CHAPTER 2

ELEMENTS OF CONTINUUM MECHANICS

2.1 INTRODUCTION

This chapter reviews some of the key elements of continuum mechanics that are essential to both the understanding and development of the theory of plasticity. These concepts are mainly concerned with the analysis of stress and strain, equilibrium equations and compatibility conditions, as well as elastic stress-strain relations. The reader is referred to other texts such as Prager (1961), Fung (1965), Timoshenko and Goodier (1970), Spencer (1980) and Malvern (1969) for a detailed treatment of continuum mechanics.

2.2 STRESS STATE AND EQUILIBRIUM

2.2.1 Two-dimensional elements

As shown in Figure 2.1, the stress state for a two-dimensional element is defined by four stress components \( \sigma_{xx}, \sigma_{yy}, \sigma_{xy} \) and \( \sigma_{yx} \). The moment equilibrium demands that two shear stresses are equal in magnitude, namely \( \sigma_{xy} = \sigma_{yx} \). Note that compressive stresses are treated as positive here.

These stress components can be displaced as elements of a square matrix:

\[
\begin{bmatrix}
\sigma_{xx} & \sigma_{xy} \\
\sigma_{yx} & \sigma_{yy}
\end{bmatrix}
\]
The two most frequent cases of two-dimensional engineering problems are those of plane stress and plane strain. For the case of plane stress, the stresses normal to the \( xy \) plane are assumed to be identically zero. On the other hand, the case of plane strain only has non-zero strain components in the \( xy \) plane. In this case, the normal stress in the direction normal to the \( xy \) plane may be determined from the stresses acting on the \( xy \) plane \((\sigma_{xx}, \sigma_{yy}, \sigma_{xy})\) through elastic stress-strain relations that will be discussed later in this chapter. Whilst plane stress is a good assumption for simplifying many engineering problems in structural and mechanical engineering, plane strain is most relevant in geotechnical engineering. This is because many important geotechnical problems, such as embankments and tunnels, may be adequately analysed as a two-dimensional plane strain problem.

(a) Transformation of stresses and principal stresses

Now let us investigate the stress components at the point with respect to a new coordinate system \((x'oy')\), which is obtained by rotating the original coordinate system \((xoy)\) anticlockwise by an angle of \( \theta \) (see Figure 2.1). It can be easily shown that the stresses in these two coordinate systems are related by the following equations:

\[
\sigma_{x'x'} = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{yy} - \sigma_{xx}}{2} \cos 2\theta - \sigma_{xy} \sin 2\theta \tag{2.1}
\]

\[
\sigma_{y'y'} = \frac{\sigma_{xx} + \sigma_{yy}}{2} - \frac{\sigma_{yy} - \sigma_{xx}}{2} \cos 2\theta + \sigma_{xy} \sin 2\theta \tag{2.2}
\]

\[
\sigma_{x'y'} = \sigma_{xy} \cos 2\theta + \frac{\sigma_{yy} - \sigma_{xx}}{2} \sin 2\theta \tag{2.3}
\]

The principal stresses are those acting on a principal plane where shear stress is zero. The principal planes can be determined by setting equation (2.3) to zero, which gives:

\[
\tan 2\theta = \frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}} \tag{2.4}
\]

Substituting the above solution into equations (2.1) and (2.2) leads to the expressions for the two principal stresses:

\[
\sigma_1 = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2} \tag{2.5}
\]

\[
\sigma_2 = \frac{\sigma_{xx} + \sigma_{yy}}{2} - \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2} \tag{2.6}
\]

where \( \sigma_1 \) and \( \sigma_2 \) are also known as the major and minor principal stresses respectively.
The transformation of stresses, analytically expressed by the above equations, can also be simply achieved by using a Mohr-circle construction. Assume that the positive plane normal to the x direction is denoted by A and the positive plane normal to the y direction by B. Whilst compressive normal stresses are regarded as positive, shear stresses acting clockwise are treated as positive.

![Mohr-circle construction](image)

**Figure 2.2: Transformation of stresses using a Mohr-circle construction**

Using the Mohr-circle construction shown in Figure 2.2, the stresses on the plane A and B are defined by the coordinates of points A and B in the Mohr-circle. The stresses for the corresponding planes \(A'\) and \(B'\) with respect to a new coordinate system \((x'y')\) are equal to the coordinates of the points \(A'\) and \(B'\) in the Mohr-circle. It is noted that the points \(A'\) and \(B'\) are arrived by rotating the points A and B respectively by two times the angle between the coordinate systems \((xoy)\) and \((x'y')\).

By definition, the principal stresses are the coordinates of the interaction points between the Mohr-circle and the normal stress axis.

**(b) Equations of interior stress equilibrium**

By accounting for stress variation with coordinates, equations of stress equilibrium can be established. It is instructive to first consider all the stresses in the x direction, as shown in Figure 2.3. The quantity \(X\) is assumed to be the body force (i.e. force per unit volume). The equation of force equilibrium in the x direction leads to the following equation of stresses:

\[
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = X
\]  

(2.7)
Similarly consideration of the force equilibrium in the y direction gives the second equation of stresses:

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = Y$$  \hspace{1cm} (2.8)

where $Y$ denotes the body force in the y direction.

Equations (2.7) and (2.8) are known as the equations of interior stress equilibrium for two-dimensional problems.

(c) Equations of boundary stress equilibrium

When some part of a boundary is subject to tractions (shear and normal components), they need to be in equilibrium with interior stresses acting surrounding that part of the boundary.
Let us assume that the orientation of the boundary with known tractions is denoted by the angle $\alpha$, as shown in Figure 2.4. The equilibrium of the triangular element requires:

\begin{align*}
\sigma_{xx} \cos \alpha + \sigma_{xy} \sin \alpha &= T_x & (2.9) \\
\sigma_{xy} \cos \alpha + \sigma_{yy} \sin \alpha &= T_y & (2.10)
\end{align*}

where $T_x$ and $T_y$ are the applied traction components in the $x$ and $y$ directions.

![Figure 2.5: Stress state for three-dimensional elements](image)

### 2.2.2 Three-dimensional elements

#### (a) Stress tensor

The state of stress for a three-dimensional point is defined by a matrix containing nine stress components shown in Figure 2.5. The nine components of the stress at any point form a second order tensor, known as the stress tensor $\sigma_{ij}$, where $i$ and $j$ take integral values 1, 2 and 3. In this way, the stress components can be expressed as elements of a square matrix:

\[
\begin{bmatrix}
\sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\
\sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\
\sigma_{zx} & \sigma_{zy} & \sigma_{zz}
\end{bmatrix}
= 
\begin{bmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} \\
\sigma_{21} & \sigma_{22} & \sigma_{23} \\
\sigma_{31} & \sigma_{32} & \sigma_{33}
\end{bmatrix}
= \sigma_{ij} & (2.11)
\]

As in the two-dimensional case, moment equilibrium demands the following relationships on shear stresses:

\[
\sigma_{xy} = \sigma_{yx} \quad \sigma_{xz} = \sigma_{zx} \quad \sigma_{yz} = \sigma_{zy}
\] & (2.12)

As a result there are only six independent stress components: three normal stresses $(\sigma_{xx}, \sigma_{yy}, \sigma_{zz})$ and three shear stresses $(\sigma_{xy}, \sigma_{yz}, \sigma_{xz})$. 

(b) Principal stresses

The state of stress at a point in three dimensions can also be defined by three principal stresses $\sigma_1, \sigma_2$ and $\sigma_3$. These principal stresses are linked to the components of the stress tensor by the following cubic equation:

$$\sigma^3 - I_1\sigma + I_2\sigma + I_3 = 0 \quad (2.13)$$

where $I_1, I_2$ and $I_3$ are known as the first, second and third stress invariant respectively, which are defined as follows:

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \quad (2.14)$$

$$I_2 = \sigma_{xx} \sigma_{yy} \sigma_{yy} + \sigma_{zz} \sigma_{xx} - \sigma_{xy}^2 - \sigma_{yx}^2 - \sigma_{xz}^2 \quad (2.15)$$

$$I_3 = \sigma_{xx} \sigma_{yy} \sigma_{zz} - \sigma_{xx} \sigma_{yz}^2 - \sigma_{yy} \sigma_{xz}^2 - \sigma_{zz} \sigma_{yx}^2 + 2\sigma_{xy} \sigma_{yz} \sigma_{xz} \quad (2.16)$$

The stress tensor in terms of the principal stresses takes the form

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} = \sigma_{ij} \quad (2.17)$$

In this case the stress invariants are linked to the principal stresses as follows

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3 \quad (2.18)$$

$$I_2 = \sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_1 \quad (2.19)$$

$$I_3 = \sigma_1 \sigma_2 \sigma_3 \quad (2.20)$$

(c) The mean stress and deviatoric stresses

The mean stress of a stressed point is defined as the average of normal stresses in three directions, which can be expressed as follows:

$$p = \frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) = \frac{1}{3}I_1 \quad (2.21)$$

The deviatoric components of the stress are defined by

$$s_{ij} = \sigma_{ij} - p\delta_{ij} \quad (2.22)$$

where $\delta_{ij}$ is the Kronecker delta whose value is 1 when $i = j$ and is equal to 0 otherwise.

The three invariants of deviatoric stress are...
\[ J_1 = s_{kk} = 0 \]  
\[ J_2 = \frac{1}{2} s_{ij} s_{ij} = \frac{1}{3} (I_1^2 + 2I_3) \]
\[ = \frac{1}{6} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right] \]  
\[ = \frac{1}{6} \left[ (\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2 \right] \]
\[ + \sigma_{yy}^2 + \sigma_{zz}^2 + \sigma_{xx}^2 \]  
\[ J_3 = \frac{1}{3} s_{ij} s_{jk} s_{ki} = \frac{1}{27} (2I_1^3 + 9I_1 I_2 + 27I_3) \]

It is noted that in the theory of soil plasticity, the most useful stress invariants are \( I_1 \), \( J_2 \) and \( J_3 \). Physically, \( I_1 \) indicates the effect of mean stress, \( J_2 \) represents the magnitude of shear stress, and \( J_3 \) determines the direction of shear stress. As will be discussed in the rest of this book, all these three quantities (mean stress, shear stress and shear stress direction) have a key role to play in the theory of elastic-plastic stress-strain relations.

![Figure 2.6: Lode angle on a deviatoric plane](image)

The three principal stresses can be determined from the stress invariants as follows

\[ \sigma_1 = \frac{1}{3} I_1 + \frac{2}{\sqrt{3}} \sqrt{J_2} \sin (\theta_l + 120^0) \]  
\[ \sigma_2 = \frac{1}{3} I_1 + \frac{2}{\sqrt{3}} \sqrt{J_2} \sin (\theta_l) \]  
\[ \sigma_3 = \frac{1}{3} I_1 + \frac{2}{\sqrt{3}} \sqrt{J_2} \sin (\theta_l - 120^0) \]
where $\theta_l$ is known as the Lode angle defined in Figure 2.6 as

$$\theta_l = \tan^{-1} \left[ \frac{1}{\sqrt{3}} \left( \frac{2\sigma_3 - \sigma_1 - \sigma_2}{\sigma_1 - \sigma_2} \right) \right]$$

or

$$\theta_l = -\frac{1}{3} \sin^{-1} \left[ \frac{3\sqrt{3}}{2} \left( \frac{J_3}{J_2^{3/2}} \right) \right]$$

which ranges between $-30^0$ and $30^0$.

In soil mechanics (Roscoe and Burland, 1968; Muir Wood, 1990), the mean stress $\mu$ is often used in pair with a generalised shear stress $\varphi$ defined below

$$\varphi = \sqrt{\frac{1}{4} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]^{1/2}}$$

$$\varphi = \sqrt{3J_2}$$

which reduces to

$$\varphi = \sigma_1 - \sigma_3$$

for the triaxial loading condition where $\sigma_2 = \sigma_3$.

In terms of $\mu$ and $\varphi$, the principal stresses are

$$\sigma_1 = \mu + \frac{2}{3} \varphi \sin (\theta_l + 120^0)$$

$$\sigma_2 = \mu + \frac{2}{3} \varphi \sin (\theta_l)$$

$$\sigma_3 = \mu + \frac{2}{3} \varphi \sin (\theta_l - 120^0)$$

(d) Equations of stress equilibrium

By taking account of stress variation with coordinates, the force equilibrium conditions in three directions will lead to the following well-known equation of stress equilibrium:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = X$$

$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = Y$$

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = Z$$
where \( X, Y \) and \( Z \) are the body forces acting in the \( x, y \) and \( z \) directions respectively.

2.3 STRAIN AND COMPATIBILITY

2.3.1 Two-dimensional elements

Let us use \( u \) and \( v \) to denote the displacement components in the \( x \) and \( y \) directions of a point in two dimensions. It can be easily shown that the normal strains in both directions are linked to the displacements by the following relationship:

\[
\varepsilon_{xx} = \frac{\partial u}{\partial x} \tag{2.41}
\]

\[
\varepsilon_{yy} = \frac{\partial v}{\partial y} \tag{2.42}
\]

In addition, the shear strain is given by

\[
\varepsilon_{xy} = \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \tag{2.43}
\]

These strain components can be displayed as elements of a square matrix:

\[
\begin{bmatrix}
\varepsilon_{xx} & \varepsilon_{xy} \\
\varepsilon_{yx} & \varepsilon_{yy}
\end{bmatrix}
\]

It is noted that the definition of a well-used shear strain (termed the engineering shear strain) is given by

\[
\gamma_{xy} = 2\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{2.44}
\]

The strain components \( \varepsilon_{xx}, \varepsilon_{yy} \) and \( \varepsilon_{xy} \) are not independent and they are linked by the following condition (known as the compatibility condition):

\[
\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2 \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} \tag{2.45}
\]

which is obtained from equations (2.41) to (2.43) by eliminating \( u \) and \( v \).

2.3.2 Three-dimensional elements

In three dimensions, the displacement components in the \( x, y \) and \( z \) directions are denoted by \( u, v \) and \( w \) respectively. The components of strain can be expressed by a strain tensor:

\[
\begin{bmatrix}
\varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\
\varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\
\varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz}
\end{bmatrix}
= 
\begin{bmatrix}
\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\
\varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\
\varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33}
\end{bmatrix}
= \varepsilon_{ij} \tag{2.46}
\]
The strain components are related to the displacement field as follows:

\[ \varepsilon_{xx} = \frac{\partial u}{\partial x} \]  \hspace{1cm} (2.48)
\[ \varepsilon_{yy} = \frac{\partial v}{\partial y} \]  \hspace{1cm} (2.49)
\[ \varepsilon_{zz} = \frac{\partial w}{\partial z} \]  \hspace{1cm} (2.50)
\[ \varepsilon_{xy} = \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \]  \hspace{1cm} (2.51)
\[ \varepsilon_{xz} = \frac{1}{2} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \]  \hspace{1cm} (2.52)
\[ \varepsilon_{yz} = \frac{1}{2} \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \]  \hspace{1cm} (2.53)

In addition to equation (2.45), two more conditions of strain compatibility can be obtained:

\[ \frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} = 2 \frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z} \]  \hspace{1cm} (2.54)
\[ \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial x^2} = 2 \frac{\partial^2 \varepsilon_{xz}}{\partial x \partial z} \]  \hspace{1cm} (2.55)

### 2.4 ELASTIC STRESS-STRAIN RELATIONS

Although this book is concerned with stress-strain relations in the plastically deforming region, it is fundamental to understand the stress-strain relations in the elastic region (widely known as Hooke’s law).

#### 2.4.1 Plane stress conditions

Some engineering practical problems can be simplified as a plane stress problem in which the stress in one direction (e.g. the z direction) is so small that it can be ignored, namely \( \sigma_{zz} = 0 \). In this case, the strain components can be related to the stress components in the following way:

\[ \varepsilon_{xx} = \frac{1}{E} (\sigma_{xx} - \nu \sigma_{yy}) \]  \hspace{1cm} (2.56)
\[ \varepsilon_{yy} = \frac{1}{E} (\sigma_{yy} - \nu \sigma_{xx}) \]  \hspace{1cm} (2.57)
\[ \varepsilon_{xy} = \frac{1 + \nu}{E} \sigma_{xy} \quad (2.58) \]

where \( E \) and \( \nu \) are material constants known as the Young’s modulus and Poisson’s ratio respectively. These linear stress-strain relations can also be expressed in the following way:

\[ \sigma_{xx} = \frac{E}{1 - \nu^2} (\varepsilon_{xx} + \nu \varepsilon_{yy}) \quad (2.59) \]

\[ \sigma_{yy} = \frac{E}{1 - \nu^2} (\varepsilon_{yy} + \nu \varepsilon_{xx}) \quad (2.60) \]

\[ \sigma_{xy} = \frac{E}{1 - \nu^2} (1 - \nu) \varepsilon_{xy} \quad (2.61) \]

### 2.4.2 Plane strain conditions

Many geotechnical engineering problems can be adequately analysed as a plane strain problem in which the strain in one direction, say the \( z \) direction, is very small so that it can be ignored, namely \( \varepsilon_{zz} = 0 \). In this case, the stress-strain relations are given by:

\[ \sigma_{xx} = \frac{(1 - \nu)E}{(1 + \nu)(1 - \nu^2)} (\varepsilon_{xx} + \nu \varepsilon_{yy}) \quad (2.62) \]

\[ \sigma_{yy} = \frac{(1 - \nu)E}{(1 + \nu)(1 - \nu^2)} (\varepsilon_{yy} + \nu \varepsilon_{xx}) \quad (2.63) \]

\[ \sigma_{xy} = \frac{E}{1 + \nu} \varepsilon_{xy} \quad (2.64) \]

### 2.4.3 Three-dimensional conditions

In three dimensions, the elastic normal stress-strain relations take the following simple form:

\[ \varepsilon_{xx} = \frac{1}{E} [\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})] \quad (2.65) \]

\[ \varepsilon_{yy} = \frac{1}{E} [\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})] \quad (2.66) \]

\[ \varepsilon_{zz} = \frac{1}{E} [\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})] \quad (2.67) \]

The shear stresses are related to the shear strains by the following relations

\[ \varepsilon_{xy} = \frac{1 + \nu}{E} \sigma_{xy} = \frac{\sigma_{xy}}{2G} \quad (2.68) \]
\[
\varepsilon_{yz} = \frac{1 + \nu}{E} \sigma_{yz} = \frac{\sigma_{yz}}{2G} \tag{2.69}
\]
\[
\varepsilon_{xz} = \frac{1 + \nu}{E} \sigma_{xz} = \frac{\sigma_{xz}}{2G} \tag{2.70}
\]
in which \( G \) is shear modulus of the material.

2.5 SUMMARY

(1) The state of stress at a point is defined by a stress tensor with nine stress components. However only six of them, three normal stresses and three shear stresses, are independent due to moment equilibrium. These stresses need to satisfy three equations of equilibrium.

(2) Deformation of a point can be described by strains. In three dimensions, there are also three normal strains and three shear strains. These strains are also linked by compatibility conditions.

(3) The relationship between stresses and strains can be very complex and mainly depends on material types and loading conditions. For linear elastic materials, the relationship between stresses and strains is governed by Hooke's law.

REFERENCES


