# 2

# Euclid's approach to geometry

PREVIEW

Length is the fundamental concept of Euclid's geometry, but several important theorems seem to be "really" about angle or area—for example, the theorem on the sum of angles in a triangle and the Pythagorean theorem on the sum of squares. Also, Euclid often uses area to prove theorems about length, such as the Thales theorem.

In this chapter, we retrace some of Euclid's steps in the theory of angle and area to show how they lead to the Pythagorean theorem and the Thales theorem. We begin with his theory of angle, which shows most clearly the influence of his *parallel axiom*, the defining axiom of what is now called *Euclidean geometry*.

Angle is linked with length from the beginning by the so-called SAS ("side angle side") criterion for equal triangles (or "congruent triangles," as we now call them). We observe the implications of SAS for isosceles triangles and the properties of angles in a circle, and we note the related criterion, ASA ("angle side angle").

The theory of area depends on ASA, and it leads directly to a proof of the Pythagorean theorem. It leads more subtly to the Thales theorem and its consequences that we saw in Chapter 1. The theory of angle then combines nicely with the Thales theorem to give a second proof of the Pythagorean theorem.

In following these deductive threads, we learn more about the scope of straightedge and compass constructions, partly in the exercises. Interesting spinoffs from these investigations include a process for cutting any polygon into pieces that form a square, a construction for the square root of any length, and a construction of the regular pentagon.

## 2.1 The parallel axiom

In Chapter 1, we saw how useful it is to have *rectangles*: four-sided polygons whose angles are all right angles. Rectangles owe their existence to *parallel lines*—lines that do not meet—and fundamentally to the *parallel axiom* that Euclid stated as follows.

**Euclid's parallel axiom.** *If a straight line crossing two straight lines makes the interior angles on one side together less than two right angles, then the two straight lines will meet on that side.*

Figure 2.1 illustrates the situation described by Euclid's parallel axiom, which is what happens when the two lines are *not* parallel. If $\alpha+\beta$ is less than two right angles, then $\mathscr{L}$ and $\mathscr{M}$ meet somewhere on the right.

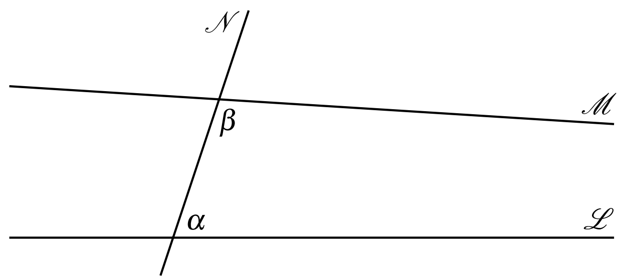


Figure 2.1: When lines are not parallel

It follows that if $\mathscr{L}$ and $\mathscr{M}$ do not meet on either side, then $\alpha+\beta=\pi$. In other words, if $\mathscr{L}$ and $\mathscr{M}$ *are* parallel, then $\alpha$ and $\beta$ together make a straight angle and the angles made by $\mathscr{L}$, $\mathscr{M}$, and $\mathscr{N}$ are as shown in Figure 2.2.

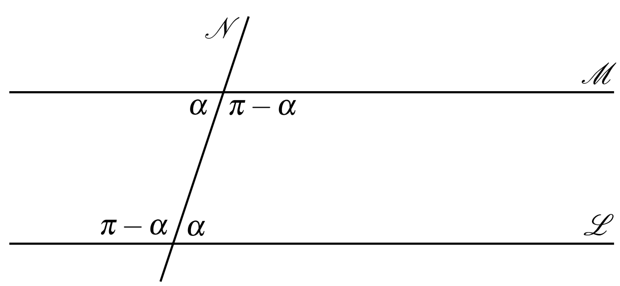


Figure 2.2: When lines are parallel

It also follows that any line through the intersection of $\mathscr{N}$ and $\mathscr{M}$, not meeting $\mathscr{L}$, makes the angle $\pi - \alpha$ with $\mathscr{N}$. Hence, this line equals $\mathscr{M}$. That is, *if a parallel to $\mathscr{L}$ through a given point exists, it is unique*.

It is a little more subtle to show the existence of a parallel to $\mathscr{L}$ through a given point $P$, but one way is to appeal to a principle called ASA ("angle side angle"), which will be discussed in Section 2.2.

Suppose that the lines $\mathscr{L}$, $\mathscr{M}$, and $\mathscr{N}$ make angles as shown in Figure 2.2, and that $\mathscr{L}$ and $\mathscr{M}$ are *not* parallel. Then, on at least one side of $\mathscr{N}$, there is a triangle whose sides are the segment of $\mathscr{N}$ between $\mathscr{L}$ and $\mathscr{M}$ and the segments of $\mathscr{L}$ and $\mathscr{M}$ between $\mathscr{N}$ and the point where they meet. According to ASA, this triangle is completely determined by the angles $\alpha$, $\pi - \alpha$ and the segment of $\mathscr{N}$ between them. But then an identical triangle is determined on the other side of $\mathscr{N}$, and hence $\mathscr{L}$ and $\mathscr{M}$ also meet on the other side. This result contradicts Euclid's assumption (implicit in the construction axioms discussed in Section 1.1) that *there is a unique line through any two points*. Hence, the lines $\mathscr{L}$ and $\mathscr{M}$ are in fact parallel when the angles are as shown in Figure 2.2.

Thus, both the existence and the uniqueness of parallels follow from Euclid's parallel axiom (existence "follows trivially," because Euclid's parallel axiom is not required). It turns out that they also imply it, so the parallel axiom can be stated equivalently as follows.

**Modern parallel axiom.** *For any line $\mathscr{L}$ and point $P$ outside $\mathscr{L}$, there is exactly one line through $P$ that does not meet $\mathscr{L}$.*

This form of the parallel axiom is often called "Playfair's axiom," after the Scottish mathematician John Playfair who used it in a textbook in 1795. Playfair's axiom is simpler in form than Euclid's, because it does not involve angles, and this is often convenient. However, we often need parallel lines *and* the equal angles they create, the so-called *alternate interior angles* (for example, the angles marked $\alpha$ in Figure 2.2). In such situations, we prefer to use Euclid's parallel axiom.

### Angles in a triangle

The existence of parallels and the equality of alternate interior angles imply a beautiful property of triangles.

**Angle sum of a triangle.** *If $\alpha$, $\beta$, and $\gamma$ are the angles of any triangle, then $\alpha + \beta + \gamma = \pi$.*

To prove this property, draw a line $\mathscr{L}$ through one vertex of the triangle, parallel to the opposite side, as shown in Figure 2.3.

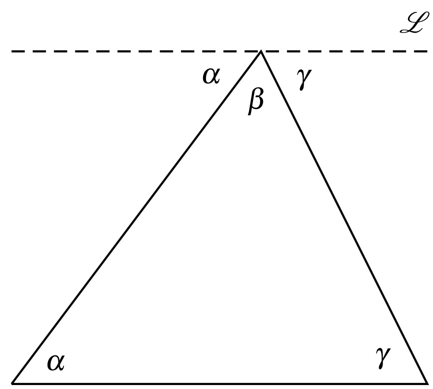


Figure 2.3: The angle sum of a triangle

Then the angle on the left beneath $\mathscr{L}$ is alternate to the angle $\alpha$ in the triangle, so it is equal to $\alpha$. Similarly, the angle on the right beneath $\mathscr{L}$ is equal to $\gamma$. But then the straight angle $\pi$ beneath $\mathscr{L}$ equals $\alpha+\beta+\gamma$, the angle sum of the triangle. □

## Exercises

The triangle is the most important polygon, because any polygon can be built from triangles. For example, the angle sum of any quadrilateral (polygon with four sides) can be worked out by cutting the quadrilateral into two triangles.

**2.1.1** Show that the angle sum of any quadrilateral is $2\pi$.

A polygon $\mathscr{P}$ is called *convex* if the line segment between any two points in $\mathscr{P}$ lies entirely in $\mathscr{P}$. For these polygons, it is also easy to find the angle sum.

**2.1.2** Explain why a convex $n$-gon can be cut into $n-2$ triangles.

**2.1.3** Use the dissection of the $n$-gon into triangles to show that the angle sum of a convex $n$-gon is $(n-2)\pi$.

**2.1.4** Use Exercise 2.1.3 to find the angle at each vertex of a *regular* $n$-gon (an $n$-gon with equal sides and equal angles).

**2.1.5** Deduce from Exercise 2.1.4 that copies of a regular $n$-gon can tile the plane only for $n=3,4,6$.

## 2.2 Congruence axioms

Euclid says that two geometric figures *coincide* when one of them can be moved to fit exactly on the other. He uses the idea of moving one figure to coincide with another in the proof of Proposition 4 of Book I: *If two triangles have two corresponding sides equal, and the angles between these sides equal, then their third sides and the corresponding two angles are also equal.*

His proof consists of moving one triangle so that the equal angles of the two triangles coincide, and the equal sides as well. But then the third sides necessarily coincide, because their endpoints do, and hence, so do the other two angles.

Today we say that two triangles are *congruent* when their corresponding angles and side lengths are equal, and we no longer attempt to prove the proposition above. Instead, we *take it as an axiom* (that is, an unproved assumption), because it seems simpler to assume it than to introduce the concept of motion into geometry. The axiom is often called SAS (for "side angle side").

**SAS axiom.** *If triangles $ABC$ and $A'B'C'$ are such that*

$$|AB| = |A'B'|, \quad \text{angle } ABC = \text{angle } A'B'C', \quad |BC| = |B'C'|$$

*then also*

$$|AC| = |A'C'|, \quad \text{angle } BCA = \text{angle } B'C'A', \quad \text{angle } CAB = \text{angle } C'A'B'.$$

For brevity, one often expresses SAS by saying that two triangles are congruent if two sides and the included angle are equal. There are similar conditions, ASA and SSS, which also imply congruence (but SSA does not—can you see why?). They can be deduced from SAS, so it is not necessary to take them as axioms. However, we will assume ASA here to save time, because it seems just as natural as SAS.

One of the most important consequences of SAS is Euclid's Proposition 5 of Book I. It says that a triangle with two equal sides has two equal angles. Such a triangle is called *isosceles*, from the Greek for "equal sides." The spectacular proof below is not from Euclid, but from the Greek mathematician Pappus, who lived around 300 CE.

**Isosceles triangle theorem.** *If a triangle has two equal sides, then the angles opposite to these sides are also equal.*

Suppose that triangle $ABC$ has $|AB| = |AC|$. Then triangles $ABC$ and $ACB$, which of course are the *same* triangle, are congruent by SAS (Figure 2.4). Their left sides are equal, their right sides are equal, and so are the angles between their left and right sides, because they are the *same* angle (the angle at $A$).

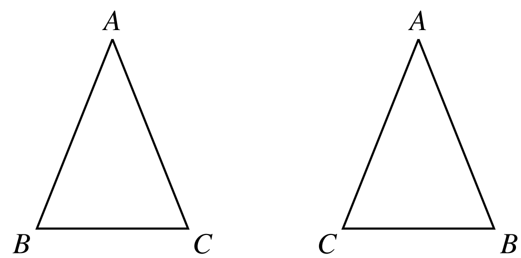


Figure 2.4: Two views of an isosceles triangle

But then it follows from SAS that all corresponding angles of these triangles are equal: for example, the bottom left angles. In other words, the angle at $B$ equals the angle at $C$, so the angles opposite to the equal sides are equal. □

A useful consequence of ASA is the following theorem about parallelograms, which enables us to determine the area of triangles. (Remember, a parallelogram is defined as a figure bounded by two pairs of parallel lines—the definition does not say anything about the lengths of its sides.)

**Parallelogram side theorem.** *Opposite sides of a parallelogram are equal.*

To prove this theorem we divide the parallelogram into triangles by a diagonal (Figure 2.5), and try to prove that these triangles are congruent. They are, because

- they have the common side $AC$,
- their corresponding angles $\alpha$ are equal, being alternate interior angles for the parallels $AD$ and $BC$,
- their corresponding angles $\beta$ are equal, being alternate interior angles for the parallels $AB$ and $DC$.

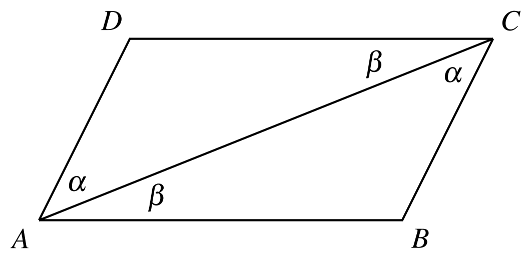


Figure 2.5: Dividing a parallelogram into triangles

Therefore, the triangles are congruent by ASA, and in particular we have the equalities $|AB| = |DC|$ and $|AD| = |BC|$ between corresponding sides. But these are also the opposite sides of the parallelogram. □

### Exercises

**2.2.1** Using the parallelogram side theorem and ASA, find congruent triangles in Figure 2.6. Hence, show that the diagonals of a parallelogram bisect each other.

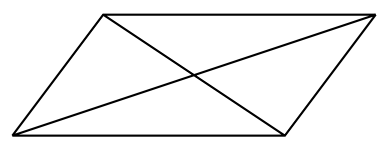

Figure 2.6: A parallelogram and its diagonals

**2.2.2** Deduce that the diagonals of a rhombus—a parallelogram whose sides are all equal—meet at right angles. (*Hint*: You may find it convenient to use SSS, which says that triangles are congruent when their corresponding sides are equal.)

**2.2.3** Prove the isosceles triangle theorem differently by bisecting the angle at $A$.

## 2.3 Area and equality

The principle of logic used in Section 1.2—that things equal to the same thing are equal to each other—is one of five principles that Euclid calls *common notions*. The common notions he states are particularly important for his theory of area, and they are as follows:

1. Things equal to the same thing are also equal to one another.
2. If equals are added to equals, the wholes are equal.
3. If equals are subtracted from equals, the remainders are equal.
4. Things that coincide with one another are equal to one another.
5. The whole is greater than the part.

The word "equal" here means "equal in some specific respect." In most cases, it means "equal in length" or "equal in area," although Euclid's idea of "equal in area" is not exactly the same as ours, as I will explain below. Likewise, "addition" can mean addition of lengths or addition of areas, but Euclid never adds a length to an area because this has no meaning in his system.

A simple but important example that illustrates the use of "equals" is Euclid's Proposition 15 of Book I: *Vertically opposite angles are equal*. Vertically opposite angles are the angles $\alpha$ shown in Figure 2.7.

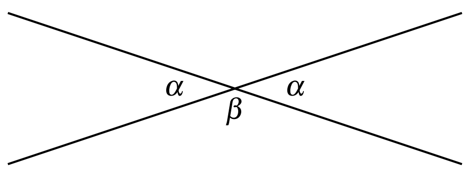


Figure 2.7: Vertically opposite angles

They are equal because each of them equals a straight angle minus $\beta$.

## The square of a sum

Proposition 4 of Book II is another interesting example. It states a property of squares and rectangles that we express by the algebraic formula

$$(a+b)^2 = a^2 + 2ab + b^2.$$

Euclid does *not* have algebraic notation, so he has to state this equation in words: *If a line is cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments.* Whichever way you say it, Figure 2.8 explains why it is true.

The line is $a+b$ because it is cut into the two segments $a$ and $b$, and hence

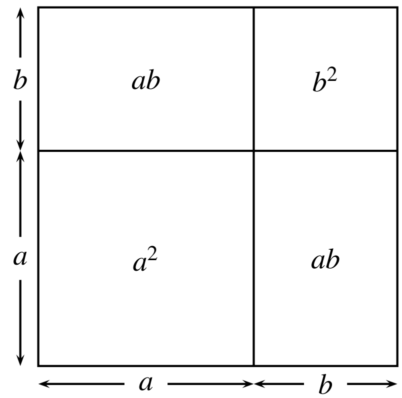


Figure 2.8: The square of a sum of line segments

- The square on the line is what we write as $(a+b)^2$.
- The squares on the two segments $a$ and $b$ are $a^2$ and $b^2$, respectively.
- The rectangle "contained" by the segments $a$ and $b$ is $ab$.
- The square $(a+b)^2$ equals (in area) the sum of $a^2$, $b^2$, and two copies of $ab$.

It should be emphasized that, in Greek mathematics, the *only* interpretation of $ab$, the "product" of line segments $a$ and $b$, is the rectangle with perpendicular sides $a$ and $b$ (or "contained in" $a$ and $b$, as Euclid used to say). This rectangle could be shown "equal" to certain other regions, but only by cutting the regions into identical pieces by straight lines. The Greeks did not realize that this "equality of regions" was the same as equality of numbers—the numbers we call the *areas* of the regions—partly because they did not regard irrational lengths as numbers, and partly because they did not think the product of lengths should be a length.

As mentioned in Section 1.5, this belief was not necessarily an obstacle to the development of geometry. To find the area of nonrectangular regions, such as triangles or parallelograms, one has to think about cutting regions into pieces in any case. For such simple regions, there is no particular advantage in thinking of the area as a number, as we will see in Section 2.4. But first we need to investigate the concept mentioned in Euclid's Common Notion number 4. What does it mean for one figure to "coincide" with another?

### Exercises

In Figure 2.8, the large square is subdivided by two lines: one of them perpendicular to the bottom side of the square and the other perpendicular to the left side of the square.

**2.3.1** Use the parallel axiom to explain why all other angles in the figure are necessarily right angles.

Figure 2.8 presents the algebraic identity $(a+b)^2 = a^2 + 2ab + b^2$ in geometric form. Other well-known algebraic identities can also be given a geometric presentation.

**2.3.2** Give a diagram for the identity $a(b+c) = ab+ac$.

**2.3.3** Give a diagram for the identity $a^2 - b^2 = (a+b)(a-b)$.

Euclid does not give a geometric theorem that explains the identity $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. But it is not hard to do so by interpreting $(a+b)^3$ as a cube with edge length $a+b$, $a^3$ as a cube with edge $a$, $a^2b$ as a box with perpendicular edges $a$, $a$, and $b$, and so on.

**2.3.4** Draw a picture of a cube with edges $a+b$, and show it cut by planes (parallel to its faces) that divide each edge into a segment of length $a$ and a segment of length $b$.

**2.3.5** Explain why these planes cut the original cube into eight pieces:

- a cube with edges $a$,
- a cube with edges $b$,
- three boxes with edges $a, a, b$,
- three boxes with edges $a, b, b$.

## 2.4 Area of parallelograms and triangles

The first nonrectangular region that can be shown "equal" to a rectangle in Euclid's sense is a parallelogram. Figure 2.9 shows how to use straight lines to cut a parallelogram into pieces that can be reassembled to form a rectangle.

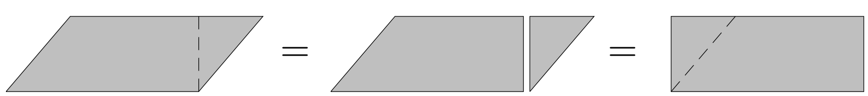

Figure 2.9: Assembling parallelogram and rectangle from the same pieces

Only one cut is needed in the example of Figure 2.9, but more cuts are needed if the parallelogram is more sheared, as in Figure 2.10.

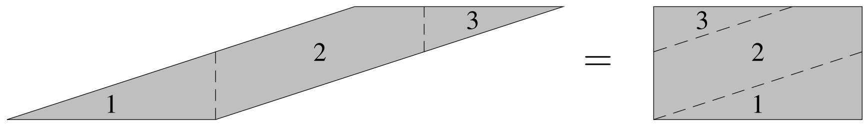


Figure 2.10: A case in which more cuts are required

In Figure 2.10 we need two cuts, which produce the pieces labeled 1, 2, 3. The number of cuts can become arbitrarily large as the parallelogram is sheared further. We can avoid large numbers of cuts by allowing *subtraction* of pieces as well as addition. Figure 2.11 shows how to convert any rectangle to any parallelogram with the same *base* $OR$ and the same *height* $OP$. We need only add a triangle, and then subtract an equal triangle.

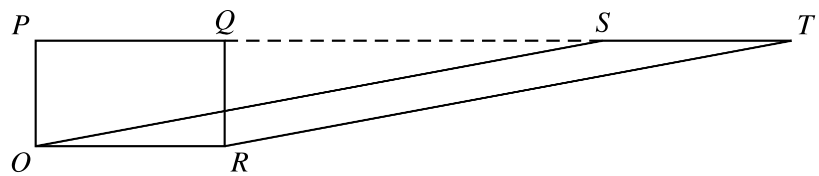


Figure 2.11: Rectangle and parallelogram with the same base and height

To be precise, if we start with rectangle $OPQR$ and add triangle $RQT$, then subtract triangle $OPS$ (which equals triangle $RQT$ by the parallelogram side theorem of Section 2.2), the result is parallelogram $OSTR$. Thus, the parallelogram is equal (in area) to a rectangle with the same base and height. We write this fact as

$$\text{area of parallelogram} = \text{base} \times \text{height}.$$

To find the area of a triangle $ABC$, we notice that it can be viewed as "half" of a parallelogram by adding to it the congruent triangle $ACD$ as shown in Figure 2.5, and again in Figure 2.12.

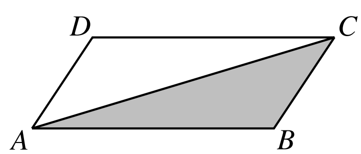


Figure 2.12: A triangle as half a parallelogram

Clearly,

area of triangle $ABC +$ area of triangle $ACD =$ area of parallelogram $ABCD$,

and the two triangles "coincide" (because they are congruent) and so they have equal area by Euclid's Common Notion 4. Thus,

$$\text{area of triangle} = \frac{1}{2}\,\text{base} \times \text{height}.$$

This formula is important in two ways:

- *As a statement about area.* From a modern viewpoint, the formula gives the area of the triangle as a product of numbers. From the ancient viewpoint, it gives a rectangle "equal" to the triangle, namely, the rectangle with the same base and half the height of the triangle.

- *As a statement about proportionality.* For triangles with the same height, the formula shows that their areas are proportional to their bases. This statement turns out to be crucial for the proof of the Thales theorem (Section 2.6).

The proportionality statement follows from the assumption that each line segment has a real number length, which depends on the acceptance of irrational numbers. As mentioned in the previous section, the Greeks did not accept this assumption. Euclid got the proportionality statement by a lengthy and subtle "theory of proportion" in Book V of the *Elements*.

## Exercises

To back up the claim that the formula $\frac{1}{2}$ base $\times$ height gives a way to find the area of the triangle, we should explain how to find the height.

**2.4.1** Given a triangle with a particular side specified as the "base," show how to find the height by straightedge and compass construction.

The equality of triangles $OPS$ and $RQT$ follows from the parallelogram side theorem, as claimed above, but a careful proof would explain what other axioms are involved.

**2.4.2** By what Common Notion does $|PQ| = |ST|$?

**2.4.3** By what Common Notion does $|PS| = |QT|$?

**2.4.4** By what congruence axiom is triangle $OPS$ congruent to triangle $RQT$?

## 2.5 The Pythagorean theorem

The Pythagorean theorem is about areas, and indeed Euclid proves it immediately after he has developed the theory of area for parallelograms and triangles in Book I of the *Elements*. First let us recall the statement of the theorem.

**Pythagorean theorem.** *For any right-angled triangle, the sum of the squares on the two shorter sides equals the square on the hypotenuse.*

We follow Euclid's proof, in which he divides the square on the hypotenuse into the two rectangles shown in Figure 2.13. He then shows that the light gray square equals the light gray rectangle and that the dark gray square equals the dark gray rectangle, so the sum of the light and dark squares is the square on the hypotenuse, as required.

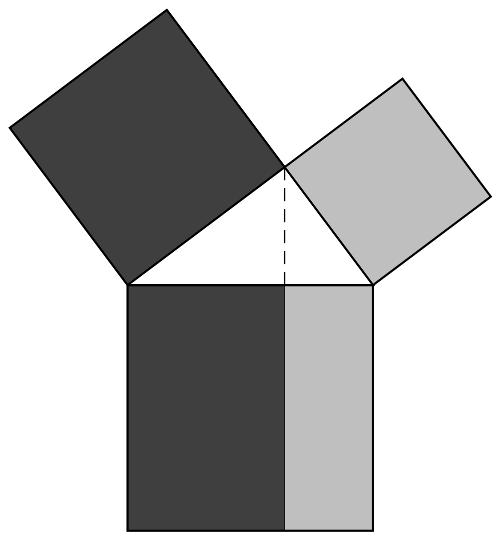

Figure 2.13: Dividing the square for Euclid's proof

First we show equality for the light gray regions in Figure 2.13, and in fact we show that *half* of the light gray square equals half of the light gray rectangle. We start with a light gray triangle that is obviously half of the light gray square, and we successively replace it with triangles of the same base or height, ending with a triangle that is obviously half of the light gray rectangle (Figure 2.14).

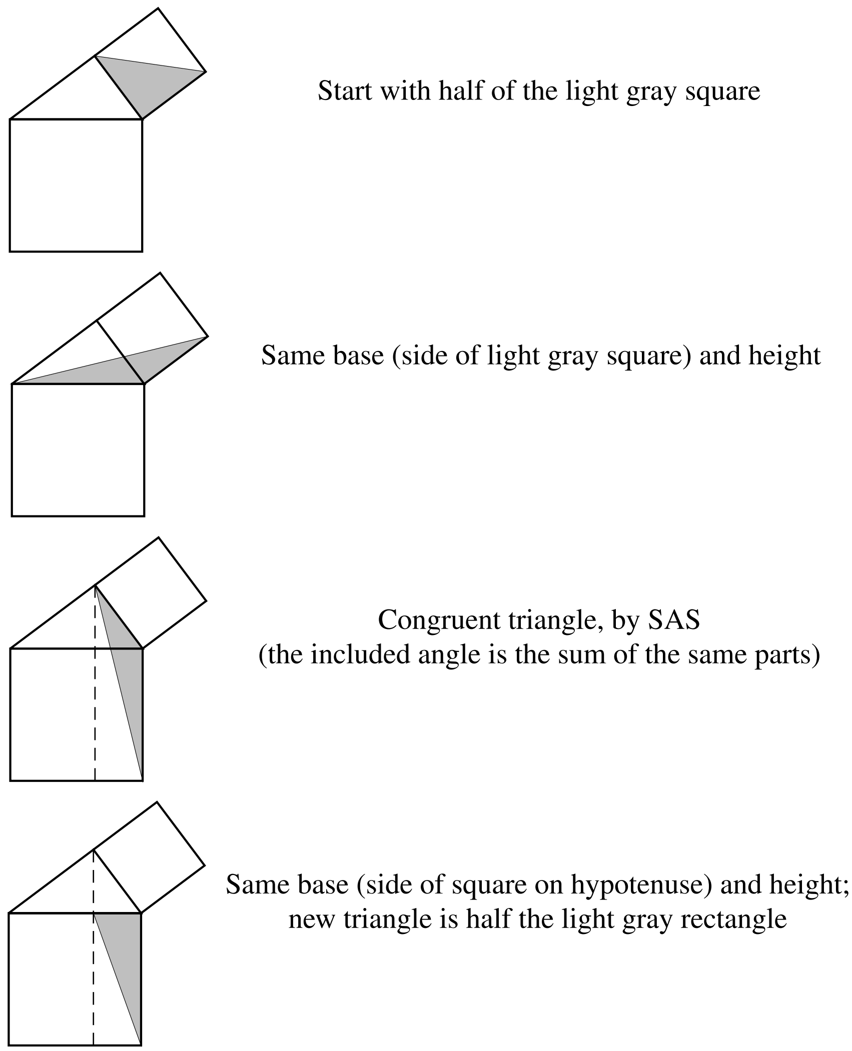


Figure 2.14: Changing the triangle without changing its area

The same argument applies to the dark gray regions, and thus, the Pythagorean theorem is proved. $\square$

Figure 2.13 suggests a natural way to construct a square equal in area to a given rectangle. Given the light gray rectangle, say, the problem is to reconstruct the rest of Figure 2.13.

We can certainly extend a given rectangle to a square and hence reconstruct the square on the hypotenuse. The main problem is to reconstruct the right-angled triangle, from the hypotenuse, so that the other vertex lies on the dashed line. See whether you can think of a way to do this; a really elegant solution is given in Section 2.7. Once we have the right-angled triangle, we can certainly construct the squares on its other two sides—in particular, the gray square equal in area to the gray rectangle.

### Exercises

It follows from the Pythagorean theorem that a right-angled triangle with sides 3 and 4 has hypotenuse $\sqrt{3^2+4^2} = \sqrt{25} = 5$. But there is *only one* triangle with sides 3, 4, and 5 (by the SSS criterion mentioned in Exercise 2.2.2), so putting together lengths 3, 4, and 5 always makes a right-angled triangle. This triangle is known as the $(3,4,5)$ triangle.

**2.5.1** Verify that the $(5,12,13)$, $(8,15,17)$, and $(7,24,25)$ triangles are right-angled.

**2.5.2** Prove the converse Pythagorean theorem: If $a,b,c > 0$ and $a^2+b^2 = c^2$, then the triangle with sides $a,b,c$ is right-angled.

**2.5.3** How can we be sure that lengths $a,b,c > 0$ with $a^2+b^2 = c^2$ actually fit together to make a triangle? (*Hint:* Show that $a+b > c$.)

Right-angled triangles can be used to construct certain irrational lengths. For example, we saw in Section 1.5 that the right-angled triangle with sides 1, 1 has hypotenuse $\sqrt{2}$.

**2.5.4** Starting from the triangle with sides 1, 1, and $\sqrt{2}$, find a straightedge and compass construction of $\sqrt{3}$.

**2.5.5** Hence, obtain constructions of $\sqrt{n}$ for $n = 2,3,4,5,6,\ldots$.

## 2.6 Proof of the Thales theorem

We mentioned this theorem in Chapter 1 as a fact with many interesting consequences, such as the proportionality of similar triangles. We are now in a position to prove the theorem as Euclid did in his Proposition 2 of Book VI. Here again is a statement of the theorem.

**The Thales theorem.** *A line drawn parallel to one side of a triangle cuts the other two sides proportionally.*

The proof begins by considering triangle $ABC$, with its sides $AB$ and $AC$ cut by the parallel $PQ$ to side $BC$ (Figure 2.15). Because $PQ$ is parallel to $BC$, the triangles $PQB$ and $PQC$ on base $PQ$ have the same height, namely the distance between the parallels. They therefore have the same area.

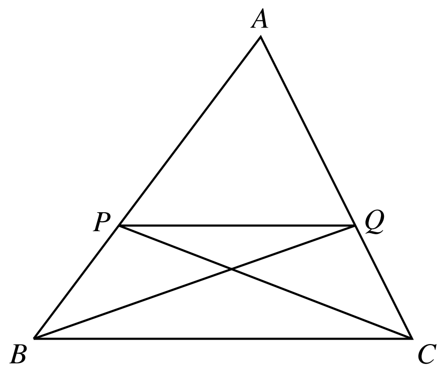


Figure 2.15: Triangle sides cut by a parallel

If we add triangle $APQ$ to each of the equal-area triangles $PQB$ and $PQC$, we get the triangles $AQB$ and $APC$, respectively. Hence, the latter triangles are also equal in area.

Now consider the two triangles—$APQ$ and $PQB$—that make up triangle $AQB$ as triangles with bases on the line $AB$. They have the same height relative to this base (namely, the perpendicular distance of $Q$ from $AB$). Hence, their bases are in the ratio of their areas:

$$\frac{|AP|}{|PB|} = \frac{\text{area } APQ}{\text{area } PQB}.$$

Similarly, considering the triangles $APQ$ and $PQC$ that make up the triangle $APC$, we find that

$$\frac{|AQ|}{|QC|} = \frac{\text{area } APQ}{\text{area } PQC}.$$

Because area $PQB$ equals area $PQC$, the right sides of these two equations are equal, and so are their left sides. That is,

$$\frac{|AP|}{|PB|} = \frac{|AQ|}{|QC|}.$$

In other words, the line $PQ$ cuts the sides $AB$ and $AC$ proportionally. □

### Exercises

As seen in Exercise 1.3.6, $|AP|/|PB| = |AQ|/|QC|$ is equivalent to $|AP|/|AB| = |AQ|/|AC|$. This equation is a more convenient formulation of the Thales theorem if you want to prove the following generalization:

**2.6.1** Suppose that there are several parallels $P_1Q_1, P_2Q_2, P_3Q_3\dots$ to the side $BC$ of triangle $ABC$. Show that

$$\frac{|AP_1|}{|AQ_1|} = \frac{|AP_2|}{|AQ_2|} = \frac{|AP_3|}{|AQ_3|} = \cdots = \frac{|AB|}{|AC|}.$$

We can also drop the assumption that the parallels $P_1Q_1, P_2Q_2, P_3Q_3\dots$ fall across a triangle $ABC$.

**2.6.2** If parallels $P_1Q_1, P_2Q_2, P_3Q_3\dots$ fall across a pair of *parallel* lines $\mathscr{L}$ and $\mathscr{M}$, what can we say about the lengths they cut from $\mathscr{L}$ and $\mathscr{M}$?

## 2.7 Angles in a circle

The isosceles triangle theorem of Section 2.2, simple though it is, has a remarkable consequence.

**Invariance of angles in a circle.** *If A and B are two points on a circle, then, for all points C on one of the arcs connecting them, the angle ACB is constant.*

To prove invariance we draw lines from $A, B, C$ to the center of the circle, $O$, along with the lines making the angle $ACB$ (Figure 2.16).

Because all radii of the circle are equal, $|OA| = |OC|$. Thus triangle $AOC$ is isosceles, and the angles $\alpha$ in it are equal by the isosceles triangle theorem. The angles $\beta$ in triangle $BOC$ are equal for the same reason.

Because the angle sum of any triangle is $\pi$ (Section 2.1), it follows that the angle at $O$ in triangle $AOC$ is $\pi - 2\alpha$ and the angle at $O$ in triangle $BOC$ is $\pi - 2\beta$. It follows that the third angle at $O$, angle $AOB$, is $2(\alpha+\beta)$, because the total angle around any point is $2\pi$. But angle $AOB$ is *constant*, so $\alpha+\beta$ is also constant, and $\alpha+\beta$ is precisely the angle at $C$. □

An important special case of this theorem is when $A$, $O$, and $B$ lie in a straight line, so $2(\alpha+\beta) = \pi$. In this case, $\alpha+\beta = \pi/2$, and thus we have the following theorem (which is also attributed to Thales).

**Angle in a semicircle theorem.** *If A and B are the ends of a diameter of a circle, and C is any other point on the circle, then angle ACB is a right angle.* □

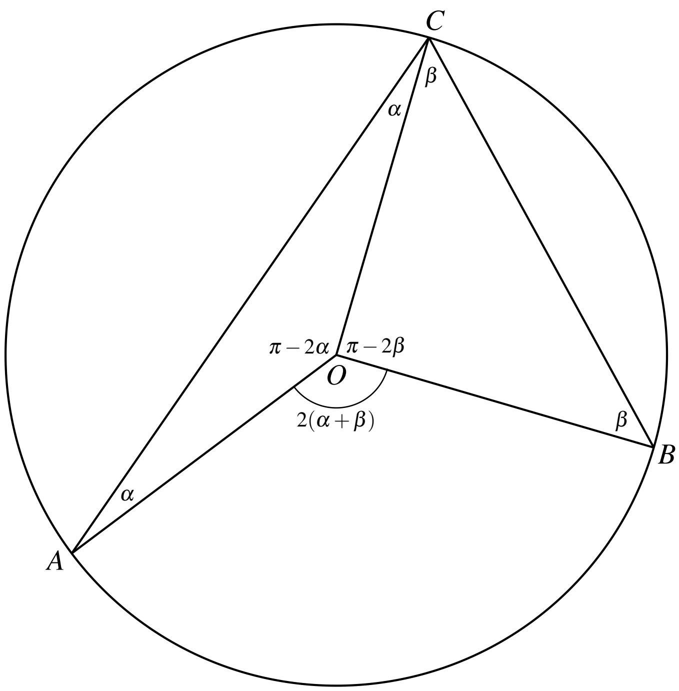


Figure 2.16: Angle $\alpha + \beta$ in a circle

This theorem enables us to solve the problem left open at the end of Section 2.5: Given a hypotenuse $AB$, how do we construct the right-angled triangle whose other vertex $C$ lies on a given line? Figure 2.17 shows how.

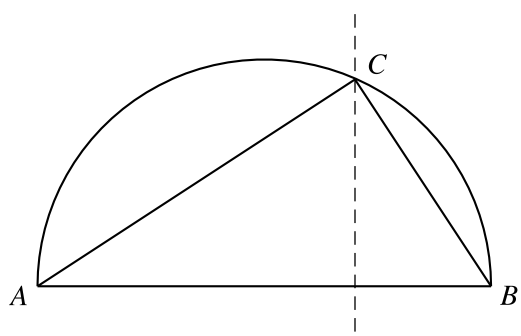


Figure 2.17: Constructing a right-angled triangle with given hypotenuse

The trick is to draw the semicircle on diameter $AB$, which can be done by first bisecting $AB$ to obtain the center of the circle. Then the point where the semicircle meets the given line (shown dashed) is necessarily the other vertex $C$, because the angle at $C$ is a right angle.

This construction completes the solution of the problem raised at the end of Section 2.5: finding a square equal in area to a given rectangle. In Section 2.8 we will show that Figure 2.17 also enables us to construct the *square root* of an arbitrary length, and it gives a new proof of the Pythagorean theorem.

### Exercises

**2.7.1** Explain how the angle in a semicircle theorem enables us to construct a right-angled triangle with a given hypotenuse $AB$.

**2.7.2** Then, by looking at Figure 2.13 from the bottom up, find a way to construct a square equal in area to a given rectangle.

**2.7.3** Given any two squares, we can construct a square that equals (in area) the sum of the two given squares. Why?

**2.7.4** Deduce from the previous exercises that any polygon may be "squared"; that is, there is a straightedge and compass construction of a square equal in area to the given polygon. (You may assume that the given polygon can be cut into triangles.)

The possibility of "squaring" any polygon was apparently known to Greek mathematicians, and this may be what tempted them to try "squaring the circle": constructing a square equal in area to a given circle. There is no straightedge and compass solution of the latter problem, but this was not known until 1882.

Coming back to angles in the circle, here is another theorem about invariance of angles:

**2.7.5** If a quadrilateral has its vertices on a circle, show that its opposite angles sum to $\pi$.

## 2.8 The Pythagorean theorem revisited

In Book VI, Proposition 31 of the *Elements*, Euclid proves a generalization of the Pythagorean theorem. From it, we get a new proof of the ordinary Pythagorean theorem, based on the proportionality of similar triangles.

Given a right-angled triangle with sides $a$, $b$, and hypotenuse $c$, we divide it into two smaller right-angled triangles by the perpendicular to the hypotenuse through the opposite vertex (the dashed line in Figure 2.18).

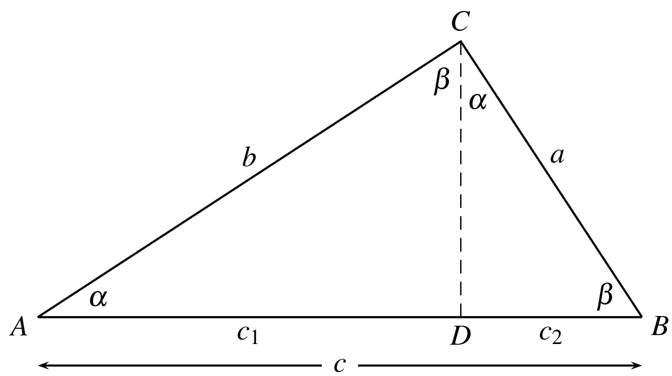


Figure 2.18: Subdividing a right-angled triangle into similar triangles

All three triangles are similar because they have the same angles $\alpha$ and $\beta$. If we look first at the angle $\alpha$ at $A$ and the angle $\beta$ at $B$, then

$$\alpha + \beta = \frac{\pi}{2}$$

because the angle sum of triangle $ABC$ is $\pi$ and the angle at $C$ is $\pi/2$. But then it follows that angle $ACD = \beta$ in triangle $ACD$ (to make its angle sum $= \pi$) and angle $DCB = \alpha$ in triangle $DCB$ (to make its angle sum $= \pi$).

Now we use the proportionality of these triangles, calling the side opposite $\alpha$ in each triangle "short" and the side opposite $\beta$ "long" for convenience. Comparing triangle $ABC$ with triangle $ADC$, we get

$$\frac{\text{long side}}{\text{hypotenuse}} = \frac{b}{c} = \frac{c_1}{b}, \quad \text{hence} \quad b^2 = cc_1.$$

Comparing triangle $ABC$ with triangle $DCB$, we get

$$\frac{\text{short side}}{\text{hypotenuse}} = \frac{a}{c} = \frac{c_2}{a}, \quad \text{hence} \quad a^2 = cc_2.$$

Adding the values of $a^2$ and $b^2$ just obtained, we finally get

$$a^2 + b^2 = cc_2 + cc_1 = c(c_1 + c_2) = c^2 \quad \text{because } c_1 + c_2 = c,$$

and this is the Pythagorean theorem. □

This second proof is not really shorter than Euclid's first (given in Section 2.5) when one takes into account the work needed to prove the proportionality of similar triangles. However, we often need similar triangles, so they are a standard tool, and a proof that uses standard tools is generally preferable to one that uses special machinery. Moreover, the splitting of a right-angled triangle into similar triangles is itself a useful tool—it enables us to construct the square root of any line segment.

### Straightedge and compass construction of square roots

Given any line segment $l$, construct the semicircle with diameter $l+1$, and the perpendicular to the diameter where the segments 1 and $l$ meet (Figure 2.19). Then *the length $h$ of this perpendicular is $\sqrt{l}$.*

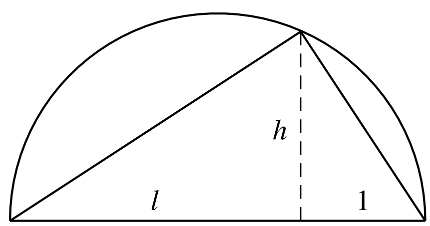


Figure 2.19: Construction of the square root

To see why, construct the right-angled triangle with hypotenuse $l+1$ and third vertex where the perpendicular meets the semicircle. We know that the perpendicular splits this triangle into two similar, and hence proportional, triangles. In the triangle on the left,

$$\frac{\text{long side}}{\text{short side}} = \frac{l}{h}.$$

In the triangle on the right,

$$\frac{\text{long side}}{\text{short side}} = \frac{h}{1}.$$

Because these ratios are equal by proportionality of the triangles, we have

$$\frac{l}{h} = \frac{h}{1},$$

hence $h^2 = l$; that is, $h = \sqrt{l}$. □

This result complements the constructions for the rational operations $+, -, \times$, and $\div$ we gave in Chapter 1. The constructibility of these and $\sqrt{}$ was first pointed out by Descartes in his book *Géométrie* of 1637. Rational operations and $\sqrt{}$ are in fact *precisely* what can be done with straightedge and compass. When we introduce coordinates in Chapter 3 we will see that any "constructible point" has coordinates obtainable from the unit length 1 by $+, -, \times, \div$, and $\sqrt{}$.

## Exercises

Now that we know how to construct the $+, -, \times, \div$, and $\sqrt{}$ of given lengths, we can use algebra as a shortcut to decide whether certain figures are constructible by straightedge and compass. If we know that a certain figure is constructible from the length $(1+\sqrt{5})/2$, for example, then we know that the figure is constructible—period—because the length $(1+\sqrt{5})/2$ is built from the unit length by the operations $+, \times, \div$, and $\sqrt{}$.

This is precisely the case for the regular pentagon, which was constructed by Euclid in Book IV, Proposition 11, using virtually all of the geometry he had developed up to that point. We also need nearly everything we have developed up to this point, but it fills less space than four books of the *Elements*!

The following exercises refer to the regular pentagon of side 1 shown in Figure 2.20 and its diagonals of length $x$.

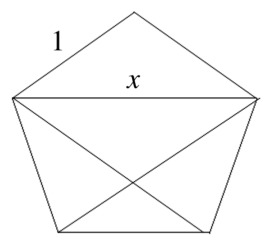


Figure 2.20: The regular pentagon

**2.8.1** Use the symmetry of the regular pentagon to find similar triangles implying

$$\frac{x}{1} = \frac{1}{x-1},$$

that is, $x^2 - x - 1 = 0$.

**2.8.2** By finding the positive root of this quadratic equation, show that each diagonal has length $x = (1+\sqrt{5})/2$.

**2.8.3** Now show that the regular pentagon is constructible.

## 2.9 Discussion

Euclid found the most important axiom of geometry—the parallel axiom—and he also identified the basic theorems and traced the logical connections between them. However, his approach misses certain fine points and is not logically complete. For example, in his very first proof (the construction of the equilateral triangle), he assumes that certain circles have a point in common, but none of his axioms guarantee the existence of such a point. There are many such situations, in which Euclid assumes something is true because it *looks* true in the diagram.

Euclid's theory of area is a whole section of his geometry that seems to have no geometric support. Its concepts seem more like arithmetic—addition, subtraction, and proportion—but its concept of multiplication is not the usual one, because multiplication of more than three lengths is not allowed.

These gaps in Euclid's approach to geometry were first noticed in the 19th century, and the task of filling them was completed by David Hilbert in his *Grundlagen der Geometrie* (Foundations of Geometry) of 1899. On the one hand, Hilbert introduced axioms of *incidence* and *order*, giving the conditions under which lines (and circles) meet. These justify the belief that "geometric objects behave as the pictures suggest." On the other hand, Hilbert replaced Euclid's theory of area with a genuine arithmetic, which he called *segment arithmetic*. He defined the sum and product of segments as we did in Section 1.4 and proved that these operations on segments have the same properties as ordinary sum and product. For example,

$$a+b=b+a, \quad ab=ba, \quad a(b+c)=ab+ac, \quad \text{and so on.}$$

In the process, Hilbert discovered that the Pappus and Desargues theorems (Exercises 1.4.3 and 1.4.4) play a decisive role.

The downside of Hilbert's completion of Euclid is that it is lengthy and difficult. Nearly 20 axioms are required, and some key theorems are hard to prove. To some extent, this hardship occurs because Hilbert insists on geometric definitions of $+$ and $\times$. He wants numbers to come from "inside" geometry rather than from "outside". Thus, to prove that $ab=ba$ he needs the theorem of Pappus, and to prove that $a(bc)=(ab)c$ he needs the theorem of Desargues.

Even today, the construction of segment arithmetic is an admirable feat. As Hilbert pointed out, it shows that Euclid was right to believe that the theory of proportion could be developed without new geometric axioms. Still, it is somewhat quixotic to build numbers "inside" Euclid's geometry when they are brought from "outside" into nearly every other branch of geometry. It is generally easier to build geometry on numbers than the reverse, and Euclidean geometry is no exception, as I hope to show in Chapters 3 and 4.

This is one reason for bypassing Hilbert's approach, so I will merely list his axioms here. They are thoroughly investigated in Hartshorne's *Geometry: Euclid and Beyond* or Hilbert's own book, which is available in English translation. Hartshorne's book has the clearest available derivation of ordinary geometry and segment arithmetic from the Hilbert axioms, so it should be consulted by anyone who wants to see Euclid's approach taken to its logical conclusion.

There is another reason to bypass Hilbert's axioms, apart from their difficulty. In my opinion, Hilbert's greatest geometric achievement was to build arithmetic, not in *Euclidean* geometry, but in *projective* geometry. As just mentioned, Hilbert found that the keys to segment arithmetic are the Pappus and Desargues theorems. These two theorems do not involve the concept of length, and so they really belong to a more primitive kind of geometry. This primitive geometry (projective geometry) has only a handful of axioms—*fewer than the usual axioms for arithmetic*—so it is more interesting to build arithmetic inside it. It is also less trouble, because we do not have to *prove* the Pappus and Desargues theorems. We will explain how projective geometry contains arithmetic in Chapters 5 and 6.

### Hilbert's axioms

The axioms concern undefined objects called "points" and "lines," the related concepts of "line segment," "ray," and "angle," and the relations of "betweenness" and "congruence." Following Hartshorne, we simplify Hilbert's axioms slightly by stating some of them in a stronger form than necessary.

The first group of axioms is about *incidence*: conditions for points to lie on lines or for lines to pass through points.

I1. For any two points $A$, $B$, a unique line passes through $A$, $B$.

I2. Every line contains at least two points.

I3. There exist three points not all on the same line.

I4. For each line $\mathscr{L}$ and point $P$ not on $\mathscr{L}$ there is a unique line through $P$ not meeting $\mathscr{L}$ (parallel axiom).

The next group is about *betweenness* or *order*: a concept overlooked by Euclid, probably because it is too "obvious." The first to draw attention to betweenness was the German mathematician Moritz Pasch, in the 1880s. We write $A * B * C$ to denote that $B$ is between $A$ and $C$.

B1. If $A * B * C$, then $A, B, C$ are three points on a line and $C * B * A$.

B2. For any two points $A$ and $B$, there is a point $C$ with $A * B * C$.

B3. Of three points on a line, exactly one is between the other two.

B4. Suppose $A, B, C$ are three points not in a line and that $\mathscr{L}$ is a line not passing through any of $A, B, C$. If $\mathscr{L}$ contains a point $D$ between $A$ and $B$, then $\mathscr{L}$ contains either a point between $A$ and $C$ or a point between $B$ and $C$, but not both (Pasch's axiom).

The next group is about *congruence of line segments* and *congruence of angles*, both denoted by $\cong$. Thus, $AB \cong CD$ means that $AB$ and $CD$ have equal length and $\angle ABC \cong \angle DEF$ means that $\angle ABC$ and $\angle DEF$ are equal angles. Notice that C2 and C5 contain versions of Euclid's Common Notion 1: "Things equal to the same thing are equal to each other."

C1. For any line segment $AB$, and any ray $\mathscr{R}$ originating at a point $C$, there is a unique point $D$ on $\mathscr{R}$ with $AB \cong CD$.

C2. If $AB \cong CD$ and $AB \cong EF$, then $CD \cong EF$. For any $AB$, $AB \cong AB$.

C3. Suppose $A * B * C$ and $D * E * F$. If $AB \cong DE$ and $BC \cong EF$, then $AC \cong DF$. (Addition of lengths is well-defined.)

C4. For any angle $\angle BAC$, and any ray $\overrightarrow{DF}$, there is a unique ray $\overrightarrow{DE}$ on a given side of $\overrightarrow{DF}$ with $\angle BAC \cong \angle EDF$.

C5. For any angles $\alpha, \beta, \gamma$, if $\alpha \cong \beta$ and $\alpha \cong \gamma$, then $\beta \cong \gamma$. Also, $\alpha \cong \alpha$.

C6. Suppose that $ABC$ and $DEF$ are triangles with $AB \cong DE$, $AC \cong DF$, and $\angle BAC \cong \angle EDF$. Then, the two triangles are congruent, namely $BC \cong EF$, $\angle ABC \cong \angle DEF$, and $\angle ACB \cong \angle DFE$. (This is SAS.)

Then there is an axiom about the intersection of circles. It involves the concept of points *inside* the circle, which are those points whose distance from the center is less than the radius.

E. Two circles meet if one of them contains points both inside and outside the other.

Next there is the so-called *Archimedean axiom*, which says that no length can be "infinitely large" relative to another.

A. For any line segments $AB$ and $CD$, there is a natural number $n$ such that $n$ copies of $AB$ are together greater than $CD$.

Finally, there is the so-called *Dedekind axiom*, which says that the line is *complete*, or has *no gaps*. It implies that its points correspond to real numbers. Hilbert wanted an axiom like this to force the plane of Euclidean geometry to be the same as the plane $\mathbb{R}^2$ of pairs of real numbers.

D. Suppose the points of a line $\mathscr{L}$ are divided into two nonempty subsets $\mathscr{A}$ and $\mathscr{B}$ in such a way that no point of $\mathscr{A}$ is between two points of $\mathscr{B}$ and no point of $\mathscr{B}$ is between two points of $\mathscr{A}$. Then, a unique point $P$, either in $\mathscr{A}$ or $\mathscr{B}$, lies between any other two points, of which one is in $\mathscr{A}$ and the other is in $\mathscr{B}$.

Axiom D is not needed to derive any of Euclid's theorems. They do not involve all real numbers but only the so-called *constructible* numbers originating from straightedge and compass constructions. However, who can be sure that we will never need nonconstructible points? One of the most important numbers in geometry, $\pi$, is nonconstructible! (Because the circle cannot be squared.) Thus, it seems prudent to use Axiom D so that the line is complete from the beginning.

In Chapter 3, we will take the real numbers as the starting point of geometry, and see what advantages this may have over the Euclid–Hilbert approach. One clear advantage is *access to algebra*, which reduces many geometric problems to simple calculations. Algebra also offers some conceptual advantages, as we will see.

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