$$M = \begin{bmatrix} a_0 & a_1 & \cdots & \cdots & a_m & & & \\ & \ddots & \ddots & & & \ddots & & \\ & & a_0 & a_1 & \ldots & \ldots & a_m & \\ b_0 & b_1 & \cdots & b_n & & & & \\ & b_0 & b_1 & \cdots & b_n & & & \\ & & \ddots & \ddots & & \ddots & & \\ & & & b_0 & b_1 & \ldots & b_n & \end{bmatrix}$$

(n staggered rows of a_i's, m staggered rows of b_j's, all other entries 0). The resultant eliminates x, leaving a polynomial in the coefficients that vanishes if and only if f and g share a root. Let $p(x)$ and $q(x)$ be monic polynomials in $\mathbf{Q}[x]$. Consider the resultants

$$\tilde{q}(t,u) = R(p(s), u - s - t; s), \qquad r(u) = R(\tilde{q}(t,u), q(t); t).$$

Show that if α and β satisfy the polynomials p and q then $\alpha + \beta$ satisfies the polynomial r. Similarly, find polynomials satisfied by $\alpha\beta$ and $1/\alpha$ if $\alpha \neq 0$. (A hint for this exercise is at the end of the book.)

6.4.2. Use the methods of the section or of Exercise 6.4.1 to find a monic integer polynomial satisfied by $\sqrt{2} + \mu_3$.

6.4.3. Show that $\overline{\mathbf{Z}} \cap \mathbf{Q} = \mathbf{Z}$. (A hint for this exercise is at the end of the book.)

6.4.4. Show that every algebraic number takes the form of an algebraic integer divided by a rational integer. (A hint for this exercise is at the end of the book.)

6.4.5. Prove Theorem 6.4.5 and its corollaries.

6.5 Algebraic eigenvalues

Returning to the material of Section 6.3, recall the action of the weight-2 Hecke operators $T = T_p$ and $T = \langle d \rangle$ on the dual space as composition from the right,

$$T : \mathcal{S}_2(\Gamma_1(N))^\wedge \longrightarrow \mathcal{S}_2(\Gamma_1(N))^\wedge, \qquad \varphi \mapsto \varphi \circ T,$$

and recall that the action descends to the quotient $\mathrm{J}_1(N)$. Thus the operators act as endomorphisms on the kernel $\mathrm{H}_1(X_1(N), \mathbf{Z})$, a finitely generated Abelian group. In particular the characteristic polynomial $f(x)$ of T_p acting on $\mathrm{H}_1(X_1(N), \mathbf{Z})$ has integer coefficients, and being a characteristic polynomial it is monic. Since an operator satisfies its characteristic polynomial, $f(T_p) = 0$ on $\mathrm{H}_1(X_1(N), \mathbf{Z})$. Since T_p is $\mathbf{C}$-linear, also $f(T_p) = 0$ on $\mathcal{S}_2(\Gamma_1(N))^\wedge$ and so $f(T_p) = 0$ on $\mathcal{S}_2(\Gamma_1(N))$. Therefore the characteristic polynomial of T_p on $\mathcal{S}_2(\Gamma_1(N))$ divides $f(x)$ and the eigenvalues of T_p satisfy $f(x)$, making them algebraic integers. Since p is arbitrary this proves

Theorem 6.5.1. *Let $f \in \mathcal{S}_2(\Gamma_1(N))$ be a normalized eigenform for the Hecke operators T_p. Then the eigenvalues $a_n(f)$ are algebraic integers.*

To refine this result we need to view the Hecke operators as lying within an algebraic structure, not merely as a set.

Definition 6.5.2. *The* ***Hecke algebra over*** $\mathbf{Z}$ *is the algebra of endomorphisms of $\mathcal{S}_2(\Gamma_1(N))$ generated over $\mathbf{Z}$ by the Hecke operators,*

$$\mathbb{T}_{\mathbf{Z}} = \mathbf{Z}[\{T_n, \langle n \rangle : n \in \mathbf{Z}^+\}].$$

The ***Hecke algebra*** $\mathbb{T}_{\mathbf{C}}$ ***over*** $\mathbf{C}$ *is defined similarly.*

Each level has its own Hecke algebra, but N is omitted from the notation since it is usually written somewhere nearby. Clearly any $f \in \mathcal{S}_2(\Gamma_1(N))$ is an eigenform for all of $\mathbb{T}_{\mathbf{C}}$ if and only if f is an eigenform for all Hecke operators T_p and $\langle d \rangle$.

For the remainder of this chapter the methods will shift to working with algebraic structure rather than thinking about objects such as Hecke operators one at a time. In particular modules will figure prominently, and so in this context Abelian groups will often be called $\mathbf{Z}$-modules. For example, viewing the $\mathbf{Z}$-module $\mathbb{T}_{\mathbf{Z}}$ as a ring of endomorphisms of the finitely generated free $\mathbf{Z}$-module $\mathrm{H}_1(X_1(N), \mathbf{Z})$ shows that it is finitely generated as well (Exercise 6.5.1). Again letting $f(\tau) = \sum_{n=1}^{\infty} a_n(f) q^n$ be a normalized eigenform, the homomorphism

$$\lambda_f : \mathbb{T}_{\mathbf{Z}} \longrightarrow \mathbf{C}, \quad Tf = \lambda_f(T) f$$

therefore has as its image a finitely generated $\mathbf{Z}$-module. Since the image is $\mathbf{Z}[\{a_n(f) : n \in \mathbf{Z}^+\}]$ this shows that even though there are infinitely many eigenvalues $a_n(f)$, the ring they generate has finite rank as a $\mathbf{Z}$-module. More specifically, letting

$$I_f = \ker(\lambda_f) = \{T \in \mathbb{T}_{\mathbf{Z}} : Tf = 0\}$$

gives a ring and $\mathbf{Z}$-module isomorphism (Exercise 6.5.2)

$$\mathbb{T}_{\mathbf{Z}}/I_f \xrightarrow{\sim} \mathbf{Z}[\{a_n(f)\}]. \tag{6.12}$$

The image ring sits inside some finite-degree extension field of $\mathbf{Q}$, i.e., a number field. The rank of $\mathbb{T}_{\mathbf{Z}}/I_f$ is the degree of this number field as an extension of $\mathbf{Q}$.

Definition 6.5.3. *Let $f \in \mathcal{S}_2(\Gamma_1(N))$ be a normalized eigenform, $f(\tau) = \sum_{n=1}^{\infty} a_n q^n$. The field $\mathbf{K}_f = \mathbf{Q}(\{a_n\})$ generated by the Fourier coefficients of f is called* ***the number field of*** *f.*

The reader is referred to William Stein's web site [Ste] and John Cremona's tables [Cre97] for examples.

Any embedding $\sigma : \mathbf{K}_f \hookrightarrow \mathbf{C}$ conjugates f by acting on its coefficients. That is, if $f(\tau) = \sum_{n=1}^{\infty} a_n q^n$ then notating the action with a superscript,

$$f^\sigma(\tau) = \sum_{n=1}^{\infty} a_n^\sigma q^n.$$

In fact this action produces another eigenform.

Theorem 6.5.4. *Let f be a weight 2 normalized eigenform of the Hecke operators, so that $f \in \mathcal{S}_2(N, \chi)$ for some N and χ. Let $\mathbf{K}_f$ be its number field. For any embedding $\sigma : \mathbf{K}_f \hookrightarrow \mathbf{C}$ the conjugated f^σ is also a normalized eigenform in $\mathcal{S}_2(N, \chi^\sigma)$ where $\chi^\sigma(n) = \chi(n)^\sigma$. If f is a newform then so is f^σ.*

The proof will require two beginning results from commutative algebra, so these are stated first.

Proposition 6.5.5 (Nakayama's Lemma). *Suppose that A is a commutative ring with unit and $J \subset A$ is an ideal contained in every maximal ideal of A, and suppose that M is a finitely generated A-module such that $JM = M$. Then $M = \{0\}$.*

Proof. Suppose that $M \neq \{0\}$ and let $m_1, \ldots, m_n$ be a minimal set of generators for M over A. Since $JM = M$, in particular $m_n \in JM$, giving it the form $m_n = a_1 m_1 + \cdots + a_n m_n$ with all $a_i \in J$. Thus

$$(1 - a_n)m_n = a_1 m_1 + \cdots + a_{n-1} m_{n-1}.$$

But $1 - a_n$ is invertible in A, else it sits in a maximal ideal, which necessarily contains a_n as well and therefore is all of A, impossible. Thus $m_1, \ldots, m_{n-1}$ is a smaller generating set, and this contradiction proves the lemma. □

Again suppose that A is a commutative ring with unit and $J \subset A$ is an ideal, and suppose that M is an A-module and a finite-dimensional vector space over some field $\mathbf{k}$. The dual space $M^\wedge = \mathrm{Hom}_{\mathbf{k}}(M, \mathbf{k})$ is an A-module in the natural way, $a\varphi = \varphi \circ a$ for $a \in A$ and $\varphi \in M^\wedge$ (i.e., $(a\varphi)(m) = \varphi(am)$ for $m \in M$), and similarly for $(M/JM)^\wedge$. Let $M[J]$ be the elements of M annihilated by J, and similarly for $M^\wedge[J]$. Then there exist natural isomorphisms of A-modules (Exercise 6.5.3)

$$(M/JM)^\wedge \cong M^\wedge[J], \qquad M^\wedge/JM^\wedge \cong M[J]^\wedge. \tag{6.13}$$

Now we can prove Theorem 6.5.4.

Proof. The Fourier coefficients $\{a_n\}$ are a system of eigenvalues for the operators T_n acting on $\mathcal{S}_2(\Gamma_1(N))$. We need to show that the conjugated coefficients $\{a_n^\sigma\}$ are again a system of eigenvalues.

As explained at the beginning of the section, the action of $\mathbb{T}_{\mathbf{Z}}$ on $\mathcal{S}_2(\Gamma_1(N))$ transfers to the dual space $\mathcal{S}_2(\Gamma_1(N))^\wedge$ as composition on the right and then descends to the Jacobian $\mathrm{J}_1(N)$, inducing an action on the homology group $\mathrm{H}_1(X_1(N),\mathbf{Z})$. This $\mathbf{Z}$-module is free of rank $2g$ where g is the genus of $X_1(N)$ and the dimension of $\mathcal{S}_2(\Gamma_1(N))$. Take a homology basis $\{\varphi_1,\dots,\varphi_{2g}\}\subset\mathcal{S}_2(\Gamma_1(N))^\wedge$, so that

$$\mathrm{H}_1(X_1(N),\mathbf{Z})=\mathbf{Z}\varphi_1\oplus\cdots\oplus\mathbf{Z}\varphi_{2g}.$$

With respect to this basis, each group element $\sum_{j=1}^{2g}n_j\varphi_j$ is represented as an integral row vector $\vec{v}=[n_j]\in\mathbf{Z}^{2g}$ and each $T\in\mathbb{T}_{\mathbf{Z}}$ is represented by an integral $2g$-by-$2g$ matrix $[T]\in\mathrm{M}_{2g}(\mathbf{Z})$, so that the action of T as composition from the right is multiplication by $[T]$,

$$T:\vec{v}\mapsto\vec{v}[T]. \tag{6.14}$$

This action of $\mathbb{T}_{\mathbf{Z}}$ extends linearly to the free $\mathbf{C}$-module generated by the set $\{\varphi_1,\dots,\varphi_{2g}\}$, the $2g$-dimensional complex vector space

$$V=\mathbf{C}\varphi_1\oplus\cdots\oplus\mathbf{C}\varphi_{2g}.$$

Each element $(z_1\varphi_1,\dots,z_{2g}\varphi_{2g})$ is represented by a complex row vector $\vec{v}=[z_j]\in\mathbf{C}^{2g}$, and the action of each T is still described by (6.14). Suppose $\{\lambda(T):T\in\mathbb{T}_{\mathbf{Z}}\}$ is a system of eigenvalues of $\mathbb{T}_{\mathbf{Z}}$ on V, i.e., some nonzero $\vec{v}\in\mathbf{C}^{2g}$ satisfies

$$\vec{v}[T]=\lambda(T)\vec{v},\quad T\in\mathbb{T}_{\mathbf{Z}}.$$

Let $\sigma:\mathbf{C}\longrightarrow\mathbf{C}$ be any automorphism extending the given embedding $\sigma:\mathbf{K}_f\hookrightarrow\mathbf{C}$. Then σ acts on $\vec{v}$ elementwise and fixes the elements of each matrix $[T]$ since it fixes $\mathbf{Q}$. Thus

$$\vec{v}^\sigma[T]=(\vec{v}[T])^\sigma=(\lambda(T)\vec{v})^\sigma=\lambda(T)^\sigma\vec{v}^\sigma,\quad T\in\mathbb{T}_{\mathbf{Z}},$$

showing that if $\{\lambda(T):T\in\mathbb{T}_{\mathbf{Z}}\}$ is a system of eigenvalues on V then so is $\{\lambda(T)^\sigma:T\in\mathbb{T}_{\mathbf{Z}}\}$. To prove the theorem, this result needs to be transferred from V to $\mathcal{S}_2(\Gamma_1(N))$.

For convenience, abbreviate $\mathcal{S}_2(\Gamma_1(N))$ to $\mathcal{S}_2$ for the duration of this proof. The space $\mathcal{S}_2$ is isomorphic as a complex vector space to its dual space

$$\mathcal{S}_2^\wedge=\mathbf{C}\varphi_1+\cdots+\mathbf{C}\varphi_{2g}.$$

The dual space is not V but a g-dimensional quotient of V under the map $(z_1\varphi_1,\dots,z_{2g}\varphi_{2g})\mapsto\sum z_j\varphi_j$. The map has a g-dimensional kernel because the $\{\varphi_j\}$ are linearly independent over $\mathbf{R}$ but dependent over $\mathbf{C}$. The proof will construct a complementary space $\overline{\mathcal{S}_2^\wedge}$ isomorphic to $\mathcal{S}_2$ such that V is isomorphic to the direct sum $\mathcal{S}_2^\wedge\oplus\overline{\mathcal{S}_2^\wedge}$ as a $\mathbb{T}_{\mathbf{Z}}$-module and therefore the desired result transfers to the sum, and then the proof will show that the systems of eigenvalues on the sum are the systems of eigenvalues on $\mathcal{S}_2$, transferring the result to $\mathcal{S}_2$ as necessary.

To construct the complementary space, recall the operator $w_N = [\left[\begin{smallmatrix} 0 & 1 \\ -N & 0 \end{smallmatrix}\right]]_2$ from Section 5.5, satisfying the relation $w_N T = T^* w_N$ for all $T \in \mathbb{T}_{\mathbf{Z}}$ where as usual T^* is the adjoint of T. For any cusp form $g \in \mathcal{S}_2$ consider an associated map ψ_g from cusp forms to scalars,

$$\psi_g : \mathcal{S}_2 \longrightarrow \mathbf{C}, \qquad \psi_g(h) = \langle w_N g, h \rangle.$$

Then $\psi_g(h + \tilde{h}) = \psi_g(h) + \psi_g(\tilde{h})$ but since the Petersson inner product is conjugate-linear in its second factor,

$$\psi_g(zh) = \bar{z}\psi_g(h), \quad z \in \mathbf{C}.$$

Thus ψ_g belongs to the set $\overline{\mathcal{S}_2^\wedge}$ of conjugate-linear functions on $\mathcal{S}_2$, the complex conjugates of the dual space $\mathcal{S}_2^\wedge$ (Exercise 6.5.4). This set forms a complex vector space with the obvious operations. It is immediate that $\psi_{g+\tilde{g}} = \psi_g + \psi_{\tilde{g}}$ and $\psi_{zg} = z\psi_g$ for $g, \tilde{g} \in \mathcal{S}_2$ and $z \in \mathbf{C}$, and therefore the map

$$\Psi : \mathcal{S}_2 \longrightarrow \overline{\mathcal{S}_2^\wedge}, \qquad g \mapsto \psi_g$$

is $\mathbf{C}$-linear. It has trivial kernel, making it an isomorphism. The vector space $\overline{\mathcal{S}_2^\wedge}$ is a $\mathbb{T}_{\mathbf{Z}}$-module with the Hecke operators acting from the right as composition, and the linear isomorphism Ψ is also $\mathbb{T}_{\mathbf{Z}}$-linear since

$$\psi_{Tg}(h) = \langle w_N T g, h \rangle = \langle T^* w_N g, h \rangle = \langle w_N g, Th \rangle = (\psi_g \circ T)(h).$$

That is, $\mathcal{S}_2$ and $\overline{\mathcal{S}_2^\wedge}$ are isomorphic as complex vector spaces and as $\mathbb{T}_{\mathbf{Z}}$-modules. In particular, every system of eigenvalues $\{\lambda(T) : T \in \mathbb{T}_{\mathbf{Z}}\}$ on $\mathcal{S}_2$ is a system of eigenvalues on $\overline{\mathcal{S}_2^\wedge}$ and conversely.

Also, every system of eigenvalues on $\mathcal{S}_2$ is a system of eigenvalues on the dual space $\mathcal{S}_2^\wedge$ and conversely. To see this, let $f \in \mathcal{S}_2$ be a normalized eigenform. Similarly to before there is a map

$$\lambda_f : \mathbb{T}_{\mathbf{C}} \longrightarrow \mathbf{C}, \qquad Tf = \lambda_f(T).$$

(We need the complex Hecke algebra $\mathbb{T}_{\mathbf{C}}$ in this paragraph.) Let $J_f = \ker(\lambda_f) = \{T \in \mathbb{T}_{\mathbf{C}} : Tf = 0\}$, a prime ideal of $\mathbb{T}_{\mathbf{C}}$. An application of Nakayama's Lemma shows that $J_f \mathcal{S}_2 \neq \mathcal{S}_2$ (Exercise 6.5.5), making the quotient $\mathcal{S}_2/J_f\mathcal{S}_2$ nontrivial. It follows that the subspace of the dual space annihilated by J_f,

$$\mathcal{S}_2^\wedge[J_f] = \{\varphi \in \mathcal{S}_2^\wedge : \varphi \circ T = 0 \text{ for all } T \in J_f\},$$

is nonzero since it is isomorphic to $(\mathcal{S}_2/J_f\mathcal{S}_2)^\wedge$ by the first isomorphism in (6.13). Since T_1 is the identity operator, $T - \lambda_f(T)T_1 \in J_f$ for any $T \in \mathbb{T}_{\mathbf{C}}$, and so any nonzero $\varphi \in \mathcal{S}_2^\wedge[J_f]$ satisfies

$$\varphi \circ T = \varphi \circ (T - \lambda_f(T)T_1) + \lambda_f(T)\varphi = \lambda_f(T)\varphi, \quad T \in \mathbb{T}_{\mathbf{C}}.$$

Restricting our attention to $\mathbb{T}_{\mathbf{Z}}$ again, this shows that $\{\lambda_f(T) : T \in \mathbb{T}_{\mathbf{Z}}\}$ is a system of eigenvalues on $\mathcal{S}_2^\wedge$ as claimed. The converse follows by replacing $\mathcal{S}_2$ and $\mathcal{S}_2^\wedge$ with their duals, since the finite-dimensional vector space $\mathcal{S}_2$ is naturally isomorphic to its double dual as a $\mathbb{T}_{\mathbf{Z}}$-module. Thus the cusp forms $\mathcal{S}_2$ and the sum $\mathcal{S}_2^\wedge \oplus \overline{\mathcal{S}_2^\wedge}$ have the same systems of eigenvalues.

Consider the $\mathbf{C}$-linear map

$$V \longrightarrow \mathcal{S}_2^\wedge \oplus \overline{\mathcal{S}_2^\wedge}, \qquad (z_1\varphi_1, \dots, z_{2g}\varphi_{2g}) \mapsto \big(\sum z_j\varphi_j, \sum z_j\bar{\varphi}_j\big).$$

This is also a $\mathbb{T}_{\mathbf{Z}}$-module map since $\overline{\varphi_j \circ T} = \bar{\varphi}_j \circ T$. The map has trivial kernel since if $\sum z_j\varphi_j = 0$ in $\mathcal{S}_2^\wedge$ and $\sum z_j\bar{\varphi}_j = 0$ in $\overline{\mathcal{S}_2^\wedge}$ then both $\sum z_j\varphi_j = 0$ and $\sum \bar{z}_j\varphi_j = 0$ in $\mathcal{S}_2^\wedge$, i.e., $\sum \mathrm{Re}(z_j)\varphi_j = 0$ and $\sum \mathrm{Im}(z_j)\varphi_j = 0$ in $\mathcal{S}_2^\wedge$; but the $\{\varphi_j\}$ are linearly independent over $\mathbf{R}$, so this implies $z_j = 0$ for all j. Since the domain and codomain have the same dimension the map is a linear isomorphism of $\mathbb{T}_{\mathbf{Z}}$-modules. The result that if $\{\lambda(T) : T \in \mathbb{T}_{\mathbf{Z}}\}$ is a system of eigenvalues then so is $\{\lambda(T)^\sigma : T \in \mathbb{T}_{\mathbf{Z}}\}$ now transfers from V to $\mathcal{S}_2^\wedge \oplus \overline{\mathcal{S}_2^\wedge}$ and then to $\mathcal{S}_2$. Thus if $f(\tau) = \sum a_n q^n$ is a normalized eigenform in $\mathcal{S}_2(N,\chi)$ then its conjugate $f^\sigma(\tau) = \sum a_n^\sigma q^n$ is a normalized eigenform in $\mathcal{S}_2(N,\chi^\sigma)$ as desired.

It remains to prove the last statement of the theorem, that if f is a newform then so is f^σ. By Theorem 5.8.3, f^σ takes the form $f^\sigma(\tau) = \sum_i a_i f_i(n_i\tau)$ where each f_i is a newform at level M_i with $n_iM_i \mid N$. (Note that this uses only the part of that theorem that we have proved, that the set of such f_i spans $\mathcal{S}_2(\Gamma_1(N))$.) Let $\tau = \sigma^{-1} : \mathbf{C} \longrightarrow \mathbf{C}$, an extension of another embedding $\tau : \mathbf{K}_f \hookrightarrow \mathbf{C}$. Then $f = (f^\sigma)^\tau = \sum_i a_i^\tau f_i^\tau(n_i\tau)$. If f^σ is not new then by Exercise 5.8.4 it is old and all M_i are strictly less than N. Since each f_i^τ is also a modular form at level M_i this shows that f is old as well. The result follows by contraposition. □

Linearly combining the normalized eigenforms gives modular forms with coefficients in $\mathbf{Z}$.

Corollary 6.5.6. *The space $\mathcal{S}_2(\Gamma_1(N))$ has a basis of forms with rational integer coefficients.*

Proof. Let f be any newform at level M where $M \mid N$. Let $\mathbf{K} = \mathbf{K}_f$ be the number field of f. Let $\{\alpha_1, \dots, \alpha_d\}$ be a basis of $\mathcal{O}_{\mathbf{K}}$ as a $\mathbf{Z}$-module and let $\{\sigma_1, \dots, \sigma_d\}$ be the embeddings of $\mathbf{K}$ into $\mathbf{C}$. Consider the matrix from the end of the previous section and the vector

$$A = \begin{bmatrix} \alpha_1^{\sigma_1} & \cdots & \alpha_1^{\sigma_d} \\ \vdots & \ddots & \vdots \\ \alpha_d^{\sigma_1} & \cdots & \alpha_d^{\sigma_d} \end{bmatrix}, \qquad \vec{f} = \begin{bmatrix} f^{\sigma_1} \\ \vdots \\ f^{\sigma_d} \end{bmatrix},$$

and let $\vec{g} = A\vec{f}$, i.e.,

$$g_i = \sum_{j=1}^{d} \alpha_i^{\sigma_j} f^{\sigma_j}, \quad i = 1, \ldots, d.$$

Then $\mathrm{span}(\{g_1, \ldots, g_d\}) = \mathrm{span}(\{f^{\sigma_1}, \ldots, f^{\sigma_d}\})$ since A is invertible. Each g_i takes the form $g_i(\tau) = \sum_n a_n(g_i)q^n$ with all $a_n(g_i) \in \overline{\mathbf{Z}}$. For any automorphism $\sigma : \mathbf{C} \longrightarrow \mathbf{C}$, as σ_j runs through the embeddings of $\mathbf{K}_f$ into $\mathbf{C}$ so does $\sigma_j\sigma$ (composing left to right), and so

$$g_i^{\sigma} = \sum_{j=1}^{d} \alpha_j^{\sigma_j\sigma} f^{\sigma_j\sigma} = g.$$

That is, each $a_n(g_i)$ is fixed by all automorphisms of $\mathbf{C}$, showing that each $a_n(g_i)$ lies in $\overline{\mathbf{Z}} \cap \mathbf{Q} = \mathbf{Z}$. Repeating this argument for each newform f whose level divides N gives the result. □

Exercises

6.5.1. Let M be a free $\mathbf{Z}$-module of rank r. Show that the ring of endomorphisms of M is a free $\mathbf{Z}$-module of rank r^2, and so any subring is a free $\mathbf{Z}$-module of finite rank.

6.5.2. Let $f \in \mathcal{S}_2(\Gamma_1(N))$ be a normalized eigenform. Thus $f \in \mathcal{S}_2(N, \chi)$ for some Dirichlet character $\chi : (\mathbf{Z}/N\mathbf{Z})^* \longrightarrow \mathbf{C}^*$ and $\lambda_f(\langle d \rangle) = \chi(d)$ for all $d \in (\mathbf{Z}/N\mathbf{Z})^*$. Show that there is a ring and $\mathbf{Z}$-module isomorphism $\mathbb{T}_{\mathbf{Z}}/I_f \xrightarrow{\sim} \mathbf{Z}[\{a_n(f), \chi(d)\}]$. Show that adjoining the $\chi(d)$ values is redundant, making (6.12) in the text correct. (A hint for this exercise is at the end of the book.)

6.5.3. Prove the isomorphisms (6.13). (A hint for this exercise is at the end of the book.)

6.5.4. Let V be any complex vector space with dual space $V^\wedge$. Show that the set $\overline{V^\wedge} = \{\bar{\varphi} : \varphi \in V^\wedge\}$ is the set of functions $\psi : V \longrightarrow \mathbf{C}$ such that $\psi(v + v') = \psi(v) + \psi(v')$ and $\psi(zv) = \bar{z}\psi(v)$ for all $v, v' \in V$ and $z \in \mathbf{C}$.

6.5.5. Let J_f, $\mathbb{T}_{\mathbf{C}}$, and $\mathcal{S}_2$ be as in the proof of Theorem 6.5.4. Show that J_f is a prime ideal. Define the *local ring of* $\mathbb{T}_{\mathbf{C}}$ *at* J_f as a set of equivalence classes of formal elements

$$A = \{T/U : T \in \mathbb{T}_{\mathbf{C}}, U \in \mathbb{T}_{\mathbf{C}} - J_f\}/\sim$$

where the equivalence relation is

$$T/U \sim T'/U' \quad \text{if} \quad V(U'T - UT') = 0 \text{ for some nonzero } V \in \mathbb{T}_{\mathbf{C}} - J_f.$$

http://www.springer.com/978-0-387-23229-4

A First Course in Modular Forms
Diamond, F.; Shurman, J.
2005, XVI, 450 p. 57 illus., Hardcover
ISBN: 978-0-387-23229-4