Chapter 2

PRELIMINARIES

Principieremo col definire un termine di cui è comodo fare uso per scansare lungaggini. Diremo che i componenti di una collettività godono, in una certo posizione, del MASSIMO DI OFELIMITÀ, *quando è imposible allontanarsi pochissimo da quella posizione giovando, o nuocendo, a tutti i componenti la collettività; ogni piccolissimo spostamento da quella posizione avendo necessariamente per effetto di giovare a parte dei componenti la collettività e di nuocere ad altri.*

[*We will begin by defining a term which is desirable to use in order to avoid prolixity. We will say that the members of a collectivity enjoy* MAXIMUM OPHELIMITY *in a certain position when it is impossible to find a way of moving from that position very slightly in such a manner that the ophelimity enjoyed by each of the individuals of that collectivity increases or decreases. That is to say, any small displacement in departing from that position necessarily has the effect of increasing the ophelimity which certain individuals enjoy, and decreasing that which others enjoy, of being agreeable to some and disagreeable to others.*]

Vilfredo Pareto,
Manuale di Economia Politica.

1. This Chapter Is About ...

... comparing alternatives. We compare alternatives with respect to criteria. Since, as a rule, we admit more than one criterion, the matter is not straightforward. But first, let us observe that even with two or more criteria there are situations when in a set of feasible alternatives one alternative is preferred to any other with respect to all criteria. Such an alternative is called *utopian*. Table 2.1 and related Figure 2.1 present the case where only two feasible alternatives are available and one of them (alternative A) is utopian. Figure 2.2. shows the situation where a utopian alternative does not exist (for there is no feasible alternative corresponding to the most preferred criteria values as represented by $\hat{y}$).

Table 2.1. Two alternatives, one of which is utopian.

	Criterion type	*Alternative A*	*Alternative B*
Criterion 1	"better if more"	75	25
Criterion 2	"better if more"	75	25

In reality utopian alternatives happen infrequently. On the other hand, it quite often happens that one of a pair of alternatives is preferred with respect to all criteria. This is called *dominance*. With the notion of dominance we easily arrive at the notion of *efficiency*. We also recall two related notions, namely *weak efficiency* and *proper efficiency*, which we will need in subsequent chapters.

To avoid ambiguities we need a dose of formalism and so we define MCDM problems in terms of sets of feasible alternatives, criteria, and *outcomes*.

Finally, we revisit the idea of trade-off and we define two particular instances of this notion.

In MCDM problems both efficiency and trade-off notions serve to differentiate alternatives.

2. Dominance and Efficiency

At the core of MCDM is the notion of efficiency. This is the bridge between informal definition (1.1) and Multiple Criteria Decision Mak-

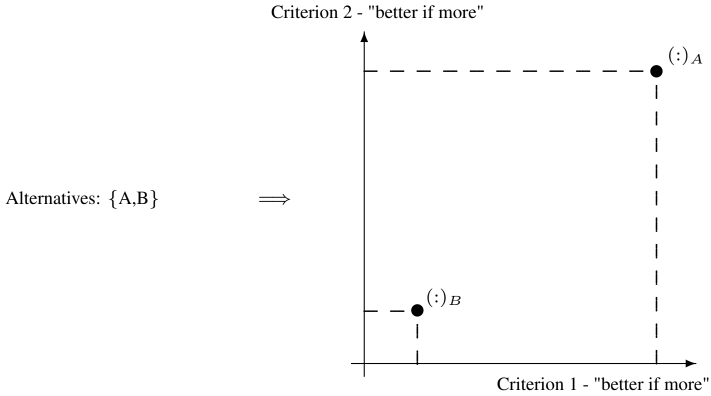

Figure 2.1. An example of two alternatives.

ing. The notion emerges immediately and naturally when one evaluates alternatives with respect to more than just one criterion.

If alternatives are compared, for example, with respect to the costs they incur (imagine buying a house), then quite naturally the cheapest seems to be the most preferred. But what if an additional criterion comes into play? In the case of buying a house size is never to be forgotten. A popular belief is that the bigger a house the better (within, of course, certain sensible limits). But what if the two criteria, cost and size, are analyzed jointly? Can the cost be the lowest for the biggest house? If so, this would be a "free lunch", but the reality of real estate markets shows us that bigger the house, the greater the cost. It does not, however, exclude the possibility that we can sometimes discover on a local market a bigger house being cheaper than a smaller one (we ignore any other possible criteria). Such situations are formalized by the notion of dominance.

Given a set of alternatives, a feasible alternative x is called *dominated* if there is another feasible alternative in the set, say alternative x', such that:

- x' is equally or more preferred than x with respect to all criteria, and
- x' is more preferred than x for at least one criterion.

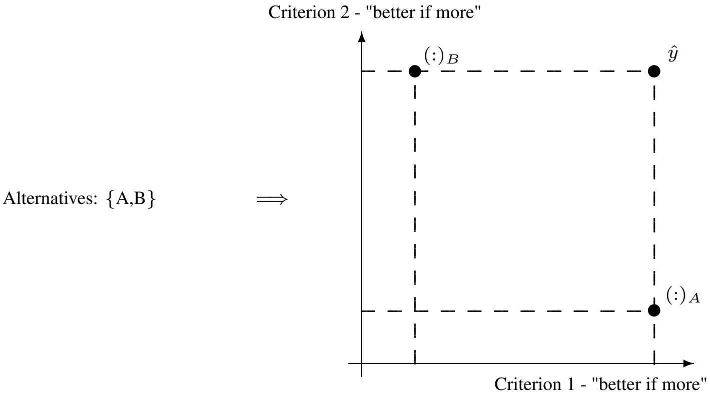

Figure 2.2. An example where a utopian alternative does not exist.

If the above holds, the alternative x' is called *dominating*. A pair of alternatives x and x', where x is dominated and x' is dominating, is said to be in *Pareto dominance relation*. Clearly, in a set of more than two alternatives, one alternative can be dominating and at the same time dominated.

Throughout the book we adopt the convention that all criteria are of "better if more" type. We also make the assumption that criteria are represented by numerical values. Then, if a criterion is of "better if less" type, we can always change it to "better if more" type by multiplying all possible values of this criterion by -1.

Given a set of feasible alternatives, an alternative which is not dominated by any other alternative of this set is called *efficient*. In other words, an alternative is efficient if there is no other alternative in the set:

- equally or more preferred with respect to all criteria,

and

- more preferred for at least one criterion.

To be efficient an alternative is required much less than to be utopian. Consequently, efficient alternatives are more common than utopian. An utopian alternative is necessarily efficient but not vice versa.

Alternatives which are not efficient are called *nonefficient*.

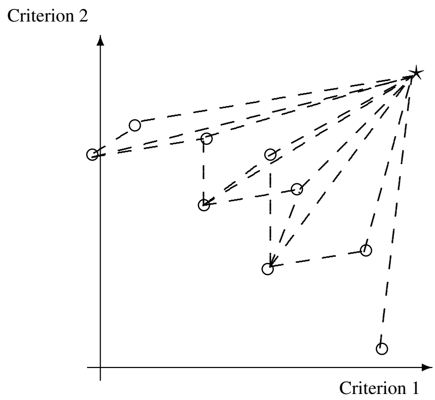

Figure 2.3. Pairs of alternatives in Pareto dominance relationships - case I.

With the convention that stars represent utopian alternatives, solid disks represent efficient but not utopian alternatives, and circles represent dominated alternatives, Figure 2.3 and Figure 2.4 show a handful of alternatives represented by values of two criteria. Dashed lines between pairs of alternatives indicate that those alternatives are in Pareto dominance relationship. In Figure 2.3 one alternative is clearly utopian (and therefore it is efficient). That alternative, by definition, is in Pareto dominance relationship with all the remaining alternatives. Figure 2.4 shows the same set of alternatives with the utopian alternative removed. In this case several alternatives are efficient.

If a set of alternatives is given implicitly by a number of conditions (constraints), the number of alternatives can be infinite. It is impossible then to represent graphically all Pareto dominance relationships but we can still do that for selected pairs of alternatives. Figure 2.5 gives an example of a set of alternatives represented by values of two criteria, where the set of criteria values has the shape of a polygon. In this case efficient alternatives are those whose criteria vectors form a part of the polygon border, as marked in the figure by the thick line.

It is common that nonefficient alternatives are neglected in the decision making processes by being clearly unreasonable candidates for the most preferred alternative. In our house buying example, if one can buy more for less, why do otherwise? Why not consume a "free lunch"? But it

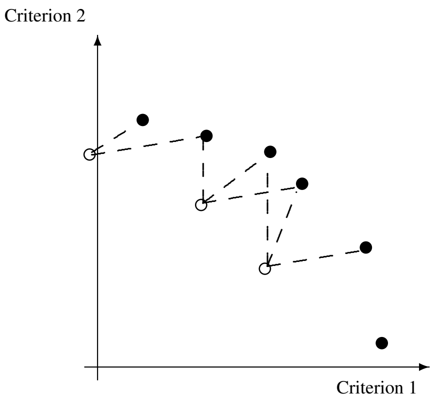

Figure 2.4. Pairs of alternatives in Pareto dominance relationships - case II.

is reasonable not to remove nonefficient alternatives from consideration permanently. In a changed decision making environment (e.g. the set of criteria changed) nonefficient (i.e. dominated) alternatives may become efficient. Therefore, it is wise to tag dominated alternatives as only temporarily neglected and keep them for possible future analysis.

With the above remark on the possible non-permanent non-efficiency status of alternatives in mind, we can now safely state that the majority of MCDM methods consist in selecting the most preferred alternative from efficient alternatives.

3. Definitions And Problem Settings

In what follows we shall need a dose of formalism to structure decision making problem data.

In the MCDM framework decision making problem (1.1) is formalized as follows:

$$\begin{gathered}\text{choose an alternative } x \text{ for which vector } f(x),\ x \in X_0 \subseteq \mathcal{X}, \\ \text{is the most preferred,}\end{gathered} \tag{2.1}$$

where $\mathcal{X}$ is the set (space) of potential alternatives, X_0 is the set of feasible alternatives, $f : \mathcal{X} \to \mathcal{R}^k$ is the criteria map in which $f = (f_1, ..., f_k)$,

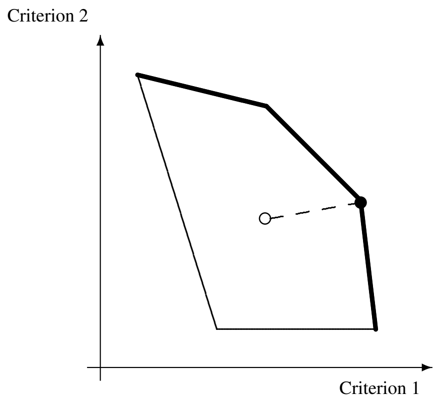

Figure 2.5. Pairs of alternatives in Pareto dominance relationships - case III.

and $f_i : \mathcal{X} \to \mathcal{R}$ are *criteria functions*, $i = 1, ..., k$, $k \geq 2$. As said before, we assume that all criteria are of "better if more" type.

An alternative x for which $f(x)$ is the most preferred vector of criteria values, is the *most preferred* alternative.

The underlying assumption for the main stream of research done in the field of MCDM is that the environment for decision making is certain (as opposed to uncertain). In this book we are concerned exclusively with decision making problems which can be fairly modeled under such an assumption.

♣ If the assumption about the certain nature of the decision making environment is not justified, one has to take appropriate measures to act in an uncertain environment. Below we present, in broad outline, how decision making problems in certain and uncertain environments are formulated and how those two decision frameworks differ.

Setting A - certain environment, single criterion.

Given a set of n alternatives, each alternative is evaluated with respect to the criterion and yields a score. Assume that the DM has value function v, which to every alternative considered assigns a numerical value v^j, $j = 1, ..., n$, representing the DM's preference with respect to the score of that alternative. The most preferred alternative is that showing the highest value of the value function. Table 2.2 shows data for Setting A.

Table 2.2. Decision making problem – Setting A.

	Alternative 1	...	*Alternative* n
Criterion	θ^1	$\cdot$	θ^n
Value	v^1	$\cdot$	v^n

This is the most basic setting of decision making problems. Very often value functions are assumed to be identity mappings, i.e. decisions are made on the base of scores.

Setting B - certain environment, multiple criteria.

Given a set of n alternatives, each alternative is evaluated with respect to the set of k criteria and yields a k-tuple of scores. Assume that the DM has value function v which to alternative j assigns numerical value $v^j,\ j = 1, ..., n$, representing the DM's preference with respect to k-tuple of scores of that alternative. The most preferred alternative is that showing the highest value of value function v. Table 2.3 shows data for Setting B.

The existence of a value function of that sort is just a theoretical postulate, with all its practical drawbacks that we discussed in Chapter 1. This book is about how to help the DM to make decisions without resorting to the concept of value function.

Setting B is a problem formulation for a finite number of alternatives. In this book, however, MCDM problems are formulated in the manner which admits a finite and infinite number of alternatives. A good example of an infinite number of alternatives occurs when one selects a percentage of a certain resource to be used or invested.

Setting C - uncertain environment, single criterion.

Given a set of n alternatives, each alternative is evaluated with respect to the criterion and yields a score. This is done for every state i of nature which may occur with probability $p_l,\ l = 1, ..., m$. It is assumed that the DM has value function v in the form of expected value of scores: $v^j = \sum_{l=1}^{m} p_l \theta_l^j,\ j = 1, ..., n$. The most preferred alternative is that

Table 2.3. Decision making problem – Setting B.

	Alternative 1	$\ldots$	*Alternative* n
Criterion 1	θ_1^1	$\cdot$	θ_1^n
$\vdots$	$\vdots$	$\vdots$	$\vdots$
Criterion k	θ_k^1	$\ldots$	θ_k^n
Value	v^1	$\ldots$	v^n

showing the highest value of the value function. Table 2.4 shows data for Setting C.

Table 2.4. Decision making problem – Setting C.

	Alternative 1	$\ldots$	*Alternative* n	*Probability*
State 1	θ_1^1	$\cdot$	θ_1^n	p_1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
State m	θ_m^1	$\ldots$	θ_m^n	p_m
Expected value	v^1	$\ldots$	v^n	

Setting D - uncertain environment, multiple criteria.

Setting D can be arrived at either from Setting B by admitting uncertainty or from Setting D by admitting multiple criteria.

Given a set of n alternatives, each alternative is evaluated with respect to the set of k criteria and yields k-tuple of scores. This is done for every state i of nature which may occur with probability $p_l,\ l = 1, ..., m$. DM's value function v with respect to each criterion is, as in Setting C, the expected value of scores: $v_i^j = \sum_{l=1}^{m} p_l \theta_{l,i}^j\,,\ i = 1, ..., k,\ j = 1, ..., n.$

Now the DM faces the problem of selecting from among alternatives characterized by k-tuples of numbers (expected values). Assume, as in Setting B, that the DM has value function V which to every alternative considered assigns numerical value $V^j,\ j = 1, ..., n$, representing the DM's preference to k-tuple of expected values of that alternative. The most preferred alternative is that showing the highest value of value function V. Table 2.5 shows data for Setting D.

Research on decision making problems conforming to Setting D is limited. This is due to the involved structure of this framework.

Table 2.5. Decision making problem – Setting D.

	Alternative 1	$\dots$	*Alternative* n	*Probability*
State 1	$\theta^1_{1,1}$ $\vdots$ $\theta^1_{1,k}$	$\vdots$	$\theta^n_{1,1}$ $\vdots$ $\theta^n_{1,k}$	p_1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
State m	$\theta^1_{m,1}$ $\vdots$ $\theta^1_{m,k}$	$\vdots$	$\theta^n_{m,1}$ $\vdots$ $\theta^n_{m,k}$	p_m
Expected values	v^1_1 $\vdots$ v^1_k	$\vdots$	v^n_1 $\vdots$ v^n_k	
Value	V^1	$\dots$	V^n	

Relations between all four formulations are represented graphically in Figure 2.6.

♠

From the algorithmic point of view problem (2.1) is ill defined. In fact, as long as we do not know what "most preferred" means precisely we are not in a position to propose any problem solving method. This information, explicit or implicit, if exists, is in the exclusive possession of the DM. The underlying assumption of MCDM methodologies we are concerned with in this book is that this information cannot be acquired from the DM up front and at once.

In the sequel we frequently exploit the fact that the following problem, called *vector optimization problem*,

$$\textit{vmax}\ f(x),\ \ x \in X_0 \subseteq \mathcal{X}, \tag{2.2}$$

where *vmax* stands for the identification of all efficient alternatives, is almost always well defined. By this we mean that under minor assumptions satisfied in practical applications the solution to (2.2) always exists.

Without any ambiguity alternatives are represented by their criteria values. With this in mind we deal mainly with elements $f(x)$ of set $f(X_0)$ and for the sake of simplicity we use y and Z to denote

$$y = f(x),\ \text{and}\ Z = f(X_0).$$

Clearly, $Z \subseteq \mathcal{R}^k$. We call elements of set Z *outcomes* and space $\mathcal{R}^k$ the *outcome space*. Under this convention, for given feasible alternative $x,\ \ y_i$ is the value of the $i-$th component of outcome $y = f(x)$. Thus, y_i is the value of $i-$th criterion for alternative x.

All properties of alternatives considered in subsequent paragraphs and chapters can be defined and operationalized in terms of outcomes. We directly refer to notation x , $X_0,\ f(x),\ f(X_0)$ only when giving examples of MCDM problems with implicitly (i.e. in the form of constraints) defined feasible alternatives.

Element $\hat{y}$ of $\mathcal{R}^k$, called *utopian,* is calculated as

$$\hat{y}_i = max_{y \in Z}\, y_i,\ i = 1, ..., k.$$

Throughout this book it is assumed that all these maxima exist.

Element $\hat{y}$ need not represent any feasible alternative.

Below we recall the definitions of efficiency, weak efficiency, and proper efficiency of outcomes.

DEFINITION 1 The outcome $\bar{y} \in Z$ is:
- *efficient* if $y_i \geq \bar{y}_i$, $i = 1, ..., k,\ y \in Z$, implies $y = \bar{y}$,
- *weakly efficient* if there is no $y,\ y \in Z$, such that $y_i > \bar{y}_i$, $i = 1, ..., k$,

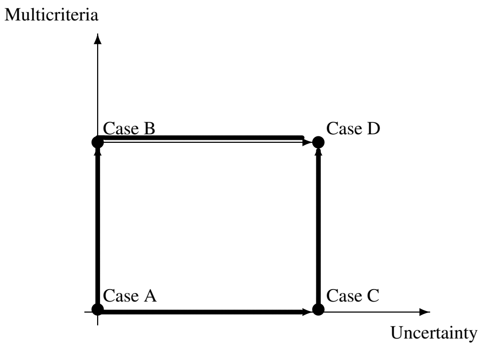

Figure 2.6. Four settings of decision making problems.

- *properly efficient* if it is efficient and there exists a finite number $M > 0$ such that for each i we have

$$\frac{y_i - \bar{y}_i}{\bar{y}_j - y_j} \leq M$$

for some j such that $y_j < \bar{y}_j$, whenever $y \in Z$ and $y_i > \bar{y}_i$. □

Outcomes which are not efficient are called *nonefficient*.

Clearly, from the definition of efficient (nonefficient) alternatives given in Section 2 it follows that an alternative is efficient (nonefficient) if and only if its outcome is efficient (nonefficient).

It is common to call the subset of all efficient outcomes of Z the *Pareto set* and below we shall often use this term.

The notion of efficiency remains at the core of MCDM. For $y \in Z$ the implications:

$$\begin{aligned} y \text{ is properly efficient } &\Rightarrow\ y \text{ is efficient,} \\ y \text{ is efficient } &\Rightarrow\ y \text{ is weakly efficient,} \end{aligned} \tag{2.3}$$

hold, but implications

$$\begin{aligned} y \text{ is weakly efficient } &\Rightarrow\ y \text{ is efficient,} \\ y \text{ is efficient } &\Rightarrow\ y \text{ is properly efficient,} \end{aligned}$$

are, in general, false.

♣ There is a direct relation between the Vilfredo Pareto definition of maximal ophelimity collectively enjoyed, given on the first page of this chapter, and the notion of efficient outcome. Under the assumption that all the criteria are of "better if more" type, and assuming that for each criterion the ophelimity of its value is measured by its value itself, each efficient outcome represents an alternative at which the collective ophelimity, represented by all criteria values jointly, is maximized.
♠

The reason for dealing with the notion of weak efficiency and proper efficiency is purely technical. There exist some algorithmically and computationally convenient methods, presented in Chapter 3, to derive weakly efficient outcomes and properly efficient outcomes (and thus the corresponding alternatives, weakly efficient and properly efficient). The following relations

$$\{\text{all properly efficient outcomes of } Z\} \subseteq \{\text{all efficient outcomes of } Z\}$$
$$\subseteq \{\text{all weakly efficient outcomes of } Z\}, \tag{2.4}$$

which result from implications (2.3), hold. Therefore, it is common to refer to the subset of all weakly efficient outcomes of problem (2.1) (or (2.2)) as an upper approximation of the Pareto set, and the subset of all properly efficient outcomes as a lower approximation of the Pareto set.

Weakly efficient outcomes which are nonefficient occur if two or more outcomes are located on a hyperplane parallel to any of the axes. This is illustrated in Figure 2.7.

Figure 2.8 shows an outcome set where all elements of the boundary marked with the thick line are properly efficient except those three marked with bullets. From this example we intuitively infer that *improperly* (i.e. not properly) efficient outcomes (alternatives) are, in a sense, rare and therefore of little significance in real decision problems.

If set of outcomes Z is polyhedral (the set represented in Figure 2.5 is polyhedral and bounded, thus it is two dimensional polytope, hence polygon), then all efficient outcomes are properly efficient. Then, in this case the notion of proper efficiency is redundant. This notion is also redundant in the case of finite sets Z, since in such sets for all efficient outcomes numbers M from Definition 1 of this chapter can be found by enumeration.

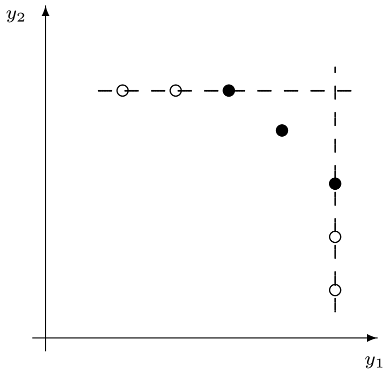

Figure 2.7. Weakly efficient and efficient outcomes.

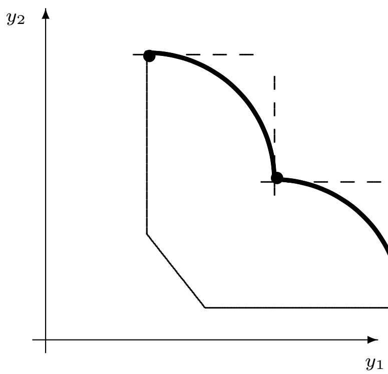

Figure 2.8. Improperly efficient outcomes.

♣ Another definition of outcome efficiency, equivalent to that given in Definition 1 of this chapter, can also be given in terms of cones. Set K is a cone if $\lambda K \in K$ for any $\lambda \geq 0$.

Let R^k_+ denote the nonnegative orthant $\{y \in \mathcal{R}^k \,|\, y_i \geq 0,\, i = 1, ..., k\}$. The nonnegative orthant is a cone. R^k_+ can be interpreted as the set of directions of improvement (recall that all criteria are of "better if more" type), i.e. given an element $y \in \mathcal{R}^k$, any element $y',\ y' \in \mathcal{R}^k,\ y' \neq y,$

such that $y' \in \{y\} + R^k_+$ has all coordinates at least as great as y and at least one coordinate strictly greater than y.

The other definition of outcome efficiency states that an outcome y (recall that an outcome is an element of Z) is efficient if there is no outcome y', $y' \neq y$, such that $y' \in \{y\} + R^k_+$. In other words, $y \in Z$ is efficient if $(\{y\} + R^k_+) \cap Z = \{y\}$.

The notion of cone is very convenient in MCDM for graphical illustrations of efficiency issues.

First of all, it is easy to check graphically whether an outcome is efficient or not. It is enough to place cone R^k_+ at an outcome whose efficiency has to be verified, say outcome y, and see if set $\{y\} + R^k_+$ contains any outcome different than y. If it does not, y is efficient, otherwise it is nonefficient. In Figure 2.9 bullet marked outcomes are efficient.

If y is nonefficient, then by definition y is *dominated* by some other outcome, say y' (y' *dominates* y), such that $y' \in \{y\} + R^k_+$, or equivalently, $y \in \{y'\} - R^k_+$.

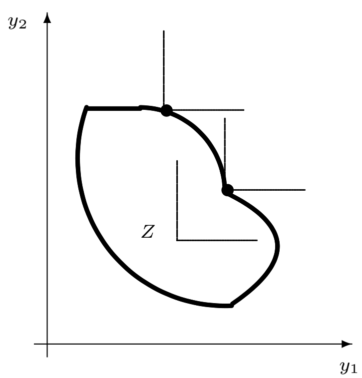

Figure 2.9. Checking efficiency with cone R^k_+.

With cone R^k_+ it is also possible to give another definition of weak efficiency, equivalent to that given in Definition 1 of this chapter. Namely, an outcome y is weakly efficient if there is no outcome y', $y' \neq y$, such that $y' \in \{y\} + int(R^k_+)$, where $int(\cdot)$ denotes the interior of a set. In other words, $y \in Z$ is weakly efficient if $(\{y\} + int(R^k_+)) \cap Z = \{\emptyset\}$.

Hence, to check graphically whether an outcome is weakly efficient or not it is enough to place cone R_+^k at the outcome whose weak efficiency has to be verified, say outcome y, and see if set $\{y\} + int(R_+^k)$ contains any outcome. If it does not, y is weakly efficient, otherwise it is not weakly efficient. In Figure 2.10 the bullet marked outcome is weakly efficient.

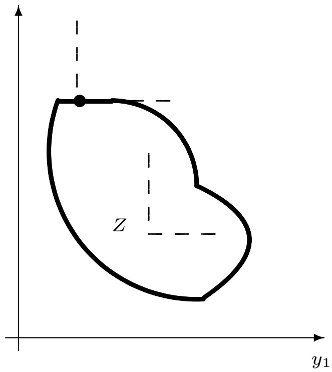

Figure 2.10. Checking weak efficiency with interior of cone R_+^k.

The definition of efficiency based on cone R_+^k provides for easy generalization of the notion of efficiency and weak efficiency. Let K be any convex cone. We say that outcome y is K-efficient if $(\{y\} + K) \cap Z = \{y\}$ and that it is K-weakly efficient if $(\{y\} + int(K)) \cap Z = \{\emptyset\}$.

With the generalized definition of efficiency we can also provide a more illustrative, but equivalent, definition of proper efficiency than that given in Definition 1 of this chapter. Namely, an outcome y is properly efficient if there exists a convex cone K such that $R_+^k \setminus \{0\} \subseteq int(K)$ and y is efficient with respect to K, i.e. $(\{y\} + K) \cap Z = \{y\}$. In Figure 2.11 the bullet marked outcome is properly efficient.

♠

4. Trade-offs

Following Webster's Ninth New Collegiate Dictionary (1987) "trade-off" is understood as: 1. a balancing of factors all of which are not

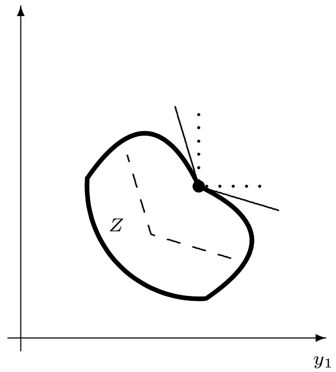

Figure 2.11. Checking proper efficiency with a convex cone containing R^k_+.

attainable at the same time, 2. a giving up of one thing in return for another.

In MCDM "trade-off" means: a loosing in one outcome component (criterion) to gain the value of another.

Given a pair of outcomes, a trade-off exists only if no outcome dominates the other (cf. Figure 2.12, right drawing). If otherwise, e.g. y' dominates y, then y' with respect to y offers only gains, and y with respect to y' offers only losses.

Moreover, it is practical to consider trade-offs only between pairs of efficient outcomes. Indeed, if an outcome y is nonefficient, then there exists an outcome, say y', which dominates y. By replacing y by y' we do not deteriorate the value of any component (criterion) value (Figure 2.12, left drawing). Hence, y' offers better trades than y. An efficient outcome offers better trades than any outcome dominated by it (cf. Figure 2.12, right drawing).

A trade-off always refers to one selected reference outcome, say $\bar{y}$. But depending on the context, this notion can refer to two different constructs.

The first construct is the so called *point-to-point* trade-off. It is defined for a pair of selected components of a pair of designated efficient outcomes.

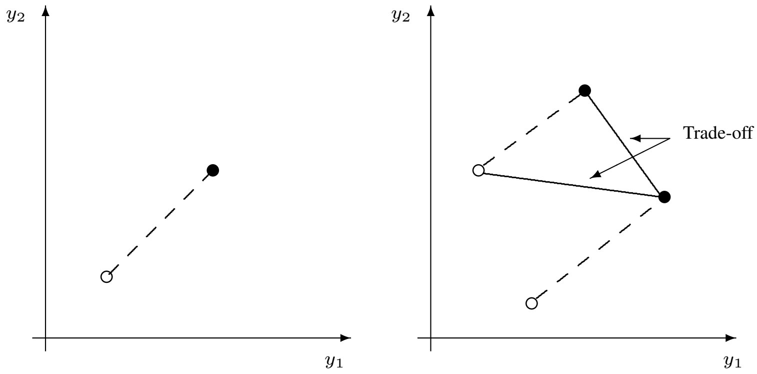

Figure 2.12. Pairs of elements for which trade-offs do not exist (dashed lines) or exist (continuous lines).

DEFINITION 2 Point-to-point trade-off $T_{i,j}^{PTP}(\bar{y}, \tilde{y})$, where $\bar{y}$ is a reference efficient outcome and $\tilde{y}$ is an efficient outcome, involving components i and j, $i, j = 1, ..., k, i \neq j$, such that

$$\tilde{y}_i - \bar{y}_i \geq 0 \text{ and } \bar{y}_j - \tilde{y}_j > 0, \tag{2.5}$$

is defined as

$$T_{i,j}^{PTP}(\bar{y}, \tilde{y}) = \frac{\tilde{y}_i - \bar{y}_i}{\bar{y}_j - \tilde{y}_j}\,. \tag{2.6}$$

□

For pairs of components i and j, $i, j = 1, ..., k$, $i \neq j$, such that the conditions (2.5) do not hold, point-to-point trade-offs are not defined.

For an efficient reference outcome $\bar{y}$ at most $k - 1$ point-to-point trade-offs exist. Indeed, since the point-to-point trade off is defined for a pair of efficient outcomes, say $\bar{y}$ and $\tilde{y}$, for at least one component, say component l, the relation $\bar{y}_l - \tilde{y}_l > 0$ must hold for otherwise $\bar{y}$ would be nonefficient. Hence, at most $k - 1$ components fulfill the relation $\tilde{y}_\cdot - \bar{y}_\cdot \geq 0$ and therefore at most $k - 1$ point-to-point trade-offs exist for an efficient reference outcome $\bar{y}$.

Suppose that one unit of component i has the same value for the DM as one unit of component j. If $T_{i,j}^{PTP}(\bar{y}, \tilde{y}) > 1$, then with respect to components i and j reference outcome $\bar{y}$ is more preferred than $\tilde{y}$. And

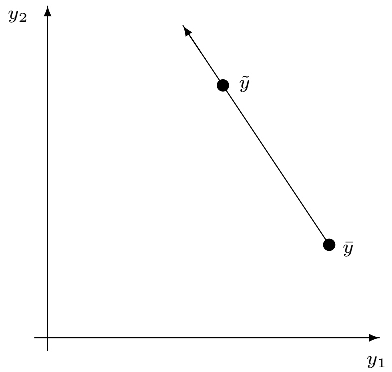

Figure 2.13. The double nature of a trade-off: value and direction.

symmetrically, $T_{i,j}^{PTP}(\bar{y}, \tilde{y}) < 1$ indicates that with respect to i and j outcome $\tilde{y}$ is more preferred than $\bar{y}$. However, this simple "which is more preferred" rule works only for a selected pair of components or in the case $k = 2$. In Chapter 6 we shall address the problem of how to make use of trade-off information when more than two components are taken into account. In any case large (note that the meaning of "large" is context-dependent) values of a point-to-point trade-offs $T_{\cdot,\cdot}^{PTP}(\bar{y}, \tilde{y})$ may be an important information for the DM for his outcome evaluations.

Point-to-point trade-offs can be associated with a direction, i.e. point-to-point trade-offs $T_{i,j}^{PTP}(\bar{y}, \tilde{y})$, $i, j = 1, ..., k$, $i \neq j$, can be read from any point of the ray leaving $\bar{y}$ and passing through $\tilde{y}$, as shown in Figure 2.13.

If outcomes are explicitly given, then point-to-point trade-offs are of little use. For a pair of efficient outcomes point-to-point trade-offs provide no additional information to that represented by components of outcomes (values of criteria). In fact they provide even less information, because not for all pairs of indices point-to-point trade-offs are defined. In that case trade-offs are nothing else than just another form of presenting information on outcome components of a pair of outcomes.

The notion of point-to-point trade-off becomes useful when sets of outcomes are infinite. Given efficient reference outcome $\bar{y}$, $\bar{y} \in Z$, and a pair of indices i and j, $i \neq j$, one can pose two questions:

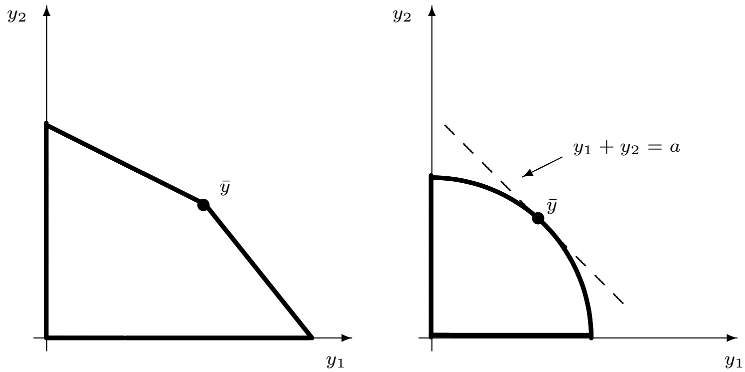

Figure 2.14. The maximum of point-to-point trade-offs $T_{\cdot,\cdot}^{PTP}(\bar{y}, y)$ may exist (left drawing) or may not exist (right drawing).

- what is the maximal value of $T_{i,j}^{PTP}(\bar{y}, y)$ over all $y \in Z$, i.e. what is the value of

$$\max_{y \in Z} T_{i,j}^{PTP}(\bar{y}, y),$$

- at which outcome (if at any) this maximum is attained.

If set Z is finite or polyhedral (Figure 2.14, the left drawing), it is guaranteed that $\max_{y \in Z} T_{i,j}^{PTP}(\bar{y}, y)$ always exists. In case Z is polyhedral, all efficient outcomes lying on a line segment starting from $\bar{y}$ have the same point-to-point trade-off values.

It is easy to see that if Z is neither finite nor polyhedral, then $\max_{y \in Z} T_{i,j}^{PTP}(\bar{y}, y)$ may not exist. In Figure 2.14, the right drawing, for efficient reference outcome $\bar{y}$ there is no efficient outcome y such that $\max_{y \in Z} T_{i,j}^{PTP}(\bar{y}, y) = 1$, but clearly $1 - T_{1,2}^{PTP}(\bar{y}, y) < \epsilon$ for some $y \in Z$, where $\epsilon > 0$ is arbitrary small.

A remedy to deal with situations like this above is offered if we replace the maximum with the supremum, where the latter means taking the least upper bound. In the example represented in the right drawing of Figure 2.14, $\sup_{y \in Z} T_{i,j}^{PTP}(\bar{y}, y) = 1$. In general taking just supremum is not very helpful, as illustrated in Figure 2.15 - for efficient reference outcome $\bar{y}$ for no pair of indices $i, j,\ i \neq j$, $\sup_{y \in Z} T_{i,j}^{PTP}(\bar{y}, y)$ is finite. To see this it is enough to project Z on any plane spanned by two out of three axes. For example, in the projection of Z on the plane spanned by axes y_1 and y_2, for any given number it is always possible to select a pair of values (y_1, y_2) such that $y_2 - \bar{y}_2 > 0,\ \bar{y}_1 - y_1 > 0$, and $\frac{y_2 - \bar{y}_2}{\bar{y}_1 - y_1}$ is greater than this number.

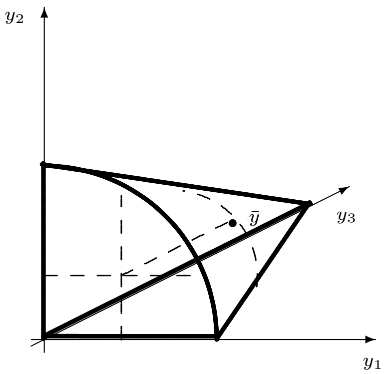

Figure 2.15. For outcome $\bar{y}$ and for any pair of indices the supremum of point-to-point trade-offs does not exist.

The problem is, at least partially, solved if we take the supremum not over set Z but over a part of it.

Let $\bar{y} \in Z,\ Z \subseteq \mathcal{R}^k$. For $i = 1, ..., k$, we denote:

$$Z_i^{<}(\bar{y}) = \{y \in Z \mid y_i < \bar{y}_i\,,\ y_l \geq \bar{y}_l\,,\ l = 1, ..., k,\ l \neq i\}.$$

DEFINITION 3 Global trade–off $T_{i,j}^G(\bar{y})$, where $\bar{y}$ is an efficient reference outcome, involving components i and $j,\ i, j = 1, ..., k,\ i \neq j$, is defined as

$$\sup_{y \in Z_j^{<}(y)} \frac{y_i - \bar{y}_i}{\bar{y}_j - y_j}\,. \tag{2.7}$$

□

By convention, if $Z_j^{<}(\bar{y}) = \emptyset$, (i.e. trade-off $T_{i,j}^G$ is not defined) then $T_{i,j}^G(\bar{y}) = -\infty,\ i = 1, ..., k,\ i \neq j.$

♣ The notion of global trade-off applies to both properly and improperly efficient outcomes.

It can be proved that for properly efficient outcomes all global trade-offs are smaller then $+\infty$. Let us recall the special cases of finite or polyhedral sets, where all outcomes are properly efficient.

For improperly efficient outcomes at least one trade-off equals $+\infty$. In Section 3 we observed, however, that in real decision making situations improperly efficient solutions occur infrequently.

In contrast to point-to-point trade-offs, which are defined with respect to two specified outcomes, deriving a global trade-off for an outcome requires calculations that relate to (parts of) outcome set Z.

♠

5. Concluding Remarks

With the material of this chapter we are well equipped to deal with issues of the next seven chapters. In fact, the notions defined up to now form a firm base for the subsequent presentations and developments and, except for some technical constructs, we shall not introduce any new concepts.

The notion of efficiency is quite intuitive. The notion of trade-off is less so (especially the notion of global trade-off) and the majority of works on MCDM exploit only the first notion without reference to the second. The rationale of using the notion of trade-off (point-to-point or global) in MCDM will be discussed in more detail in Chapter 4 and Chapter 6.

6. Annotated References

The definition of efficiency appears in almost every paper on MCDM. Some authors even decline to define this notion in research papers assuming it belongs to a common wisdom. Issues relating to all three related notions of efficiency (weak efficiency, efficiency, proper efficiency) are treated in every MCDM book. One can mention here Haimes et al. (1975), Ignizio (1976), Keeney, Raiffa (1976), Cohon (1978), Hwang et al. (1979), Rietveld (1980), Zeleny (1982), Chankong, Haimes (1983), Guddat et al. (1985), Ignizio (1985), Sawaragi et al. (1985), Yu (1985), Jahn (1986), Steuer (1986), Galas et al. (1987), Tabucanon (1988), Lewandowski, Wierzbicki (1989), Luc (1989), Haimes et al. (1990), Ringuest (1992), Vincke (1992), Kaliszewski (1994), Yoon, Hwang (1995), Skulimowski (1996), Miettinen (1999), and also (a survey paper) Stadler (1979).

The definition of proper efficiency in Definition 1 of this chapter comes from Geoffrion (1968).

The notion of trade-off has been used in the framework of vector optimization (a field with focus on problem (2.2) approached by rigid mathematical analysis) since the early 1950s and then it has penetrated MCDM. It refers to two different but related issues in MCDM. The first issue is that of properties of explicit or implicit value functions (Kuhn, Tucker 1951, Chankong, Haimes 1978, Haimes, Chankong 1979, Sakawa, Yano 1990). Under the assumption of differentiability of an utility function, trade-offs are interpreted as partial derivatives of this function with respect to criteria functions.

The second issue is benefits and costs of moving from one alternative to another measured by values of relative changes in criteria functions (Zionts, Wallenius 1976,1983, Wierzbicki 1990, Halme 1992, Henig, Buchanan 1997, Kaliszewski 1993, 1994b, Kaliszewski, Michalowski 1995, 1997,1999).

The definition of global trade-off is given first in Wierzbicki (1990), and exploited later in Kaliszewski (1993, 1994b). A method of calculating trade-offs which avoids calculating the supremum of a hyperbolic function is proposed in Kaliszewski (1993,1994b). This method is recalled briefly in Chapter 6 and later used in solving an example of a decision making problem.

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