 1.11.5, 2.8), Chapter 3 (mainly Sections 3.1, 3.3), Section 4.4, and Chapters 6 and 7. The course covers the basics of the theory with many applications in proving upper and lower bounds on the running time and space use of algorithms.

VII. Course on Randomness, Information, and Physics

Course III and Chapter 8. In physics the applications of Kolmogorov complexity include theoretical illuminations of foundational issues. For example, the approximate equality of statistical entropy and expected Kolmogorov complexity, the nature of entropy, a fundamental resolution of the Maxwell’s Demon paradox. However, also more concrete applications such as information distance, normalized information distance and its applications to phylogeny, clustering, classification, and relative semantics of words and phrases, as well as thermodynamics of computation are covered.
An Introduction to Kolmogorov Complexity and Its Applications
Li, M.; Vitányi, P.M.B.
2008, XXIV, 792 p. 21 illus., Hardcover