2

Multivariable Calculus

2.1 Limits and Continuity

Problem 2.1.1 (Fa94) *Let the function* $f : \mathbb{R}^n \to \mathbb{R}^n$ *satisfy the following two conditions:*

(i) $f(K)$ *is compact whenever* K *is a compact subset of* $\mathbb{R}^n$.

(ii) If $\{K_n\}$ *is a decreasing sequence of compact subsets of* $\mathbb{R}^n$, *then*

$$f\left(\bigcap_1^\infty K_n\right) = \bigcap_1^\infty f(K_n).$$

Prove that f *is continuous.*

Problem 2.1.2 (Sp78) *Prove that a map* $g : \mathbb{R}^n \to \mathbb{R}^n$ *is continuous only if its graph is closed in* $\mathbb{R}^n \times \mathbb{R}^n$. *Is the converse true?*
Note: See also Problem 1.2.14.

Problem 2.1.3 (Su79) *Let* $U \subset \mathbb{R}^n$ *be an open set. Suppose that the map* $h : U \to \mathbb{R}^n$ *is a homeomorphism from* U *onto* $\mathbb{R}^n$, *which is uniformly continuous. Prove* $U = \mathbb{R}^n$.

Problem 2.1.4 (Sp89) *Let* f *be a real valued function on* $\mathbb{R}^2$ *with the following properties:*

1. *For each* y_0 *in* $\mathbb{R}$, *the function* $x \mapsto f(x, y_0)$ *is continuous.*

2. *For each* x_0 *in* $\mathbb{R}$*, the function* $y \mapsto f(x_0, y)$ *is continuous.*

3. $f(K)$ *is compact whenever* K *is a compact subset of* $\mathbb{R}^2$*.*

Prove that f *is continuous.*

Problem 2.1.5 (Sp91) *Let* f *be a continuous function from the ball* $B_n = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$ *into itself. (Here,* $\|\cdot\|$ *denotes the Euclidean norm.) Assume* $\|f(x)\| < \|x\|$ *for all nonzero* $x \in B_n$*. Let* x_0 *be a nonzero point of* B_n*, and define the sequence* (x_k) *by setting* $x_k = f(x_{k-1})$*. Prove that* $\lim x_k = 0$*.*

Problem 2.1.6 (Su78) *Let* N *be a* norm *on the vector space* $\mathbb{R}^n$*; that is,* $N : \mathbb{R}^n \to \mathbb{R}$ *satisfies*

$$\begin{aligned} &N(x) \geqslant 0 \ \ \textit{and} \ \ N(x) = 0 \ \ \textit{only if} \ \ x = 0, \\ &N(x+y) \leqslant N(x) + N(y), \\ &N(\lambda x) = |\lambda| N(x) \end{aligned}$$

for all $x, y \in \mathbb{R}^n$ *and* $\lambda \in \mathbb{R}$*.*

1. *Prove that* N *is bounded on the unit sphere.*

2. *Prove that* N *is continuous.*

3. *Prove that there exist constants* $A > 0$ *and* $B > 0$*, such that for all* $x \in \mathbb{R}^n$*,* $A\|x\| \leqslant N(x) \leqslant B\|x\|$*.*

Problem 2.1.7 (Fa97) *A map* $f : \mathbb{R}^m \to \mathbb{R}^n$ *is* proper *if it is continuous and* $f^{-1}(B)$ *is compact for each compact subset* B *of* $\mathbb{R}^n$*;* f *is* closed *if it is continuous and* $f(A)$ *is closed for each closed subset* A *of* $\mathbb{R}^m$*.*

1. *Prove that every proper map* $f : \mathbb{R}^m \to \mathbb{R}^n$ *is closed.*

2. *Prove that every one-to-one closed map* $f : \mathbb{R}^m \to \mathbb{R}^n$ *is proper.*

Problem 2.1.8 (Sp83) *Suppose that* $F : \mathbb{R}^n \to \mathbb{R}^n$ *is continuous and satisfies*

$$\|F(x) - F(y)\| \geqslant \lambda \|x - y\|$$

for all $x, y \in \mathbb{R}^n$ *and some* $\lambda > 0$*. Prove that* F *is one-to-one, onto, and has a continuous inverse.*
Note: See also Problem 1.2.12.

2.2 Differential Calculus

Problem 2.2.1 (Sp93) *Prove that* $\dfrac{x^2+y^2}{4} \leqslant e^{x+y-2}$ *for* $x \geqslant 0$*,* $y \geqslant 0$*.*

Problem 2.2.2 (Fa98) *Find the minimal value of the areas of hexagons circumscribing the unit circle in* $\mathbb{R}^2$.
Note: See also Problem 1.1.12.

Problem 2.2.3 (Sp03) *Define* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *by* $f(x, 0) = 0$ *and*

$$f(x, y) = \left(1 - \cos\frac{x^2}{y}\right)\sqrt{x^2 + y^2}$$

for $y \neq 0$.

1. *Show that* f *is continuous at* $(0, 0)$.
2. *Calculate all the directional derivatives of* f *at* $(0, 0)$.
3. *Show that* f *is not differentiable at* $(0, 0)$.

Problem 2.2.4 (Fa86) *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be defined by:*

$$f(x, y) = \begin{cases} x^{4/3}\sin(y/x) & if \quad x \neq 0 \\ 0 & if \quad x = 0\,. \end{cases}$$

Determine all points at which f *is differentiable.*

Problem 2.2.5 (Sp00) *Let* F*, with components* $F_1, \dots, F_n$*, be a differentiable map of* $\mathbb{R}^n$ *into* $\mathbb{R}^n$ *such that* $F(0) = 0$*. Assume that*

$$\sum_{j,k=1}^{n} \left|\frac{\partial F_j(0)}{\partial x_k}\right|^2 = c < 1\ .$$

Prove that there is a ball B *in* $\mathbb{R}^n$ *with center* 0 *such that* $F(B) \subset B$.

Problem 2.2.6 (Fa02) *Let* p *be a polynomial over* $\mathbb{R}$ *of positive degree. Define the function* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *by* $f(x, y) = (p(x + y), p(x - y))$*. Prove that the derivative* $Df(x, y)$ *is invertible for an open dense set of points* (x, y) *in* $\mathbb{R}^2$.

Problem 2.2.7 (Fa02) *Find the most general continuously differentiable function* $g : \mathbb{R} \to (0, \infty)$ *such that the function* $h(x, y) = g(x)g(y)$ *on* $\mathbb{R}^2$ *is constant on each circle with center* $(0, 0)$.

Problem 2.2.8 (Sp80, Fa92) *Let* $f : \mathbb{R}^n \to \mathbb{R}^n$ *be continuously differentiable. Assume the Jacobian matrix* $\left(\partial f_i / \partial x_j\right)$ *has rank* n *everywhere. Suppose* f *is* proper*; that is,* $f^{-1}(K)$ *is compact whenever* K *is compact. Prove* $f(\mathbb{R}^n) = \mathbb{R}^n$.

Problem 2.2.9 (Sp89) *Suppose* f *is a continuously differentiable map of* $\mathbb{R}^2$ *into* $\mathbb{R}^2$*. Assume that* f *has only finitely many singular points, and that for each positive number* M*, the set* $\{z \in \mathbb{R}^2 \mid |f(z)| \leqslant M\}$ *is bounded. Prove that* f *maps* $\mathbb{R}^2$ *onto* $\mathbb{R}^2$.

Problem 2.2.10 (Fa81) *Let f be a real valued function on $\mathbb{R}^n$ of class C^2. A point $x \in \mathbb{R}^n$ is a critical point of f if all the partial derivatives of f vanish at x; a critical point is* nondegenerate *if the $n \times n$ matrix*

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x) \right)$$

is nonsingular.

Let x be a nondegenerate critical point of f. Prove that there is an open neighborhood of x which contains no other critical points (i.e., the nondegenerate critical points are isolated).

Problem 2.2.11 (Su80) *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function whose partial derivatives of order $\leqslant 2$ are everywhere defined and continuous.*

1. *Let $a \in \mathbb{R}^n$ be a critical point of f (i.e., $\frac{\partial f}{\partial x_j}(a) = 0,\ i = 1, \ldots, n$). Prove that a is a local minimum provided the Hessian matrix*

   $$\left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)$$

   is positive definite at $x = a$.

2. *Assume the Hessian matrix is positive definite at all x. Prove that f has, at most, one critical point.*

Problem 2.2.12 (Fa88) *Prove that a real valued C^3 function f on $\mathbb{R}^2$ whose Laplacian,*

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2},$$

is everywhere positive cannot have a local maximum.

Problem 2.2.13 (Fa01) *Let the function u on $\mathbb{R}^2$ be harmonic, not identically 0, and homogeneous of degree d, where $d > 0$. (The homogeneity condition means that $u(tx, ty) = t^d u(x, y)$ for $t > 0$.) Prove that d is an integer.*

Problem 2.2.14 (Su82) *Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ and assume that 0 is a* regular value *of f (i.e., the differential of f has rank 2 at each point of $f^{-1}(0)$). Prove that $\mathbb{R}^3 \setminus f^{-1}(0)$ is arcwise connected.*

Problem 2.2.15 (Sp87) *Let the transformation T from the subset $U = \{(u, v) \mid u > v\}$ of $\mathbb{R}^2$ into $\mathbb{R}^2$ be defined by $T(u, v) = (u + v, u^2 + v^2)$.*

1. *Prove that T is locally one-to-one.*
2. *Determine the range of T, and show that T is globally one-to-one.*

Problem 2.2.16 (Fa91) *Let f be a C^1 function from the interval $(-1, 1)$ into $\mathbb{R}^2$ such that $f(0) = 0$ and $f'(0) \neq 0$. Prove that there is a number ε in $(0, 1)$ such that $\|f(t)\|$ is an increasing function of t on $(0, \varepsilon)$.*

Problem 2.2.17 (Fa80) *For a real 2×2 matrix*

$$X = \begin{pmatrix} x & y \\ z & t \end{pmatrix},$$

let $\|X\| = x^2 + y^2 + z^2 + t^2$, and define a metric by $d(X, Y) = \|X - Y\|$. Let $\Sigma = \{X \mid \det(X) = 0\}$. Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Find the minimum distance from A to Σ and exhibit an $S \in \Sigma$ that achieves this minimum.

Problem 2.2.18 (Su80) *Let $S \subset \mathbb{R}^3$ denote the ellipsoidal surface defined by*

$$2x^2 + (y - 1)^2 + (z - 10)^2 = 1.$$

Let $T \subset \mathbb{R}^3$ be the surface defined by

$$z = \frac{1}{x^2 + y^2 + 1}.$$

Prove that there exist points $p \in S$, $q \in T$, such that the line $\overline{pq}$ is perpendicular to S at p and to T at q.

Problem 2.2.19 (Sp80) *Let P_2 denote the set of real polynomials of degree $\leqslant 2$. Define the map $J : P_2 \to \mathbb{R}$ by*

$$J(f) = \int_0^1 f(x)^2\, dx\,.$$

Let $Q = \{f \in P_2 \mid f(1) = 1\}$. Show that J attains a minimum value on Q and determine where the minimum occurs.

Problem 2.2.20 (Su79) *Let X be the space of orthogonal real $n \times n$ matrices. Let $v_0 \in \mathbb{R}^n$. Locate and describe the elements of X, where the map*

$$f : X \to \mathbb{R}, \qquad f(A) = \langle v_0, Av_0 \rangle$$

takes its maximum and minimum values.

Problem 2.2.21 (Fa78) *Let $W \subset \mathbb{R}^n$ be an open connected set and f a real valued function on W such that all partial derivatives of f are 0. Prove that f is constant.*

Problem 2.2.22 (Sp77) *In $\mathbb{R}^2$, consider the region $\mathcal{A}$ defined by $x^2 + y^2 > 1$. Find differentiable real valued functions f and g on $\mathcal{A}$ such that $\partial f/\partial x = \partial g/\partial y$ but there is no real valued function h on $\mathcal{A}$ such that $f = \partial h/\partial y$ and $g = \partial h/\partial x\cdot$*

Problem 2.2.23 (Sp77) *Suppose that $u(x, t)$ is a continuous function of the real variables x and t with continuous second partial derivatives. Suppose that u and its first partial derivatives are periodic in x with period 1, and that*

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}\cdot$$

Prove that

$$E(t) = \frac{1}{2}\int_0^1 \left(\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right) dx$$

is a constant independent of t.

Problem 2.2.24 (Su77) *Let $f(x, t)$ be a C^1 function such that $\partial f/\partial x = \partial f/\partial t$. Suppose that $f(x, 0) > 0$ for all x. Prove that $f(x, t) > 0$ for all x and t.*

Problem 2.2.25 (Fa77) *Let $f : \mathbb{R}^n \to \mathbb{R}$ have continuous partial derivatives and satisfy*

$$\left|\frac{\partial f}{\partial x_j}(x)\right| \leqslant K$$

for all $x = (x_1, \ldots, x_n)$, $j = 1, \ldots, n$. Prove that

$$|f(x) - f(y)| \leqslant \sqrt{n}K\|x - y\|$$

(where $\|u\| = \sqrt{u_1^2 + \cdots + u_n^2}$).

Problem 2.2.26 (Fa83, Sp87) *Let $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be a function which is continuously differentiable and whose partial derivatives are uniformly bounded:*

$$\left|\frac{\partial f}{\partial x_i}(x_1, \ldots, x_n)\right| \leqslant M$$

for all $(x_1, \ldots, x_n) \neq (0, \ldots, 0)$. Show that if $n \geqslant 2$, then f can be extended to a continuous function defined on all of $\mathbb{R}^n$. Show that this is false if $n = 1$ by giving a counterexample.

Problem 2.2.27 (Sp79) *Let $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be differentiable. Suppose*

$$\lim_{x\to 0} \frac{\partial f}{\partial x_j}(x)$$

exists for each $j = 1, \ldots, n$.

1. *Can f be extended to a continuous map from $\mathbb{R}^n$ to $\mathbb{R}$?*

2. *Assuming continuity at the origin, is f differentiable from $\mathbb{R}^n$ to $\mathbb{R}$?*

Problem 2.2.28 (Sp82) *Let $f : \mathbb{R}^2 \to \mathbb{R}$ have directional derivatives in all directions at the origin. Is f differentiable at the origin? Prove or give a counterexample.*

Problem 2.2.29 (Fa78) *Let $f : \mathbb{R}^n \to \mathbb{R}$ have the following properties: f is differentiable on $\mathbb{R}^n \setminus \{0\}$, f is continuous at 0, and*

$$\lim_{p \to 0} \frac{\partial f}{\partial x_i}(p) = 0$$

for $i = 1, \ldots, n$. Prove that f is differentiable at 0.

Problem 2.2.30 (Su78) *Let $U \subset \mathbb{R}^n$ be a convex open set and $f : U \to \mathbb{R}^n$ a differentiable function whose partial derivatives are uniformly bounded but not necessarily continuous. Prove that f has a unique continuous extension to the closure of U.*

Problem 2.2.31 (Fa78)

1. *Show that if $u, v : \mathbb{R}^2 \to \mathbb{R}$ are continuously differentiable and $\dfrac{\partial u}{\partial y} = \dfrac{\partial v}{\partial x}$, then $u = \dfrac{\partial f}{\partial x}$, $v = \dfrac{\partial f}{\partial y}$ for some $f : \mathbb{R}^2 \to \mathbb{R}$.*

2. *Prove there is no $f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ such that*

$$\frac{\partial f}{\partial x} = \frac{-y}{x^2 + y^2} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{x}{x^2 + y^2}.$$

Problem 2.2.32 (Su79) *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be such that*

$$f^{-1}(0) = \{v \in \mathbb{R}^3 \mid \|v\| = 1\}.$$

Suppose f has continuous partial derivatives of orders $\leqslant 2$. Is there a $p \in \mathbb{R}^3$ with $\|p\| \leqslant 1$ such that

$$\frac{\partial^2 f}{\partial x^2}(p) + \frac{\partial^2 f}{\partial y^2}(p) + \frac{\partial^2 f}{\partial z^2}(p) \geqslant 0 \ \ ?$$

Problem 2.2.33 (Sp92) *Let f be a differentiable function from $\mathbb{R}^n$ to $\mathbb{R}^n$. Assume that there is a differentiable function g from $\mathbb{R}^n$ to $\mathbb{R}$ having no critical points such that $g \circ f$ vanishes identically. Prove that the Jacobian determinant of f vanishes identically.*

Problem 2.2.34 (Fa83) *Let $f, g : \mathbb{R} \to \mathbb{R}$ be smooth functions with $f(0) = 0$ and $f'(0) \neq 0$. Consider the equation $f(x) = tg(x)$, $t \in \mathbb{R}$.*

1. *Show that in a suitably small interval $|t| < \delta$, there is a unique continuous function $x(t)$ which solves the equation and satisfies $x(0) = 0$.*

2. *Derive the first order Taylor expansion of $x(t)$ about $t = 0$.*

Problem 2.2.35 (Sp78) *Consider the system of equations*

$$\begin{aligned} 3x + y - z + u^4 &= 0 \\ x - y + 2z + u &= 0 \\ 2x + 2y - 3z + 2u &= 0 \end{aligned}$$

1. *Prove that for some $\varepsilon > 0$, the system can be solved for (x, y, u) as a function of $z \in [-\varepsilon, \varepsilon]$, with $x(0) = y(0) = u(0) = 0$. Are such functions $x(z)$, $y(z)$ and $u(z)$ continuous? Differentiable? Unique?*

2. *Show that the system cannot be solved for (x, y, z) as a function of $u \in [-\delta, \delta]$, for all $\delta > 0$.*

Problem 2.2.36 (Sp81) *Describe the two regions in (a, b)-space for which the function*

$$f_{a,b}(x, y) = ay^2 + bx$$

restricted to the circle $x^2 + y^2 = 1$, has exactly two, and exactly four critical points, respectively.

Problem 2.2.37 (Fa87) *Let u and v be two real valued C^1 functions on $\mathbb{R}^2$ such that the gradient ∇u is never 0, and such that, at each point, ∇v and ∇u are linearly dependent vectors. Given $p_0 = (x_0, y_0) \in \mathbb{R}^2$, show that there is a C^1 function F of one variable such that $v(x, y) = F(u(x, y))$ in some neighborhood of p_0.*

Problem 2.2.38 (Fa94) *Let f be a continuously differentiable function from $\mathbb{R}^2$ into $\mathbb{R}$. Prove that there is a continuous one-to-one function g from $[0, 1]$ into $\mathbb{R}^2$ such that the composite function $f \circ g$ is constant.*

Problem 2.2.39 (Su84) *Let $f : \mathbb{R} \to \mathbb{R}$ be C^1 and let*

$$\begin{aligned} u &= f(x) \\ v &= -y + xf(x). \end{aligned}$$

If $f'(x_0) \neq 0$, show that this transformation is locally invertible near (x_0, y_0) and the inverse has the form

$$\begin{aligned} x &= g(u) \\ y &= -v + ug(u). \end{aligned}$$

Problem 2.2.40 (Su78, Fa99) *Let $M_{n\times n}$ denote the vector space of real $n\times n$ matrices. Define a map $f : M_{n\times n} \to M_{n\times n}$ by $f(X) = X^2$. Find the derivative of f.*

Problem 2.2.41 (Su82) *Let $M_{2\times 2}$ be the four-dimensional vector space of all 2×2 real matrices and define $f : M_{2\times 2} \to M_{2\times 2}$ by $f(X) = X^2$.*

1. *Show that f has a local inverse near the point*

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

2. *Show that f does not have a local inverse near the point*

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Problem 2.2.42 (Fa80) *Show that there is an $\varepsilon > 0$ such that if A is any real 2×2 matrix satisfying $|a_{ij}| \leqslant \varepsilon$ for all entries a_{ij} of A, then there is a real 2×2 matrix X such that $X^2 + X^t = A$, where X^t is the transpose of X. Is X unique?*

Problem 2.2.43 (Sp96) *Let $M_{2\times2}$ be the space of 2×2 matrices over $\mathbb{R}$, identified in the usual way with $\mathbb{R}^4$. Let the function F from $M_{2\times2}$ into $M_{2\times2}$ be defined by*

$$F(X) = X + X^2.$$

Prove that the range of F contains a neighborhood of the origin.

Problem 2.2.44 (Fa78) *Let $M_{n\times n}$ denote the vector space of $n \times n$ real matrices. Prove that there are neighborhoods U and V in $M_{n\times n}$ of the identity matrix such that for every A in U, there is a unique X in V such that $X^4 = A$.*

Problem 2.2.45 (Sp79, Fa93) *Let $M_{n\times n}$ denote the vector space of $n \times n$ real matrices for $n \geqslant 2$. Let $\det : M_{n\times n} \to \mathbb{R}$ be the determinant map.*

1. *Show that* det *is C^∞.*

2. *Show that the derivative of* det *at $A \in M_{n\times n}$ is zero if and only if A has rank $\leqslant n-2$.*

Problem 2.2.46 (Fa83) *Let $F(t) = \big(f_{ij}(t)\big)$ be an $n\times n$ matrix of continuously differentiable functions $f_{ij} : \mathbb{R} \to \mathbb{R}$, and let*

$$u(t) = \operatorname{tr}\left(F(t)^3\right).$$

Show that u is differentiable and

$$u'(t) = 3\operatorname{tr}\left(F(t)^2 F'(t)\right).$$

Problem 2.2.47 (Fa81) *Let $A = \big(a_{ij}\big)$ be an $n \times n$ matrix whose entries a_{ij} are real valued differentiable functions defined on $\mathbb{R}$. Assume that the determinant* $\det(A)$ *of A is everywhere positive. Let $B = \big(b_{ij}\big)$ be the inverse matrix of A. Prove the formula*

$$\frac{d}{dt}\log(\det(A)) = \sum_{i,j=1}^{n} \frac{da_{ij}}{dt} b_{ji}.$$

Problem 2.2.48 (Sp03) 1. *Prove that there is no continuously differentiable, measure-preserving bijective function $f : \mathbb{R} \to \mathbb{R}_+$.*

2. *Find an example of a continuously differentiable, measure-preserving bijective function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}_+$.*

2.3 Integral Calculus

Problem 2.3.1 (Sp78) *What is the volume enclosed by the ellipsoid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1?$$

Problem 2.3.2 (Sp78) *Evaluate*

$$\int\int_{\mathcal{A}} e^{-x^2-y^2}\, dxdy\,,$$

where $\mathcal{A} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leqslant 1\}$.

Problem 2.3.3 (Sp98) *Given the fact that* $\displaystyle\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$*, evaluate the integral*

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+(y-x)^2+y^2)} dx\, dy\,.$$

Problem 2.3.4 (Fa86) *Evaluate*

$$\int\int_{\mathcal{R}} (x^3 - 3xy^2)\, dxdy\,,$$

where

$$\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \mid (x+1)^2 + y^2 \leqslant 9, \quad (x-1)^2 + y^2 \geqslant 1\}.$$

Problem 2.3.5 (Fa98) *Let* $\varphi(x, y)$ *be a function with continuous second order partial derivatives such that*

1. $\varphi_{xx} + \varphi_{yy} + \varphi_x = 0$ *in the punctured plane* $\mathbb{R}^2 \setminus \{0\}$,
2. $r\varphi_x \to \dfrac{x}{2\pi r}$ *and* $r\varphi_y \to \dfrac{y}{2\pi r}$ *as* $r = \sqrt{x^2 + y^2} \to 0$.

Let C_R *be the circle* $x^2 + y^2 = R^2$*. Show that the line integral*

$$\int_{C_R} e^x(-\varphi_y\, dx + \varphi_x\, dy)$$

is independent of R, and evaluate it.

Problem 2.3.6 (Sp80) *Let* $\mathcal{S} = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ *denote the unit sphere in* $\mathbb{R}^3$*. Evaluate the surface integral over* $\mathcal{S}$*:*

$$\int\int_{\mathcal{S}} (x^2 + y + z)\, dA\,.$$

Problem 2.3.7 (Sp81) *Let $\vec{\imath}$, $\vec{\jmath}$, and $\vec{k}$ be the usual unit vectors in $\mathbb{R}^3$. Let $\vec{F}$ denote the vector field*

$$(x^2 + y - 4)\vec{\imath} + 3xy\vec{\jmath} + (2xz + z^2)\vec{k}.$$

1. *Compute $\nabla \times \vec{F}$ (the curl of $\vec{F}$).*
2. *Compute the integral of $\nabla \times \vec{F}$ over the surface $x^2 + y^2 + z^2 = 16$, $z \geqslant 0$.*

Problem 2.3.8 (Sp91) *Let the vector field F in $\mathbb{R}^3$ have the form*

$$F(r) = g(\|r\|)r \qquad (r \neq (0, 0, 0)),$$

where g is a real valued smooth function on $(0, \infty)$ and $\|\cdot\|$ denotes the Euclidean norm. (F is undefined at $(0, 0, 0)$.) Prove that

$$\int_C F \cdot ds = 0$$

for any smooth closed path C in $\mathbb{R}^3$ that does not pass through the origin.

Problem 2.3.9 (Fa91) *Let $\mathcal{B}$ denote the unit ball of $\mathbb{R}^3$, $\mathcal{B} = \{r \in \mathbb{R}^3 \mid \|r\| \leqslant 1\}$. Let $J = (J_1, J_2, J_3)$ be a smooth vector field on $\mathbb{R}^3$ that vanishes outside of $\mathcal{B}$ and satisfies $\nabla \cdot \vec{J} = 0$.*

1. *For f a smooth, scalar-valued function defined on a neighborhood of $\mathcal{B}$, prove that*
$$\int_{\mathcal{B}} (\nabla f) \cdot \vec{J}\, dxdydz = 0.$$
2. *Prove that*
$$\int_{\mathcal{B}} J_1\, dxdydz = 0.$$

Problem 2.3.10 (Fa94) *Let $\mathcal{D}$ denote the open unit disc in $\mathbb{R}^2$. Let u be an* eigenfunction for the Laplacian *in $\mathcal{D}$; that is, a real valued function of class C^2 defined in $\overline{\mathcal{D}}$, zero on the boundary of $\mathcal{D}$ but not identically zero, and satisfying the differential equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \lambda u\,,$$

where λ is a constant. Prove that

$$(*) \qquad \iint_{\mathcal{D}} |\mathrm{grad}\, u|^2\, dxdy + \lambda \iint_{\mathcal{D}} u^2 dxdy = 0\,,$$

and hence that $\lambda < 0$.

Problem 2.3.11 (Fa03) *Let $\lambda, a \in \mathbb{R}$, with $a < 0$. Let $u(x, y)$ be an infinitely differentiable function defined on an open neighborhood of closed unit disc $\mathcal{D}$ such that*

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= \lambda u \qquad \text{in int}(\mathcal{D}) \\ D_n u &= a u \qquad \text{in } \partial\mathcal{D}. \end{aligned}$$

Here $D_n u$ denotes the directional derivative of u in the direction of the outward unit normal. Prove that if u is not identically zero in the interior of $\mathcal{D}$ then $\lambda < 0$.

Problem 2.3.12 (Sp92) *Let f be a one-to-one C^1 map of $\mathbb{R}^3$ into $\mathbb{R}^3$, and let J denote its Jacobian determinant. Prove that if x_0 is any point of $\mathbb{R}^3$ and $Q_r(x_0)$ denotes the cube with center x_0, side length r, and edges parallel to the coordinate axes, then*

$$|J(x_0)| = \lim_{r \to 0} r^{-3} \operatorname{vol}(f(Q_r(x_0))) \leqslant \limsup_{x \to x_0} \frac{\|f(x) - f(x_0)\|^3}{\|x - x_0\|^3}.$$

Here, $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^3$.

Springer
http://www.springer.com/978-0-387-20429-1

Berkeley Problems in Mathematics
de Souza, P.N.; Silva, J.-N.
2004, XIV, 593 p. 3 illus., Hardcover
ISBN: 978-0-387-20429-1