4.10.8. (a) Let ψ and φ be primitive Dirichlet characters modulo u and v with $(\psi\varphi)(-1) = (-1)^k$ and $uv = N$. If $a : (\mathbf{Z}/N\mathbf{Z})^2 \longrightarrow \mathbf{C}$ is the function

$$a(\overline{cv}, \overline{d+ev}) = \psi(c)\bar{\varphi}(d), \quad a(\overline{x}, \overline{y}) = 0 \text{ otherwise,}$$

show that its Fourier transform is

$$\hat{a}(-\overline{cu}, -\overline{(d+eu)}) = (g(\bar{\varphi})/v)g(\psi)\varphi(c)\bar{\psi}(d), \quad \hat{a}(\overline{x}, \overline{y}) = 0 \text{ otherwise.}$$

(Hints for this exercise are at the end of the book.)

(b) Define an Eisenstein series with parameter,

$$G_k^{\psi,\varphi}(\tau, s) = \sum_{c=0}^{u-1} \sum_{d=0}^{v-1} \sum_{e=0}^{u-1} \psi(c)\bar{\varphi}(d) G_k^{\overline{(cv, d+ev)}}(\tau, s).$$

Show that the sum (4.46) of Eisenstein series for the function a in this problem is

$$G_k^a(\tau, s) = G_k^{\psi,\varphi}(\tau, s) + (-1)^k (g(\bar{\varphi})/v) g(\psi) G_k^{\varphi,\psi}(\tau, s).$$

Thus the functional equations for the eigenspaces $\mathcal{E}_k(N, \chi)$ involve only two series at a time.

(c) Define a function $b : (\mathbf{Z}/N\mathbf{Z})^2 \longrightarrow \mathbf{C}$ and a series $E_k^{\psi,\varphi}(\tau, s)$ by the conditions

$$a = \frac{g(\bar{\varphi})}{v} b, \qquad G_k^{\psi,\varphi} = \frac{g(\bar{\varphi})}{v} E_k^{\psi,\varphi}.$$

Show that the sum (4.46) of Eisenstein series for b is more nicely symmetrized than the one for a,

$$G_k^b(\tau, s) = E_k^{\psi,\varphi}(\tau, s) + \varphi(-1) E_k^{\varphi,\psi}(\tau, s).$$

4.11 Modular forms via theta functions

This chapter ends by using theta functions to construct a modular form that both connects back to the preface and adumbrates the ideas at the end of the book. The construction is one case of a general method due to Hecke [Hec26].

Recall that the preface used Quadratic Reciprocity to motivate the Modularity Theorem via a simple analog, counting the solutions modulo p to the quadratic equation $x^2 = d$. Now consider a cubic equation instead,

$$C : x^3 = d, \qquad d \in \mathbf{Z}^+, \ d \text{ cubefree,}$$

and for each prime p let

$$a_p(C) = (\text{the number of solutions modulo } p \text{ of equation } C) - 1.$$

Results from elementary number theory show that (Exercise 4.11.1)

$$a_p(C) = \begin{cases} 2 & \text{if } p \equiv 1 \pmod 3 \text{ and } d \text{ is a nonzero cube modulo } p, \\ -1 & \text{if } p \equiv 1 \pmod 3 \text{ and } d \text{ is not a cube modulo } p, \\ 0 & \text{if } p \equiv 2 \pmod 3 \text{ or } p \mid 3d. \end{cases} \tag{4.47}$$

This section will use Poisson summation and the Cubic Reciprocity Theorem from number theory to construct a modular form θ_χ with Fourier coefficients $a_p(\theta_\chi) = a_p(C)$. Section 5.9 will show that these Fourier coefficients are eigenvalues. That is, the solution-counts of the cubic equation C are a system of eigenvalues arising from a modular form. Chapter 9 will further place this example in the context of Modularity.

Introduce the notation

$$\mathbf{e}\,(z) = e^{2\pi i z}, \quad z \in \mathbf{C}.$$

Let $A = \mathbf{Z}[\mu_3]$, let $\alpha = i\sqrt{3}$, and let $B = \frac{1}{\alpha}A$. Thus $A \subset B \subset \frac{1}{3}A$. Note that

$$|x|^2 = x_1^2 - x_1x_2 + x_2^2 \quad \text{for any } x = x_1 + x_2\mu_3 \in \mathbf{R}[\mu_3].$$

We will frequently use the formula $|x+y|^2 = |x|^2 + \operatorname{tr}(xy^*) + |y|^2$ for $x, y \in \mathbf{C}$, where y^* is the complex conjugate of y and $\operatorname{tr}(z) = z + z^*$. For any positive integer N and any $\overline{u}$ in the quotient group $\frac{1}{3}A/NA$ define a theta function,

$$\theta^{\overline{u}}(\tau, N) = \sum_{n \in A} \mathbf{e}\left(N|u/N + n|^2\tau\right), \quad \tau \in \mathcal{H}. \tag{4.48}$$

An argument similar to Exercise 4.9.2 shows that $\theta^{\overline{u}}$ is holomorphic. The following lemma establishes its basic transformation properties. From now until near the end of the section the symbol d is unrelated to the d of the cubic equation C.

Lemma 4.11.1. *Let N be a positive integer. Then*

$$\begin{aligned} \theta^{\overline{u}}(\tau+1, N) &= \mathbf{e}\left(\tfrac{|u|^2}{N}\right)\theta^{\overline{u}}(\tau, N), && \overline{u} \in B/NA, \\ \theta^{\overline{u}}(\tau, N) &= \sum_{\substack{\overline{v} \in B/dNA \\ \overline{v} \equiv \overline{u}\ (NA)}} \theta^{\overline{v}}(d\tau, dN), && \overline{u} \in B/NA,\ d \in \mathbf{Z}^+, \\ \theta^{\overline{v}}(-1/\tau, N) &= \frac{-i\tau}{N\sqrt{3}} \sum_{\overline{w} \in B/NA} \mathbf{e}\left(-\tfrac{\operatorname{tr}(vw^*)}{N}\right)\theta^{\overline{w}}(\tau, N), && \overline{v} \in B/NA. \end{aligned}$$

Proof. For the first statement compute that for $u \in B$ and $n \in A$,

$$N\left|\tfrac{u}{N} + n\right|^2 \equiv \tfrac{|u|^2}{N} \pmod{\mathbf{Z}}.$$

For the second statement note that $\theta^{\overline{u}}(\tau, N) = \sum_{n \in A} \mathbf{e}\left(dN|\frac{u/N+n}{d}|^2 d\tau\right)$. Let $n = r + dm$, making the fraction $\frac{u+Nr}{dN} + m$. Thus

$$\theta^{\overline{u}}(\tau, N) = \sum_{\substack{\overline{r} \in A/dA \\ m \in A}} \mathbf{e}\left(dN \left|\tfrac{u+Nr}{dN} + m\right|^2 d\tau\right) = \sum_{\overline{r} \in A/dA} \theta^{\overline{u+Nr}}(d\tau, dN),$$

where the reduction $\overline{u+Nr}$ is taken modulo dN. This gives the result.

The third statement is shown by Poisson summation. Recall that the defining equation (4.42) from the previous section was extended to $k=0$ in Exercise 4.10.6(a),

$$\vartheta_0^{\overline{v}}(\gamma) = \sum_{n \in \mathbf{Z}^2} e^{-\pi |(v/N+n)\gamma|^2}, \quad \overline{v} \in (\mathbf{Z}/N\mathbf{Z})^2,\ \gamma \in \mathrm{GL}_2(\mathbf{R}).$$

To apply this let $\gamma \in \mathrm{SL}_2(\mathbf{R})$ be the positive square root of $\frac{1}{\sqrt{3}}\left[\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right]$, satisfying

$$|(x_1, x_2)\gamma|^2 = \tfrac{2}{\sqrt{3}}|x_1 + x_2\mu_3|^2, \quad x_1, x_2 \in \mathbf{R}.$$

Note that $B/NA \subset \frac{1}{3}A/NA \cong A/3NA$. Identify A with $\mathbf{Z}^2$ so that if $v \in B$ then its multiple $3v \in \alpha A \subset A$ can also be viewed as an element of $\mathbf{Z}^2$. Compute with $3N$ in place of N and with γ as above that for any $t \in \mathbf{R}^+$,

$$\begin{aligned} \vartheta_0^{\overline{3v}}(\gamma(3N)^{1/2}(t/\sqrt{3})^{1/2}) &= \sum_{n \in \mathbf{Z}^2} e^{-\pi\sqrt{3}N|\left(\frac{3v}{3N}+n\right)\gamma|^2 t} \\ &= \sum_{n \in A} e^{-2\pi N|\frac{v}{N}+n|^2 t} = \theta^{\overline{v}}(it). \end{aligned}$$

This shows that the identity (4.43) with $k=0$, with $3N$ in place of N, with γ as above, and with $r = (\sqrt{3}\,t)^{-1/2}$ is (Exercise 4.11.2)

$$\theta^{\overline{v}}(-1/(it), N) = \frac{t}{N\sqrt{3}} \sum_{\overline{u} \in \frac{1}{3}A/NA} \mathbf{e}\left(-\tfrac{3(vu^*)_2}{N}\right) \theta^{\overline{u}}(3it, N), \quad \overline{v} \in B/NA,$$

where $vu^* = (vu^*)_1 + (vu^*)_2\mu_3$. Generalize from $t \in \mathbf{R}^+$ to $-i\tau$ for $\tau \in \mathcal{H}$ by the Uniqueness Theorem of complex analysis to get

$$\theta^{\overline{v}}(-1/\tau, N) = \frac{-i\tau}{N\sqrt{3}} \sum_{\overline{u} \in \frac{1}{3}A/NA} \mathbf{e}\left(-\tfrac{(\alpha v(\alpha u)^*)_2}{N}\right) \theta^{\overline{u}}(3\tau, N), \quad \overline{v} \in B/NA.$$

The sum on the right side is

$$\sum_{\overline{w} \in B/NA} \mathbf{e}\left(-\tfrac{(\alpha v w^*)_2}{N}\right) \sum_{\substack{\overline{u} \in \frac{1}{3}A/NA \\ \overline{\alpha u} \equiv \overline{w}\ (NA)}} \theta^{\overline{u}}(3\tau, N).$$

To simplify the inner sum note that $\theta^{\overline{w}}(\tau, N) = \sum_{n \in A} \mathbf{e}\left(N|\frac{w/N+n}{\alpha}|^2 3\tau\right)$ for any $\overline{w} \in B/NA$, and similarly to the proof of the second statement this works out to

$$\theta^{\overline{w}}(\tau,N) = \sum_{\substack{\overline{r}\in A/\alpha A\\ m\in A}} \mathbf{e}\left(N\left|\tfrac{(w+Nr)/\alpha}{N}+m\right|^2 3\tau\right) = \sum_{\overline{r}\in A/\alpha A} \theta^{\overline{(w+Nr)/\alpha}}(3\tau,N).$$

That is, the inner sum is

$$\sum_{\substack{\overline{u}\in\frac{1}{3}A/NA\\ \alpha\overline{u}\equiv\overline{w}\,(NA)}} \theta^{\overline{u}}(3\tau,N) = \theta^{\overline{w}}(\tau,N), \quad \overline{w}\in B/NA.$$

The proof is completed by noting that $(\alpha v w^*)_2 = \mathrm{tr}\,(vw^*)$. □

The next result shows how the theta function transforms under the group

$$\Gamma_0(3N,N) = \left\{\begin{bmatrix} a & b\\ c & d\end{bmatrix} \in \mathrm{SL}_2(\mathbf{Z}) : b\equiv 0 \pmod N,\ c\equiv 0 \pmod{3N}\right\}.$$

Proposition 4.11.2. *Let N be a positive integer. Then*

$$(\theta^{\overline{u}}[\gamma]_1)(\tau,N) = \left(\tfrac{d}{3}\right)\theta^{\overline{au}}(\tau,N), \quad \overline{u}\in A/NA,\ \gamma = \left[\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right]\in\Gamma_0(3N,N).$$

Here $(d/3)$ is the Legendre symbol.

Proof. Since $\theta^{-\overline{au}} = \theta^{\overline{au}}$ we can assume $d>0$ by replacing γ with $-\gamma$ if necessary. Write

$$\frac{a\tau+b}{c\tau+d} = \frac{1}{d}\left(\frac{1}{d/\tau+c}+b\right).$$

Apply the second statement of the lemma and then the first statement to get

$$\theta^{\overline{u}}(\gamma(\tau),N) = \sum_{\substack{\overline{v}\in B/dNA\\ \overline{v}\equiv\overline{u}\,(NA)}} \mathbf{e}\left(\tfrac{b|v|^2}{dN}\right)\theta^{\overline{v}}\left(-\tfrac{1}{-d/\tau-c},dN\right).$$

The third statement and again the first now give

$$\begin{aligned}\theta^{\overline{u}}(\gamma(\tau),N) &= \frac{i(d/\tau+c)}{dN\sqrt{3}} \sum_{\substack{\overline{v},\overline{w}\in B/dNA\\ \overline{v}\equiv\overline{u}\,(NA)}} \mathbf{e}\left(\tfrac{b|v|^2-\mathrm{tr}\,(vw^*)}{dN}\right)\theta^{\overline{w}}(-d/\tau-c,dN)\\ &= \frac{i(c\tau+d)}{dN\sqrt{3}\,\tau} \sum_{\substack{\overline{v},\overline{w}\in B/dNA\\ \overline{v}\equiv\overline{u}\,(NA)}} \mathbf{e}\left(\tfrac{b|v|^2-\mathrm{tr}\,(vw^*)-c|w|^2}{dN}\right)\theta^{\overline{w}}(-d/\tau,dN).\end{aligned}$$

Note that $cw\in NA$ for $w\in B$ since $c\equiv 0 \pmod{3N}$. It follows that

$$\begin{aligned}\sum_{\substack{\overline{v}\in B/dNA\\ \overline{v}\equiv\overline{u}\,(NA)}} \mathbf{e}\left(\tfrac{b|v|^2-\mathrm{tr}\,(vw^*)-c|w|^2}{dN}\right) &= \sum_{\substack{\overline{v}\in B/dNA\\ \overline{v}\equiv\overline{u}\,(NA)}} \mathbf{e}\left(\tfrac{b|v-cw|^2-\mathrm{tr}\,((v-cw)w^*)-c|w|^2}{dN}\right)\\ &= \mathbf{e}\left(-\tfrac{\mathrm{tr}\,(auw^*)}{N}\right) \sum_{\substack{\overline{v}\in B/dNA\\ \overline{v}\equiv\overline{u}\,(NA)}} \mathbf{e}\left(\tfrac{b|v|^2}{dN}\right),\end{aligned}$$

where the last equality uses the relation $ad - bc = 1$. Since $b \equiv 0 \pmod N$ the summand $\mathbf{e}\left(b|v|^2/(dN)\right)$ depends only on $v \pmod{dA}$, and since $(d, N) = 1$ and $\overline{u} \in A/NA$ the sum is $\sum_{\overline{v} \in A/dA} \mathbf{e}\left(b|v|^2/(dN)\right)$. This takes the value $(d/3)d$ (Exercise 4.11.3). Substitute this into the transformation formula and apply the second and third statements of the lemma to continue,

$$\begin{aligned}
\theta^{\overline{u}}(\gamma(\tau), N) &= \frac{i(c\tau + d)}{N\sqrt{3}\,\tau}\left(\frac{d}{3}\right) \sum_{\overline{w} \in B/dNA} \mathbf{e}\left(-\tfrac{\mathrm{tr}\,(auw^*)}{N}\right) \theta^{\overline{w}}(d(-1/\tau), dN) \\
&= \frac{i(c\tau + d)}{N\sqrt{3}\,\tau}\left(\frac{d}{3}\right) \sum_{\overline{v} \in B/NA} \mathbf{e}\left(-\tfrac{\mathrm{tr}\,(auv^*)}{N}\right) \theta^{\overline{v}}(-1/\tau, N) \\
&= \frac{c\tau + d}{3N^2}\left(\frac{d}{3}\right) \sum_{\overline{v}, \overline{w} \in B/NA} \mathbf{e}\left(-\tfrac{\mathrm{tr}\,(vw^* + auv^*)}{N}\right) \theta^{\overline{w}}(\tau, N).
\end{aligned}$$

Exercise 4.11.3(b) shows that the inner sum is

$$\sum_{\overline{v} \in B/NA} \mathbf{e}\left(-\frac{\mathrm{tr}\,(v(w^* + au^*))}{N}\right) = \begin{cases} 3N^2 & \text{if } \overline{w} = -\overline{au}, \\ 0 & \text{if } \overline{w} \neq -\overline{au}, \end{cases}$$

completing the proof since $\theta^{-\overline{au}} = \theta^{\overline{au}}$. □

To construct a modular form from the theta functions we need to conjugate and then symmetrize. To conjugate, let $\delta = \left[\begin{smallmatrix} N & 0 \\ 0 & 1 \end{smallmatrix}\right]$ so that

$$\delta \Gamma_0(3N^2)\delta^{-1} = \Gamma_0(3N, N)$$

and the conjugation preserves matrix entries on the diagonal. Recall that the weight-k operator was extended to $\mathrm{GL}_2^+(\mathbf{Q})$ in Exercise 1.2.11. Thus for any $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_0(3N^2)$,

$$\begin{aligned}
(\theta^{\overline{u}}[\delta\gamma]_1)(\tau) &= (\theta^{\overline{u}}[\gamma'\delta]_1)(\tau) && \text{where } \gamma' = \delta\gamma\delta^{-1} \in \Gamma_0(3N, N) \\
&= (d/3)(\theta^{\overline{au}}[\delta]_1)(\tau) && \text{since } d = d_{\gamma'}.
\end{aligned} \tag{4.49}$$

The construction is completed by symmetrizing:

Theorem 4.11.3. *Let N be a positive integer and let $\chi : (A/NA)^* \longrightarrow \mathbf{C}^*$ be a character, extended multiplicatively to A. Define*

$$\theta_\chi(\tau) = \tfrac{1}{6} \sum_{\overline{u} \in A/NA} \chi(u)\theta^{\overline{u}}(N\tau, N).$$

Then

$$\theta_\chi[\gamma]_1 = \chi(d)(d/3)\theta_\chi, \quad \gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_0(3N^2).$$

Therefore

$$\theta_\chi \in \mathcal{M}_1(3N^2, \psi), \qquad \psi(d) = \chi(d)(d/3).$$

The desired transformation of θ_χ under $\Gamma_0(3N^2)$ follows from (4.49) since $\theta_\chi = \sum_{\overline{u}} \chi(u)\theta^{\overline{u}}[\delta]_1$ (Exercise 4.11.4). To finish proving the theorem note that

$$\theta_\chi(\tau) = \tfrac{1}{6}\sum_{n\in A} \chi(n)\mathbf{e}\left(|n|^2\tau\right) = \sum_{m=0}^{\infty} a_m(\theta_\chi)e^{2\pi i\tau}$$

where

$$a_m(\theta_\chi) = \tfrac{1}{6}\sum_{\substack{n\in A\\ |n|^2=m}} \chi(n). \tag{4.50}$$

This shows that $a_n(\theta) = \mathcal{O}(n)$, i.e., the Fourier coefficients are small enough to satisfy the condition in Proposition 1.2.4.

For example, when $N = 1$ the theta function

$$\theta_{\mathbf{1}}(\tau) = \tfrac{1}{6}\sum_{n\in A} e^{2\pi i|n|^2\tau}, \quad \tau \in \mathcal{H}$$

is a constant multiple of $E_1^{\psi,\mathbf{1}}$, the Eisenstein series mentioned at the end of Section 4.8 (Exercise 4.11.5). This fact is equivalent to a representation number formula like those in Exercise 4.8.7.

Along with Poisson summation, the other ingredient for constructing a modular form to match the cubic equation C from the beginning of the section is the Cubic Reciprocity Theorem. The unit group of A is $A^* = \{\pm 1, \pm\mu_3, \pm\mu_3^2\}$. Note that formula (4.50) shows that $\theta_\chi = 0$ unless χ is trivial on A^*. Let p be a rational prime, $p \equiv 1 \pmod 3$. Then there exists an element $\pi = a + b\mu_3 \in A$ such that

$$\{n \in A : |n|^2 = p\} = A^*\pi \cup A^*\overline{\pi}.$$

The choice of π can be normalized, e.g., to $\pi = a + b\mu_3$ where $a \equiv 2 \pmod 3$ and $b \equiv 0 \pmod 3$. On the other hand a rational prime $p \equiv 2 \pmod 3$ does not take the form $p = |n|^2$ for any $n \in A$, as is seen by checking $|n|^2$ modulo 3. (See 9.1–9.6 of [IR92] for more on the arithmetic of A.) A weak form of Cubic Reciprocity is: *Let $d \in \mathbf{Z}^+$ be cubefree and let $N = 3\prod_{p|d} p$. Then there exists a character*

$$\chi : (A/NA)^* \longrightarrow \{1, \mu_3, \mu_3^2\}$$

such that the multiplicative extension of χ to all of A is trivial on A^* and on primes $p \nmid N$, while on elements π of A such that $\pi\overline{\pi}$ is a prime $p \nmid N$ it is trivial if and only if d is a cube modulo p. See Exercise 4.11.6 for simple examples.

For this character, $\theta_\chi(\tau, N) \in \mathcal{M}_1(3N^2, \psi)$ where ψ is the quadratic character with conductor 3. Formula (4.50) shows that the Fourier coefficients of prime index are

$$a_p(\theta_\chi) = \begin{cases} 2 & \text{if } p \equiv 1 \pmod 3 \text{ and } d \text{ is a nonzero cube modulo } p, \\ -1 & \text{if } p \equiv 1 \pmod 3 \text{ and } d \text{ is not a cube modulo } p, \\ 0 & \text{if } p \equiv 2 \pmod 3 \text{ or } p \mid 3d. \end{cases} \tag{4.51}$$

That is, the Fourier coefficients are the solution-counts (4.47) of equation C as anticipated at the beginning of the section.

Exercises

4.11.1. Show that for any prime p the map $x \mapsto x^3$ is an endomorphism of the multiplicative group $(\mathbf{Z}/p\mathbf{Z})^*$. Show that the map is 3-to-1 if $p \equiv 1 \pmod 3$ and is 1-to-1 if $p \equiv 2 \pmod 3$. Use this to establish (4.47).

4.11.2. Confirm that under the identification $(x_1, x_2) \leftrightarrow x_1 + x_2\mu_3$, the exponent $-\langle u, vS\rangle$ from Section 4.10 becomes $-(vu^*)_2$. Use this to verify the application of Poisson summation with $3u$, $3v$, and $3N$ in the proof of Lemma 4.11.1.

4.11.3. For $b, d \in \mathbf{Z}$ with $(3b, d) = 1$ let $\varphi_{b,d} = \sum_{\overline{v} \in A/dA} \mathbf{e}\left(b|v|^2/d\right)$. This exercise proves the formula $\varphi_{b,d} = (d/3)d$.

(a) Prove the formula when $d = p$ where $p \neq 3$ is prime. For $p = 2$ compute directly. For $p > 3$ use the isomorphism $\mathbf{Z}[\sqrt{-3}]/p\mathbf{Z}[\sqrt{-3}] \longrightarrow A/pA$ to show that

$$\varphi_{b,d} = \sum_{r_1, r_2 \in \mathbf{Z}/p\mathbf{Z}} \mathbf{e}\left(b(r_1^2 + 3r_2^2)/p\right).$$

Show that if m is not divisible by p then

$$\sum_{r \in \mathbf{Z}/p\mathbf{Z}} \mathbf{e}\left(mr^2/p\right) = \sum_{s \in \mathbf{Z}/p\mathbf{Z}} \left(1 + \left(\tfrac{ms}{p}\right)\right) \mathbf{e}\left(s/p\right) = \left(\tfrac{m}{p}\right) g(\chi)$$

where $g(\chi)$ is the Gauss sum associated to the character $\chi(s) = (s/p)$. Show that $g(\chi)^2 = \chi(-1)|g(\chi)|^2 = (-1/p)p$ similarly to (4.12). Use Quadratic Reciprocity to complete the proof.

(b) Before continuing show that for any $x \in N^{-1}B$,

$$\sum_{\overline{w} \in A/NA} \mathbf{e}\left(\operatorname{tr}(xw^*)\right) = \begin{cases} N^2 & \text{if } x \in B, \\ 0 & \text{if } x \notin B, \end{cases}$$

and

$$\sum_{\overline{v} \in B/NA} \mathbf{e}\left(\operatorname{tr}(vx^*)\right) = \begin{cases} 3N^2 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

(c) Prove the formula for $d = p^t$ inductively by showing that for $t \geq 2$,

$$\begin{aligned}\varphi_{b,p^t} &= \sum_{\overline{v} \in A/p^{t-1}A} \sum_{\overline{w} \in A/pA} \mathbf{e}\left(b|v + p^{t-1}w|^2/p^t\right) \\ &= \sum_{\overline{v} \in A/p^{t-1}A} \mathbf{e}\left(b|v|^2/p^t\right) \sum_{\overline{w} \in A/pA} \mathbf{e}\left(\operatorname{tr}(bvw^*)/p\right) = \varphi_{b,p^{t-2}}p^2.\end{aligned}$$

(d) For arbitrary $d > 0$ suppose that $d = d_1 d_2$ with $(d_1, d_2) = 1$ and $d_1, d_2 > 0$. Use the bijection

$$A/d_1A \times A/d_2A \longrightarrow A/dA, \qquad (\overline{u_1}, \overline{u_2}) \mapsto \overline{d_2u_1 + d_1u_2}$$

to show that $\varphi_{b,d} = \varphi_{bd_2,d_1}\varphi_{bd_1,d_2}$. Deduce that this holds for $d < 0$ as well. Complete the proof of the formula.

4.11.4. Verify the transformation law in Theorem 4.11.3.

4.11.5. (a) Use results from Chapter 3 to show that $\dim(\mathcal{S}_1(\Gamma_0(3)) = 0$, so that if ψ is the quadratic character modulo 3 then $\mathcal{M}_1(3, \psi) = \mathbf{C}E_1^{\psi,\mathbf{1}}$.

(b) Let $\theta_{\mathbf{1}}(\tau)$ denote the theta function in Theorem 4.11.3 specialized to $N = 1$, making the character trivial. Thus $\theta_{\mathbf{1}} \in \mathcal{M}_1(3, \psi)$ and so $\theta_{\mathbf{1}}$ is a constant multiple of $E_1^{\psi,\mathbf{1}}$. What is the constant?

(c) The Fourier coefficients $a_m(\theta_{\mathbf{1}})$ are (up to a constant multiple) representation numbers for the quadratic form $n_1^2 - n_1n_2 + n_2^2$. Thus the representation number is a constant multiple of the arithmetic function $\sigma_0^{\psi,\mathbf{1}}(m) = \sum_{d|n} \psi(d)$ for $m \geq 1$. Check the relation between $r(p)$ and $\sigma_0^{\psi,\mathbf{1}}(p)$ for prime p by using the information about this ring given in the proof of Corollary 3.7.2. Indeed, the reader with background in number theory can work this exercise backwards by deriving the representation numbers and thus the identity $\theta_{\mathbf{1}} = cE_1^{\psi,\mathbf{1}}$ arithmetically.

4.11.6. (a) Describe the character χ provided by Cubic Reciprocity for $d = 1$.

(b) To describe χ for $d = 2$, first determine the conditions modulo 2 on $a, b \in \mathbf{Z}$ that make $a + b\mu_3 \in A$ invertible modulo $2A$, and similarly for 3. Use these to show that the multiplicative group $G = (A/6A)^*$ has order 18. Show that A^* reduces to a cyclic subgroup H of order 6 in G and that the quotient G/H is generated by $g = \overline{1 + 3\mu_3}$. Explain why χ is defined on the quotient and why up to complex conjugation it is

$$\chi(H) = 1, \qquad \chi(gH) = \zeta_3, \qquad \chi(g^2H) = \zeta_3^2.$$

(Here the symbol ζ_3 is being used to distinguish the cube root of unity in the codomain $\mathbf{C}^*$ of χ from the cube root of unity in A.)

(c) Describe the character χ provided by Cubic Reciprocity for $d = 3$.

Springer

http://www.springer.com/978-0-387-23229-4

A First Course in Modular Forms
Diamond, F.; Shurman, J.
2005, XVI, 450 p. 57 illus., Hardcover
ISBN: 978-0-387-23229-4