Preface

In the Leibniz–Newton calculus, one learns the differentiation and integration of deterministic functions. A basic theorem in differentiation is the chain rule, which gives the derivative of a composite of two differentiable functions. The chain rule, when written in an indefinite integral form, yields the method of substitution. In advanced calculus, the Riemann–Stieltjes integral is defined through the same procedure of "partition-evaluation-summation-limit" as in the Riemann integral.

In dealing with random functions such as functions of a Brownian motion, the chain rule for the Leibniz–Newton calculus breaks down. A Brownian motion moves so rapidly and irregularly that almost all of its sample paths are nowhere differentiable. Thus we cannot differentiate functions of a Brownian motion in the same way as in the Leibniz–Newton calculus.

In 1944 Kiyosi Itô published the celebrated paper "Stochastic Integral" in the Proceedings of the Imperial Academy (Tokyo). It was the beginning of the Itô calculus, the counterpart of the Leibniz–Newton calculus for random functions. In this six-page paper, Itô introduced the stochastic integral and a formula, known since then as Itô's formula.

The Itô formula is the chain rule for the Itô calculus. But it cannot be expressed as in the Leibniz–Newton calculus in terms of derivatives, since a Brownian motion path is nowhere differentiable. The Itô formula can be interpreted only in the integral form. Moreover, there is an additional term in the formula, called the Itô correction term, resulting from the nonzero quadratic variation of a Brownian motion.

Before Itô introduced the stochastic integral in 1944, informal integrals involving white noise (the nonexistent derivative of a Brownian motion) had already been used by applied scientists. It was an innovative idea of Itô to consider the product of white noise and the time differential as a Brownian motion differential, a quantity that can serve as an integrator. The method Itô used to define a stochastic integral is a combination of the techniques in the Riemann–Stieltjes integral (referring to the integrator) and the Lebesgue integral (referring to the integrand).

The Itô calculus was originally motivated by the construction of Markov diffusion processes from infinitesimal generators. The previous construction of such processes had to go through three steps via the Hille–Yosida theory, the Riesz representation theorem, and the Kolmogorov extension theorem. However, Itô constructed these diffusion processes directly in a single step as the solutions of stochastic integral equations associated with the infinitesimal generators. Moreover, the properties of these diffusion processes can be derived from the stochastic integral equations and the Itô formula.

During the last six decades the Itô theory of stochastic integration has been extensively studied and applied in a wide range of scientific fields. Perhaps the most notable application is to the Black–Scholes theory in finance, for which Robert C. Merton and Myron S. Scholes won the 1997 Nobel Prize in Economics. Since the Itô theory is the essential tool for the Black–Scholes theory, many people feel that Itô should have shared the Nobel Prize with Merton and Scholes.

The Itô calculus has a large spectrum of applications in virtually every scientific area involving random functions. But it seems to be a very difficult subject for people without much mathematical background. I have written this introductory book on stochastic integration for anyone who needs or wants to learn the Itô calculus in a short period of time. I assume that the reader has the background of advanced calculus and elementary probability theory. Basic knowledge of measure theory and Hilbert spaces will be helpful. On the other hand, I have written several sections (for example, §2.4 on conditional expectation and §3.2 on the Borel–Cantelli lemma and Chebyshev inequality) to provide background for the sections that follow. I hope the reader will find them helpful. In addition, I have also provided many exercises at the end of each chapter for the reader to further understand the material.

This book is based on the lecture notes of a course I taught at Cheng Kung University in 1998 arranged by Y. J. Lee under an NSC Chair Professorship. I have revised and implemented this set of lecture notes through the courses I have taught at Meijo University arranged by K. Saitô, University of Rome "Tor Vergata" arranged by L. Accardi under a Fulbright Lecturing grant, and Louisiana State University over the past years. The preparation of this book has also benefited greatly from my visits to Hiroshima University, Academic Frontier in Science of Meijo University, University of Madeira, Vito Volterra Center at the University of Rome "Tor Vergata," and the University of Tunis El Manar since 1999.

I am very grateful for financial support to the above-mentioned universities and the following offices: the National Science Council (Taiwan), the Ministry of Education and Science (Japan), the Luso-American Foundation (Portugal), and the Italian Fulbright Commission (Italy). I would like to give my best thanks to Dr. R. W. Pettit, Senior Program Officer of the CIES Fulbright Scholar Program, and Ms. L. Miele, Executive Director of the Italian Fulbright Commission, and the personnel in her office for giving me assistance for my visit to the University of Rome "Tor Vergata."

Many people have helped me to read the manuscript for corrections and improvements. I am especially thankful for comments and suggestions from the following students and colleagues: W. Ayed, J. J. Becnel, J. Esunge, Y. Hara-Mimachi, M. Hitsuda, T. R. Johansen, S. K. Lee, C. Macaro, V. Nigro, H. Ouerdiane, K. Saitô, A. N. Sengupta, H. H. Shih, A. Stan, P. Sundar, H. F. Yang, T. H. Yang, and H. Yin. I would like to give my best thanks to my colleague C. N. Delzell, an amazing TEXpert, for helping me to resolve many tedious and difficult TEXnical problems. I am in debt to M. Regoli for drawing the flow chart to outline the chapters on the next page. I thank W. Ayed for her suggestion to include this flow chart. I am grateful to M. Spencer of Springer for his assistance in bringing out this book.

I would like to give my deepest appreciation to L. Accardi, L. Gross, T. Hida, I. Kubo, T. F. Lin, and L. Streit for their encouragement during the preparation of the manuscript. Especially, my Ph. D. advisor, Professor Gross, has been giving me continuous support and encouragement since the first day I met him at Cornell in 1966. I owe him a great deal in my career.

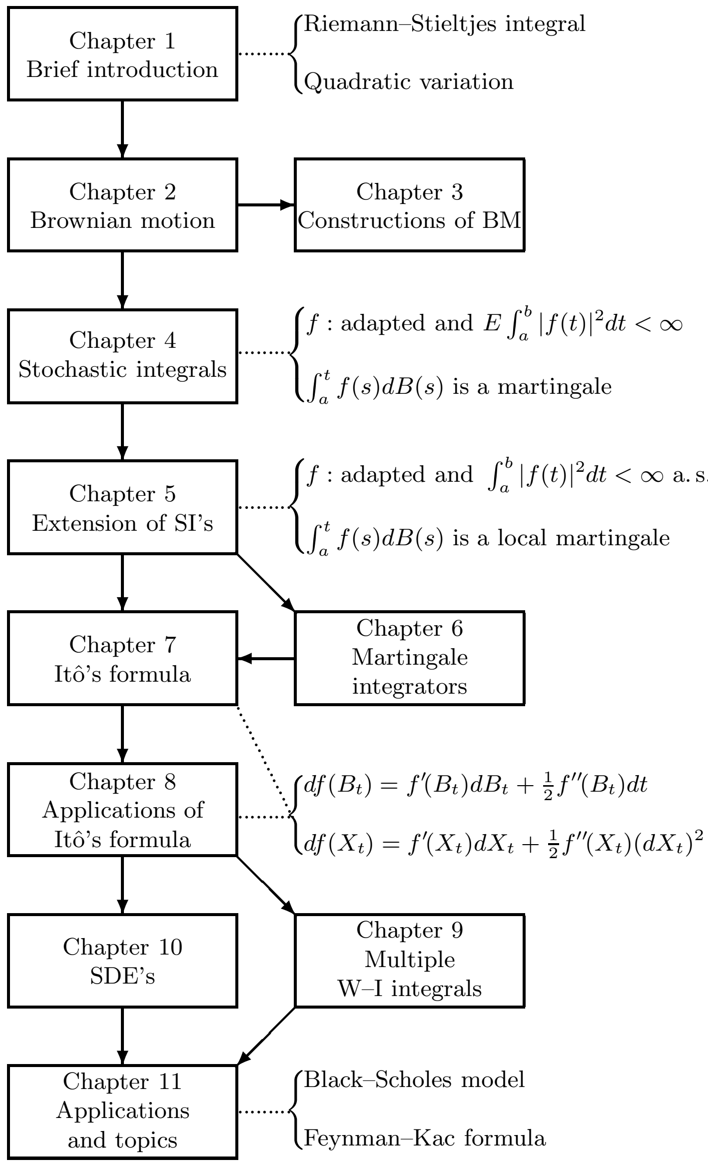
The writing style of this book is very much influenced by Professor K. Itô. I have learned from him that an important mathematical concept always starts with a simple example, followed by the abstract formulation as a definition, then properties as theorems with elaborated examples, and finally extension and concrete applications. He has given me countless lectures in his houses in Ithaca and Kyoto while his wife prepared the most delicious dinners for us. One time, while we were enjoying extremely tasty shrimp-asparagus rolls, he said to me with a proud smile "If one day I am out of a job, my wife can open a restaurant and sell only one item, the shrimp-asparagus rolls." Even today, whenever I am hungry, I think of the shrimp-asparagus rolls invented by Mrs. K. Itô. Another time about 1:30 a.m. in 1991, Professor Itô was still giving me a lecture. His wife came upstairs to urge him to sleep and then said to me, "Kuo san (Japanese for Mr.), don't listen to him." Around 1976, Professor Itô was ranked number 2 table tennis player among the Japanese probabilists. He was so strong that I just could not get any point in a game with him. His wife then said to me, "Kuo san, I will get some points for you." When she succeeded occasionally to win a point, she would joyfully shake hands with me, and Professor Itô would smile very happily.

When I visited Professor Itô in January 2005, my heart was very much touched by the great interest he showed in this book. He read the table of contents and many pages together with his daughter Keiko Kojima and me. It was like the old days when Professor Itô gave me lectures, while I was also thinking about the shrimp-asparagus rolls.

Finally, I must thank my wife, Fukuko, for her patience and understanding through the long hours while I was writing this book.

Hui-Hsiung Kuo
Baton Rouge
September 2005

Outline


Chapter 1
Brief introduction
Riemann–Stieltjes integral
Quadratic variation
Chapter 2
Brownian motion
Chapter 3
Constructions of BM
Chapter 4
Stochastic integrals
f : adapted and $E\int_a^b |f(t)|^2 dt < \infty$
$\int_a^t f(s)dB(s)$ is a martingale
Chapter 5
Extension of SI's
f : adapted and $\int_a^b |f(t)|^2 dt < \infty$ a. s.
$\int_a^t f(s)dB(s)$ is a local martingale
Chapter 7
Itô's formula
Chapter 6
Martingale
integrators
Chapter 8
Applications of
Itô's formula
$df(B_t) = f'(B_t)dB_t + \frac{1}{2}f''(B_t)dt$
$df(X_t) = f'(X_t)dX_t + \frac{1}{2}f''(X_t)(dX_t)^2$
Chapter 10
SDE's
Chapter 9
Multiple
W–I integrals
Chapter 11
Applications
and topics
Black–Scholes model
Feynman–Kac formula

http://www.springer.com/978-0-387-28720-1

Introduction to Stochastic Integration
Kuo, H.-H.
2006, XIII, 279 p. 2 illus., Softcover
ISBN: 978-0-387-28720-1