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Solutions Manual for Students

An Introduction to Mathematical Finance with Applications

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No Exercises for Chapter 1

This brief chapter is a primer on financial markets and has no exercises.

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2.1 Exercises

2.1.1 Conceptual Exercises

2.1. A physicist summed up the growth rate of an initial sum of money held over a fixed time span as follows: “If simple interest is applied during the time span, then the initial sum will grow with uniform (constant) velocity as the interest rate increases. If periodic compound interest is applied, then the growth of the initial sum will accelerate as interest rate increases.” Do you agree with this interpretation? Justify your answer.

Solution 2.1. Yes.

2.2. Theorem 2.1 on page 27 yields that k-periodic compounding of a principal $\mathcal{F}_0$ at r per annum over a time span of τ years consisting of x interest periods gives a future value,

$$\mathcal{F}_x = \left(1 + \frac{r}{k}\right)^x \mathcal{F}_0,$$

where x is a nonnegative real number and $0 \leq \frac{r}{k} < 1$ for $k = 1, 2, \ldots$. Does $\mathcal{F}_x$ increase or decrease as k increases indefinitely? Justify your answer.

Solution 2.2. Increase.

2.3. Suppose you purchase a lottery ticket for \$2. What is your return rate if you lose? What if you win \$200 million? Express your answer as a percentage.

Solution 2.3.

2.4. Consider an investment that promises a fixed sequence of future cash dividends. Briefly explain why an increase in the required return rate on the investment would decrease the current value of the investment.

Solution 2.4.

2.5. Explain what the following is stating financially about the start-up: "A start-up's NPV at 30% is $35,000."

Solution 2.5.

2.6. A friend borrows $1,000 from a lender that gives him the loan as a simple ordinary annuity at a fixed interest rate over two years with a payment of $100 per month. If your friend carries the loan to its full term, then he will have to pay more than the amount of the loan in just interest. True or false? Justify your answer.

Solution 2.6. True.

2.7. A loan with a fixed payment of $1,000 per month for five years has the stipulation that you will have to pay all the interest due on the loan even if you pay the loan off early. If immediately after you receive the loan you want to pay it off, how much do you have to pay the lender?

Solution 2.7. $60,000.

2.8. How would you modify the interpretation of the noncallable bond pricing formula (2.28) on page 69 to obtain the current price of a callable bond, i.e. a bond where the issuer has the right, but not the obligation, to redeem (in practice, cancel) the bond before maturity? Use a single call date, i.e. a date when the issuer can redeem the bond before maturity. Compare the price of a callable bond with a noncallable one. Corporations issue callable bonds because if interest rates go down, they can call their bonds and refinance their debt at a lower interest rate.

Solution 2.8.

2.9. How would you modify the interpretation of the noncallable bond pricing formula (2.78) on page 69 to obtain the current price of a puttable bond, i.e. a bond where the investor has the right, but not the obligation, to redeem the bond before maturity? Use a single put date, i.e. a date on which the investor can redeem the bond before maturity. Compare the price of a puttable bond with a noncallable one. Investors buy puttable bonds because if interest rates increase they can sell back their original bonds at the put value and invest the proceeds in a higher interest rate bond.

Solution 2.9.

2.1.2 Application Exercises

2.10. Consider a principal $\mathcal{F}_0$ that is held for n_{exact} days during a non-leap year at the simple interest rate r. By what percent is the simple interest amount using Banker's Rule greater than the simple interest amount employing exact time and exact interest?

Solution 2.10. 1.4%.

2.11. (Selling or buying a loan) On November 12, 2007, a borrower closes on a loan for $\$176,000$ at 6.25% per annum compounded daily. Repayment of the loan's maturity value (principal plus interest) is due in full on April 15, 2008. Suppose that the fine print of the original loan stipulated that the lender can sell the loan on the condition that the interest rate and maturity date remain the same. The lender sells the loan to another lender on January 5, 2008. The new lender agrees to purchase the debt for the present value of the maturity value at 10% per annum compounded daily. Assume that interest compounds daily and the borrower does not default on the loan. Use Banker's Rule when solving the following:

a) What is the maturity value of the loan?
b) What will the first lender receive for selling the loan? Is any profit made by the first lender?
c) What profit will the second lender make on the loan's maturity date if the conditions of the original loan are unchanged?
d) Though the original interest rate and maturity date are unchanged, the second lender is not prevented from reissuing the loan with a new start date set as the loan's purchase date and with the new loan's principal set as the value of the loan on the purchase date. Does the second lender make more profit by resetting the loan in this way? Explain.

Solution 2.11.

a) \$180,799.988.
b) The first lender receives \$175,798.723 and the net profit is -\$1,858.891.
c) The profit is \$1,891.771.
d) The two profits are identical, so no extra profit is made from resetting the loan.

2.12. For an interest rate of 4% per year, compare the future value 2 years from now to which \$10,000 increases under daily compounding versus continuous compounding. Assume 365 days per year and express your answer as a fractional-difference percentage of the daily compounding case.

Solution 2.12. The fractional difference as a percentage of the dailing compounding case is:

$$\frac{\mathcal{F}_{\mathrm{cts}}(2)}{\mathcal{F}(2)} - 1 = \frac{\$10,832.87}{\$10,832.82} - 1 = 0.0005\%.$$

2.13. Suppose that at the start of college you have \$1,000 to invest and would like for it to grow to \$1,250 at the end of your senior year through monthly compounding. Determine the general formula for the interest rate required for the growth and then compute the interest rate.

Solution 2.13.

2.14. Assume that college tuition is currently 30 times its cost 15 years ago. Assuming annual compounding, what is the interest rate r that gives the rate of increase in tuition?

Solution 2.14. The interest rate is 25.5% per year.

2.15. How much should you have today in an account with monthly compounding and annual interest rate of 4% to receive \$1,000 per month forever?

Solution 2.15.

2.16. (Equity in a house) A couple purchased a house 7 years ago for $\$375,000$. The house was financed by paying 20% down and signing a 30-year mortgage at 6.5% on the unpaid balance. The net market value of the house is now $\$400,000$. Assume that the couple wishes to sell the house.

a) How much equity (to the nearest dollar) does the family have in the house now, after making 84 monthly payments?
b) Find the first interest payment $\mathcal{I}_1$ and the 84th interest payment $\mathcal{I}_{84}$.

Solution 2.16.

2.17. (Social Security benefits) We present a simplified problem to illustrate Social Security benefits. A college graduate begins work at age 22. She has an annual income of $\$70,000$ until retirement (a simplification), pays 12.4% of this income into Social Security each year, and retires at age 65 with Social Security benefits of $\$20,000$ annually. How long must she live before the present value of these benefits equals the present value of her annual contributions? In other words, how long must she live after retirement to get back the full value of her contributions to Social Security? Will she get the entire value? Assume a discount rate of 4% per year, no change in her salary, and that all payments and benefits occur at the end of each year.

Solution 2.17. She never gets back the entire value of her Social Security contribution if her Social Security benefit is $\$20,000$ annually after retirement.

2.18. (Worker's compensation) The usual legal settlement for an industrial accident is the present value of the employee's lifetime earnings. If you expect to work for 10 more years, make $\$70,000$ a year in the next 2 years, and get a raise of $\$5,000$ every 2 years, what would be your settlement? Assume an annual discount rate of 4% in the first 5 years and 6% in the second 5 years, and that your paycheck is received at the end of each year.

Solution 2.18. The amount of settlement is \$625,306.76.

2.19. (Bonds) Suppose that you bought a 30-year bond with 4% annual coupon rate. You wish to sell that bond at a later date when the remaining life of the bond is 2.5 years and the current YTM of your bond has declined to 2%.

a) What is the fair value, as determined by the present value method, of the bond at the time of your sale?
b) How much would you earn if you purchased the bond for \$1,000, sold it at the fair value, and did not reinvest the coupon payments?

Solution 2.19.

a) The fair value of your bond is \$1,000.
b) The total earned is \$1,100.

2.20. (Bonds) Bonds are generally quoted as a percentage of their face value. A bond selling at 99.2% of its face value is quoted as 99.2. The following information for a Treasury bond was provided by the WSJ market data center on December 4, 2013.

Maturity	Coupon	Current price	Previous price	Change	Yield
11/30/20	2.000	99.20	99.00	0.203	2.123

The coupon column refers to the annual coupon rate. Verify that the last column indicates YTM.

Solution 2.20. *Hint*: apply Equation (2.76) and use a software to find the YTM.

Purchasing a house

The remaining Application Exercises deal with purchasing a house. Assume that you are currently renting an apartment for \$1,040 per month and you have been considering buying a house. You have saved \$10,000 towards a down payment for the house.

A salesperson informs you that he has a new house for sale, where the house and land were independently appraised at \$200,000, but are being sold by the builder at a discount price of \$185,000. The builder wants to get rid of the property quickly because the house is the last one to be sold in the development and the builder is moving on to construction of a new development.

The salesperson connects you with his in-house lender, to whom you give details about your income and grant permission to review your credit and eligibility for a loan. You inform her that you are prepared to make a down payment of \$10,000 towards the house if necessary. She gets back to you with good news that, if you put $8,100 towards the house, then they can give you a 30-year loan for the balance of $176,900 at 6.25% per annum (compounded monthly). Note that lenders require the house to appraise at or above the purchase price; otherwise, they may reject the loan or require more down payment. The lender computes the monthly mortgage payment at $1,089.20. She informs you that the remaining $1,900 of your $10,000 can be used towards costs associated with the final evaluation of the physical property and the closing of the purchase (property inspector fee, termite inspector fee, official survey, attorney fees, etc.) The builder agrees to pay for costs beyond your $1,900 and make necessary repairs you identify during the period you have to inspect the property (the due diligence period).

Hearing the news about your qualification for the loan, the salesperson asks you how much rent you are now paying. When you inform him that you pay $1,040 per month, he quickly points out that it would be a mere extra $50 per month for you to meet the mortgage payments. He emphasizes that it is better to own than to rent, especially if the mortage is just a bit more than your current rent.

You are thrilled! After the excitement subsides, however, you decide to run the numbers yourself to make sure you get a clear understanding of what you are getting into financially.[19] The problems in this project help guide you through some of this analysis.

2.21. Show that the monthly loan payment on the unpaid principal balance of $176,900 is $1,089.20.

Solution 2.21. *Hint*: apply Equation (2.62).

2.22. In addition to closing fees paid to settle the loan, there are expenses beyond the monthly mortgage payments.

First, since your deposit was less than 20% of the purchase price, you are required to take out a private mortgage insurance (PMI) to protect the lender if you default on the loan. The PMI typically lasts until the unpaid principal balance of the mortgage is paid down to 80% of the original value of the house, where the house's original value is the lesser of the purchase price and the official appraised value of the house used in closing the sale. Note that the bank may also require your payment history to be in good standing (e.g. no late payments in the past year or two) before removing PMI. Of course, if the value of the house increases nontrivially, you may be able to remove the PMI earlier. Suppose that the PMI is $141.52 per month.

Second, along with PMI, you have to pay for hazard insurance to protect the house against fire, etc. Assume that the hazard insurance is $36.50 per month.

[19] Mortgages on a house are generally modeled as simple ordinary annuities by lenders.

Third, you have to pay property taxes to the tax district (e.g. county and city) where the house is located. The property (i.e. house and land) will be valued within your tax district, which is a valuation that is separate from the appraisal done when purchasing the house. The resulting tax district's valuation is the taxable value of the house and is the amount to which the property tax rate will be applied. Suppose that the annual property tax rate is 1.3% and the taxable value of the property is $\$189,986$. For this project, the taxable property value is less than the appraised value (i.e. $\$200,000$) used for the purchase. Sometimes, however, the taxable value can be higher which was not uncommon in the aftermath of the 2008 mortgage crisis.

The PMI, hazard insurance, and property tax payments are in addition to the monthly loan payment and all together they form a single payment you make to the lender. The lender or a company hired by the lender manages these payments by taking out the portion for the loan payment (principal plus interest) and depositing the rest into an escrow account, which is used to pay the annual insurance premiums and property taxes on behalf of the borrower.

Finally, assume that the property is in a housing development that comes with a mandatory Homeowners Association (HOA) fee. The HOA fee is used to maintain the grounds, roads, etc. in the development. If you do not pay the fee, the HOA can foreclose on your property. Assume an HOA fee of $100 per month.

a) What is the estimated total monthly PITI, i.e. the minimum monthly payment covering the principal, interest, taxes, and (hazard) insurance?
b) Identify two other mandatory house expenses that are outside of the PITI payment and other basic house costs like utilities and repairs. Do exclude costs like groceries, tuition, medical expenses, etc., which are more associated with running a home. What is the minimum monthly cost of the house during the first year if you now include these two mandatory house expenses and PITI? Which of these housing costs will likely increase in the future?
c) What is your opinion about the salesperson's pitch about the cost of renting versus buying a house?

Solution 2.22.

a)
b) $1,573.04.
c)

2.23. Fill out the amortization schedule below, which is for the first five months of the loan.

Payment #	**Payment** ($\mathcal{P}$)	**Principal** ($\mathfrak{P}_\ell$)	**Interest** ($\mathcal{I}_\ell$)	**Bal.** ($\mathcal{B}_\ell$)
1	1,089.20	167.85	921.35	176,732.15
2	1,089.20			
3	1,089.20			
4	1,089.20			
5	1,089.20			

Solution 2.23.

2.24. Are there discrepancies in the above amortization table? If so, explain how to remove them mathematically.

Solution 2.24. Yes. These discrepancies are due to the rounding off of the decimals. If we work to four decimal places and then round off at the end, the discrepancies are removed.

For the remaining problems, note that only the payments towards principal and interest (PI) are relevant to the loan's balance. Costs associated with property taxes, hazard insurance, PMI, HOA, etc. are separate expenses and do not impact the balance of the loan. Such costs are typically not included in the loan's cost.

2.25. Using a software, compute the numbered payment at which the unpaid balance on the loan will first dip below 80% of the original value of the house. Roughly how many years and months does it take to reach that balance? If the value of the house has not decreased below its original value at that point in time, you would stop paying PMI henceforth.

Solution 2.25.

2.26. Determine the total amount you would pay into the mortgage, excluding escrow payments, if you make only the minimum payment over the full 30 years. What is the total cost of the mortgage? Is it more than the mortgage?

Solution 2.26.

2.27. Estimate the number of years and months it would take to pay off the mortgage if you double your monthly payments.

Solution 2.27. Eight years and ten months.

2.28. Estimate the total you would pay into the mortgage if you double your monthly payments. What is the total cost of the mortgage for doubled payments? Is it more than the mortgage?

Solution 2.28. The total cost is $54,010.40.

2.1.3 Theoretical Exercises

2.29. Suppose that an initial capital $\mathcal{F}_0$ grows to an amount $\mathcal{F}(\tau)$ over a time span τ. A mathematician modeling the growth observes that for all time spans x and h, the accumulated amount $\mathcal{F}(x)$ is a differentiable function satisfying the following:

$$\mathcal{F}(x+h) = \mathcal{F}(x) + \mathcal{F}(h) - \mathcal{F}_0, \qquad \mathcal{F}(0) = \mathcal{F}_0, \qquad \frac{d\mathcal{F}}{dx}(0) = r\,\mathcal{F}_0,$$

where $r \geq 0$. Determine the type of growth model, i.e. find $\mathcal{F}(x)$.

Solution 2.29.

$$\mathcal{F}(x) = (1 + r\,x)\mathcal{F}_0.$$

This is simple interest growth.

2.30. Derive Equation (2.18) on page 26: $G'(x) = G(x)\,G'(0)$.

Solution 2.30. *Hint*: apply the definition of the derivative of $G(x)$.

2.31. (Capital after spending, inflation, and interest) Consider the following setup:

- Begin with an initial capital $C(0)$ in an interest-bearing account and let $C(n)$ be the remaining capital at the end of the nth year.
- Assume an interest rate r is applied at the end of each year to the capital remaining on that date.
- At the end of the first year, assume that an amount S was spent from $C(0)$ on goods and services, and money will be spent on similar goods and services in each of the subsequent years.
- Suppose that the amount spent at the end of any specific year is the total amount spent by the end of the first year increased in subsequent years at the annual inflation rate $\mathtt{i}$ compounding annually until the end of the specified year. Assume that $r > \mathtt{i}$ since investors are not interested in a market interest rate that is below the inflation rate.

a) Show that the total capital at the end of the $(n+1)$st year can be expressed recursively as follows in terms of the capital at the end of the previous year, taking into account spending, inflation, and interest growth:

$$C(n+1) = (1+r)\,[C(n) - (1+\mathtt{i})^n S]. \tag{2.1}$$

b) Use induction to show that

$$C(n) = (1+r)^n \left[C_0 - \frac{1+r}{r-\mathtt{i}}S\right] + \left(\frac{1+r}{r-\mathtt{i}}\right)(1+\mathtt{i})^n\,S.$$

Solution 2.31.

2.32. Suppose that after this year your grandmother will receive regular payments from a retirement fund, but she has to choose between two options for how to receive the payments during $n+1$ years. She does not plan to spend any of the money until after the $n+1$ years. Assume that she will save all the disbursements in an account that accrues the payments as a simple ordinary annuity with k-periodic compounding at interest rate r (e.g. each payment date coincides with an interest date).

The payment start date will differ for the two plans, but both payment options will have the last payment at the start of the last interest period during the $(n+1)$st year. Your job is to help her choose between the two options.

a) **(A general future value formula)** The current problem determines a general formula that incorporates the future value of the payments into your grandmother's account. Suppose that regular payments of $\mathcal{P}$ into an interest bearing account form a simple ordinary annuity with k-periodic compounding at interest rate r. Assume that the account receives the first payment at the end of the first interest period of a certain year and the last payment at the end of the Nth interest period going forward, with no payment at the end of the $(N+1)$st interest period. Show that the amount accrued in the account at the end of the $(N+1)$st period is:

$$FV \equiv \left[\frac{(1+r/k)^{N+1} - (1+r/k)}{r/k}\right] \mathcal{P}, \tag{2.2}$$

where N is the total number of payments into the account.

b) We now explore the future values associated with the following two plans for receiving payment.

   i. *Plan A.* Assume that Plan A begins officially at the start of next year with payments of A starting at the end of the first interest period of next year. Show that the total amount she would accrue at the end of the $(n+1)$st year is:

$$\mathrm{FV}_A \equiv \left[\frac{(1+r/k)^{(n+1)k} - (1+r/k)}{r/k}\right] A.$$

   ii. *Plan B.* Under Plan B, your grandmother receives payments of B with the choice of officially starting at the beginning of the $(q+1)$st year after Plan A starts and the first payment disbursing at the end of the first interest period of the official starting year. Show that the total amount she would accrue by the end of the $(n+1)$st year is

$$\mathrm{FV}_B \equiv \left[\frac{(1+r/k)^{[(n+1)-q]k} - (1+r/k)}{r/k}\right] B,$$

   where $q = 1, 2, \ldots$. Note that for $q = 0$ the two options coincide.

c) **(Choosing between Plans A and B)** Naturally, since Plan B starts out later than Plan A and both have the same last-payment date, the payment amount of Plan B has to be higher than that of Plan A, i.e. $B > A$. Suppose that the account's interest rate exceeds a threshold as follows:

$$r > k\left[(B/A)^{1/qk} - 1\right].$$

i. Show that there is no n such that the amounts accrued under both options are equal by the end of the $(n+1)$st year.
ii. Show that Plan A is superior to Plan B, i.e. prove $\mathrm{FV}_A > \mathrm{FV}_B$.

Solution 2.32.

a) *Hint*: the value at the end of the Nth interest period is:

$$\left[\frac{(1+r/k)^N - 1}{r/k}\right]\mathcal{P}.$$

b) Apply the formula in (2.4) using the following values for N.

i. **(Plan A)** The total number of payments received is $N = (n+1)k - 1$.
ii. **(Plan B)** The total number of payments received is $N = (n+1-q)k - 1$.

c) **(Choosing between Plans A and B)** *Hint*: first prove that $\mathrm{FV}_A \neq \mathrm{FV}_B$. Then suppose that $\mathrm{FV}_A < \mathrm{FV}_B$ and disapprove it.

2.33. (Relating present and future values of a generalized annuity) Using

$$\mathcal{S}_n = \sum_{\ell=0}^{n-1}\left[\prod_{j=0}^{\ell}\left(1 + \frac{r_{n+1-j}}{k}\right)\right]\mathcal{P}_{n-\ell},$$

verify the formula

$$\mathcal{A}_n = \frac{\mathcal{S}_n}{\prod_{j=1}^{n}\left(1 + \frac{r_j}{k}\right)},$$

where $n = 1, 2, \ldots, r_j > 0$ for $j = 1, \ldots, n$, and $r_{n+1} = 0$.

Solution 2.33. *Hint*: first rewrite the product in the right hand side of the formula of $\mathcal{A}_n$ in reverse order

$$\prod_{j=1}^{n}\left(1 + \frac{r_j}{k}\right) = \prod_{j=1}^{n}\left(1 + \frac{r_{n+1-j}}{k}\right),$$

Then change the index $j \to j' = j - \ell$.

2.34. (Bonds) Given a coupon bond described by Equation (2.76) on page 68, find the future value at maturity of the bond's cash flow.

Solution 2.34.

2.35. (Bonds) Show that for a coupon bond, its yield to maturity (r_Y), current yield (r) and coupon rate (r_C) have the following relationships:

a) A bond trades at a discount if and only if $r_Y > r > r_C$.
b) A bond trades at a premium if and only if $r_Y < r < r_C$.

Solution 2.35. *Hint*: $\hat{r}\mathcal{A}_n^B = \mathcal{C} = \hat{r}_C\mathcal{M} < \hat{r}_C\mathcal{A}_n^B$, where $\mathcal{A}_n^B = B(n)$ is the present price of the bond. A bond is discount (resp., premium) iff $\mathcal{A}_n^B < \mathcal{M}$ (resp., $\mathcal{A}_n^B > \mathcal{M}$).

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Chapter 3

Selected Solutions for Chapter 3

3.1 Exercises

3.1.1 Conceptual Exercises

3.1. An investor plans to create a portfolio of ten stocks by shorting all of them. Can he use the Markowitz theory presented in this chapter? Explain your answer.

Solution 3.1.

3.2. Can you find five examples of pairs of stocks in the U.S. that are negatively correlated? Are such occurrences common?

Solution 3.2.

3.3. The volatility of the log-return rate of a portfolio over 120 days is roughly 11 times the volatility over a day. Agree or disagree? Explain your answer.

Solution 3.3. Agree.

3.4. Decide whether you agree or disagree with the statements below. Justify your answer.

a) "Investors who are not risk averse are irrational."
b) "When the risk of a portfolio vanishes, the risk of each security has to vanish."

Solution 3.4.

a) Disagree.
b) Disagree. *Hint*: consider $\rho = -1$.

3.5. A clever wealth manager constructed a portfolio of stocks such that the portfolio has no risk *and* has an expected return of 25%. What is the probability that the portfolio return rate will actually be 25%?

Solution 3.5.

3.6. Show that Problem I on page 120 is equivalent to Problem II on page 121, i.e. show that these two optimization problems have the same set of solutions for all $c > 0$.

Solution 3.6. *Hint*: prove that the solution $\boldsymbol{w}_I$ of Problem I also solves Problem II, and vice versa.

3.7. Explain the financial meaning of minimizing the function $f_P(\boldsymbol{w}) = \frac{\sigma_P^2(\boldsymbol{w})}{\mu_P(\boldsymbol{w})}$, where $\mu_P(\boldsymbol{w})$ is the portfolio expected return.

Solution 3.7.

3.8. It can be shown that the covariance of the return of the global minimum-variance N-security portfolio with the return of any other efficient N-security portfolio is always $1/A$ (Exercise 3.28). Interpret this result.

Solution 3.8.

3.9. If your initial capital increases by \$100, then we would expect an increase in your utility. To which scenario would you assign a higher utility, assuming the same risk for both? Scenario A: initial capital of \$1,000. Scenario B: initial capital of \$10,000.

Solution 3.9. Higher utility is assigned to Scenario A.

3.10. Is $u(x) = a$ a utility function for a a constant? Justify your answer.

Solution 3.10.

3.11. Is $u(x) = 1 - \mathrm{e}^{-bx}$, where $b > 0$, a risk-averse, risk-neutral, or risk-seeking utility function? Justify your answer.

Solution 3.11.

3.1.2 Application Exercises

3.12. (Two securities) The table below gives a sample of artificial historical data for an energy company (Stock A) and a phone company (Stock B). Each indicated return rate is end-of-month to end-of-month and is based on adjusted closing prices. For example, $R^A_{\text{Jan}-2012}$ is the return rate from the last trading day in December 2011, to the last trading day in January 2012. Using the table, express your answers in percentages where appropriate.

Date	R^A_{monthly}	R^B_{monthly}
Jan-2012	2.45%	2.69%
Feb-2012	3.35%	1.81%
Mar-2012	3.24%	4.94%
Apr-2012	2.93%	5.88%
May-2012	6.13%	2.51%
Jun-2012	6.19%	-0.35%
Jul-2012	0.78%	1.59%
Aug-2012	-0.19%	-3.83%
Sep-2012	4.65%	5.24%
Oct-2012	3.53%	4.85%
Nov-2012	5.03%	2.48%
Dec-2012	-1.71%	4.03%

a) Sketch the graph of the monthly total return rates of each stock as a function of time during 2012. Briefly discuss the movement of the stocks during two equal-length-time periods in 2012.
b) Estimate the expected monthly total return rate of each stock in the table. What is your answer if you use data only from Dec-2011 to Jun-2012? Compute the monthly volatility of each stock for the year 2012.
c) Estimate the monthly variance of each stock and determine the covariance and correlation coefficient between the monthly total return rates for the two stocks during 2012. Is the result what you expected? Briefly discuss.
d) For a portfolio consisting of stocks A and B, use the data in the table to determine the portfolio's expected monthly total return rate and the portfolio's monthly risk under selection I, where funds are split evenly between the two stocks, and selection II, where two-thirds of your funds are in stock A and one-third in stock B. Annualize the portfolio expected returns and risks.
e) Would you recommend portfolio selection I or II to an investor? Briefly discuss your answer.
f) Critique how the data in the table is being applied to the theoretical framework used in these exercises. Include two drawbacks with using historical data to estimate the expected returns and risks.

Solution 3.12.

3.13. (Three securities) Suppose that you have \$5,000 to invest in stocks 1, 2, and 3 with current prices $\begin{bmatrix} S_1(t_0) \\ S_2(t_0) \\ S_3(t_0) \end{bmatrix} = \begin{bmatrix} \$10.20 \\ \$53.75 \\ \$30.45 \end{bmatrix}$, covariance matrix

$$\boldsymbol{V} = \begin{bmatrix} 0.03 & -0.04 & 0.02 \\ -0.04 & 0.08 & -0.04 \\ 0.02 & -0.04 & 0.04 \end{bmatrix},$$

and expected return vector

$$\boldsymbol{\mu} = \begin{bmatrix} 0.10 \\ 0.15 \\ 0.075 \end{bmatrix}.$$

For example, stock 3 has a volatility of $\sigma_3 = 20\%$ and expected return rate of $\mu_3 = 7.5\%$. The values in $\boldsymbol{V}$ and $\boldsymbol{\mu}$ are pure numbers (not percentages). Answer the following using an appropriate software.

a) Determine the weights needed to create the global minimum-variance portfolio of these three stocks.
b) Create an efficient portfolio with an expected return rate of 18%. Explicitly state the number of shares one must hold for each stock and how you fund each position. State the portfolio risk and compare it with maximum risk among the stocks.

Solution 3.13.

a) $\mu_G = 11.25\%, \qquad \sigma_G = 5.35\%, \qquad w_G = \begin{bmatrix} 42.86\% \\ 35.71\% \\ 21.43\% \end{bmatrix}.$

b) The portfolio risk is 34.57%.

3.14. (N securities) Consider managing a portfolio with 500 risky securities and assume that the variances of the securities are robustly estimated from reliable historical data. If 1% of the remaining independent covariances of the securities are poorly estimated due to inaccuracies in the data, then determine the number of poorly estimated covariances.

Solution 3.14.

3.15. Suppose a client just inherited \$1,000,000 and has come to you seeking advice on how to split the money between two of his favorite securities so as to maximize return. Security A has expected rate of return $r_A = 0.13$ and standard deviation of $\sigma_A = 0.15$. Security B has expected rate of return $r_B = 0.14$ and standard deviation $\sigma_B = 0.20$. The correlation coefficient between their rates of return is $\rho = -0.3$. If the investor has a utility function $u(x) = \sqrt[3]{x}$, how should he invest in each stock to maximize his overall rate of return?

Solution 3.15. The investor should invest \$395,750 in Security A and \$604,250 in Security B.

3.1.3 Theoretical Exercises

3.16. Let X and Y be two random variables. Let $a_i, i = 1,2,3,4$ be real numbers with $a_2 a_4 \neq 0$. Let ρ be the correlation coefficient. Prove that

$$\rho(a_1 + a_2 X, a_3 + a_4 Y) = \begin{cases} \rho(X,Y) & \text{if } a_2 a_4 > 0 \\ -\rho(X,Y) & \text{if } a_2 a_4 < 0. \end{cases}$$

The significance of the result is to allow a change of scale to one convenient for computation.

Solution 3.16.

3.17. Prove (3.17) on page 97.

Solution 3.17.

3.18. Verify Equation (3.20) on page 103, where the portfolio log-return rates are assumed uncorrelated and identically distributed.

Solution 3.18.

3.19. (Two securities) Let $A = \boldsymbol{e}^T \boldsymbol{V}^{-1} \boldsymbol{e}$, $B = \boldsymbol{\mu}^T \boldsymbol{V}^{-1} \boldsymbol{e}$, and $C = \boldsymbol{\mu}^T \boldsymbol{V}^{-1} \boldsymbol{\mu}$. Show that if

$$(B\boldsymbol{\mu} - C\boldsymbol{e})^T \, \boldsymbol{V}^{-1} \, (B\boldsymbol{\mu} - C\boldsymbol{e}) > 0,$$

then $AC - B^2 > 0$. See (3.33) on page 107.

Solution 3.19. We have:

$$\begin{aligned} 0 &< (B\boldsymbol{\mu} - C\boldsymbol{e})^T \, \boldsymbol{V}^{-1} \, (B\boldsymbol{\mu} - C\boldsymbol{e}) \\ &= B^2 \left(\boldsymbol{\mu}^T \boldsymbol{V}^{-1} \boldsymbol{\mu}\right) \; - \; BC \left(\boldsymbol{\mu}^T \boldsymbol{V}^{-1} \boldsymbol{e}\right) \; - \; BC \left(\boldsymbol{e}^T \boldsymbol{V}^{-1} \boldsymbol{\mu}\right) \; + \; C^2 \left(\boldsymbol{e}^T \boldsymbol{V}^{-1} \boldsymbol{e}\right) \\ &= \cancel{B^2 C} \; - \; \cancel{B^2 C} \; - \; B^2 C \; + \; AC^2 \\ &= C\left(AC - B^2\right) \end{aligned}$$

Since $C > 0$, it follows $AC - B^2 > 0$.

3.20. (Two securities) Verify Equation (3.38) on page 108, i.e. show that

$$\boldsymbol{w}_\mu = \left(\frac{C - \mu B}{AC - B^2}\right) \boldsymbol{V}^{-1} \boldsymbol{e} \; + \; \left(\frac{\mu A - B}{AC - B^2}\right) \boldsymbol{V}^{-1} \boldsymbol{\mu},$$

where

$$\boldsymbol{w}_\mu = \begin{bmatrix} w_\mu \\ 1 - w_\mu \end{bmatrix} = \frac{1}{\mu_1 - \mu_2} \begin{bmatrix} \mu - \mu_2 \\ \mu_1 - \mu \end{bmatrix}.$$

Solution 3.20. *Hint*: use (3.27), (3.28), (3.29) and (3.30). It is a tedious, straightforward calculation.

3.21. Let $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers with $a \neq 0$.

a) Show that if $b^2 - 4ac < 0$ and $f(x) \geq 0$ for all $x \in \mathbb{R}$, then $f(x) > 0$ for all $x \in \mathbb{R}$.
b) Show that if $a > 0$ and $f(x) > 0$ for all $x \in \mathbb{R}$, then the global minimum point of $\sqrt{f}$ is
$$x_* = -\frac{b}{2a}$$
and the corresponding global minimum value is
$$\sqrt{f(x_*)} = \sqrt{-\frac{(b^2 - 4ac)}{4a}}.$$
c) Use the above results to give an alternative proof that the two-security portfolio variance,
$$\sigma_P^2(w) = w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\rho\sigma_1\sigma_2,$$
is strictly positive. Compute the global minimum point w_* of the portfolio risk $\sigma_P(w)$ and find $\sigma_P(w_*)$. Compare with (3.44) and (3.47); see page 110.

Solution 3.21.

a) *Hint*: the quadratic formula.
b) *Hint*: use Claims 1 and 2 below:
Claim 1: The polynomial f has one critical point, namely, $x_* = -b/(2a)$, and if $a > 0$, then x_* is a minimum and, hence, the global minimum of f.

Claim 2: If $a > 0$ and $f(x) > 0$ for all $x \in \mathbb{R}$, then the point $x = x_*$ is the global minimum of f if and only if x_* is the global minimum of $\sqrt{f}$.
c) Applying the previous results to the quadratic
$$\sigma_P^2(\mu) = \boldsymbol{w}^T \boldsymbol{V} \boldsymbol{w} = \left(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2\right) w^2 \;+\; 2\left(\rho\sigma_1\sigma_2 - \sigma_2^2\right) w \;+\; \sigma_2^2.$$

3.22. (Two securities) Given two securities S_i, $i = 1,2$. Let R_i, $i = 1,2$, be their return rates respectively. Assuming that R_i, $i = 1,2$, are independent and identically distributed continuous random variables, determine the portfolio that a risk-averse investor would select.

Solution 3.22. *Hint*: maximize

$$f(w) = E[u((wR_1 + (1-w)R_2)\mathcal{V}(t_0))]$$

where $w \in [0,1]$.

3.23. (Two securities) Consider a portfolio with two securities having returns μ_1 and μ_2, risks σ_1 and σ_2, and a correlation coefficient ρ that vanishes. To minimize this portfolio's risk-to-reward ratio, a natural quantity to minimize is:

$$f(w) = \frac{\sigma_P^2(w)}{\mu_P(w)},$$

where w is the fraction of the total investment in the security with expected return μ_1.

a) Determine an equation that any critical point of f must satisfy. What type of equation is it?
b) Show that if $\mu_1 = \mu_2$, then we obtain a linear equation for w with solution

$$w = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}.$$

This critical point coincides with the global minimum w we found for the two-security portfolio when minimizing the variance σ_P^2 with $\rho = 0$. Why are the two critical points identical?

Solution 3.23.

a) *Hint*:

$$0 = (\mu_1 - \mu_2)w^2 + 2\mu_2 w - \frac{(\mu_1 + \mu_2)\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \qquad (\star').$$

b) *Hint*: apply $\mu_1 = \mu_2$ to Equation $(\star')$.

3.24. (Three securities) Consider a portfolio with three securities having risks σ_1, σ_2, and σ_3 and correlation coefficients ρ_{12}, ρ_{13}, and ρ_{23}. Let $\boldsymbol{V}$ be the covariance matrix of the security returns. Show that $\boldsymbol{V}$ is positive definite if and only if the following hold:

a) $\sigma_1 > 0$, $\sigma_2 > 0$, and $\sigma_3 > 0$,
b) $|\rho_{12}| < 1$ and $\rho_{12}^2 + \rho_{13}^2 + \rho_{23}^2 - 2\rho_{12}\rho_{13}\rho_{23} < 1$.

Solution 3.24. *Hint*: positive definiteness of $\boldsymbol{V}$ is equivalent to all its leading principal submatrices having positive determinants.

3.25. Let A be an $n \times n$ matrix. Show that the gradient and Hessian of the quadratic $\boldsymbol{x}^T \boldsymbol{A}\boldsymbol{x}$ are:

$$\frac{\partial(\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x})}{\partial \boldsymbol{x}} = (\boldsymbol{A} + \boldsymbol{A}^T)\boldsymbol{x}, \qquad \frac{\partial^2(\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x})}{\partial \boldsymbol{x}\partial \boldsymbol{x}^T} = \boldsymbol{A} + \boldsymbol{A}^T, \qquad \boldsymbol{x} \in \mathbb{R}^n,$$

where $\left(\frac{\partial f}{\partial \boldsymbol{x}}\right) = \left[\frac{\partial f}{\partial x_1} \cdots \frac{\partial f}{\partial x_n}\right]^T$ and $\frac{\partial^2(\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x})}{\partial \boldsymbol{x}\partial \boldsymbol{x}^T} = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]_{n\times n}$.

Solution 3.25. Hints: let

$$\boldsymbol{x} = [x_1 \cdots x_n]^T, \qquad \boldsymbol{A} = \begin{bmatrix} \cdots r_1 \cdots \\ \vdots \\ \cdots r_n \cdots \end{bmatrix} = \begin{bmatrix} \vdots & & \vdots \\ c_1^T & \cdots & c_n^T \\ \vdots & & \vdots \end{bmatrix},$$

where

$$r_i = [a_{i1} \cdots a_{in}], \qquad c_j^T = [a_{1j} \cdots a_{nj}]^T$$

are the ith row and jth column of $\boldsymbol{A}$, respectively. Then:

$$\boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x} = [x_1 \cdots x_n] \begin{bmatrix} r_1 \cdot \boldsymbol{x} \\ \vdots \\ r_n \cdot \boldsymbol{x} \end{bmatrix} = x_1(r_1 \cdot \boldsymbol{x}) + \cdots + x_n(r_n \cdot \boldsymbol{x})$$

and

$$\frac{\partial(\boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x})}{\partial \boldsymbol{x}} = \begin{bmatrix} \frac{\partial(\boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x})}{\partial x_1} \\ \vdots \\ \frac{\partial(\boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x})}{\partial x_n} \end{bmatrix}$$

Determine the partials to obtain:

$$\frac{\partial(\boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x})}{\partial \boldsymbol{x}} = \left(\boldsymbol{A} + \boldsymbol{A}^T\right) \boldsymbol{x}.$$

3.26. (N securities) For an N security portfolio, show that the portfolio vector $\boldsymbol{w}$ which minimizes the variance $\sigma_P^2(\boldsymbol{w}) = \boldsymbol{w}^T \boldsymbol{V} \boldsymbol{w}$, subject to $\boldsymbol{w}^T \boldsymbol{e} = 1$, is the global minimum-variance portfolio vector. Explain why this is expected.

Solution 3.26. *Hint*: use the Lagrangian function

$$L(\boldsymbol{w}) = \boldsymbol{w}^T \boldsymbol{V} \boldsymbol{w} - \lambda(\boldsymbol{w}^T \boldsymbol{e} - 1).$$

3.27. (N-securities) Determine the equations for the lines asymptotic to the set of all minimum-variance N-securities portfolios.

Solution 3.27.

$$\mu = \pm \sqrt{\frac{AC - B^2}{A}} \sigma + \frac{B}{A}.$$

3.28. Show that the covariance of the return of the global minimum-variance N-securities portfolio with the return of any other efficient N-securities portfolio is always $1/A$.

Solution 3.28. Insert the global minimum-variance portfolio vector $\boldsymbol{w}_G = \frac{\boldsymbol{V}^{-1}\boldsymbol{e}}{\boldsymbol{e}^T \boldsymbol{V}^{-1} \boldsymbol{e}}$ in

$$\mathrm{Cov}\left(R_P(\boldsymbol{w}_G), R_P(\boldsymbol{w})\right) = \boldsymbol{w}_G^T \boldsymbol{V} \boldsymbol{w}.$$

3.29. Where does the tangent line at the diversified portfolio on the Markowitz N-securities efficient frontier intersect the μ_P-axis?

Solution 3.29. $\mu_P = \mu_G$

3.30. Determine the equation of a line asymptotic to the Markowitz N-securities efficient frontier in the (σ_P, μ_P)-plane.

Solution 3.30.

$$\mu_P = \sqrt{\frac{AC - B^2}{A}}\,\sigma_P + \frac{B}{A}.$$

3.31. Let $\boldsymbol{w}_a$ and $\boldsymbol{w}_b$ be any two distinct minimum-variance portfolio vectors. For suitable constants a and b, these vectors can be expressed in the following form:

$$\boldsymbol{w}_a = (1-a)\boldsymbol{w}_G + a\boldsymbol{w}_D, \qquad \boldsymbol{w}_b = (1-b)\boldsymbol{w}_G + b\boldsymbol{w}_D,$$

where $\boldsymbol{w}_G$ and $\boldsymbol{w}_D$ are the global minimum-variance and diversified portfolio vectors, respectively.

a) Show that any minimum-variance portfolio vector $\boldsymbol{w}$ can be expressed as

$$\boldsymbol{w} = \left(\frac{\lambda_1 A + b - 1}{b-a}\right)\boldsymbol{w}_a \; + \; \left(\frac{1 - a - \lambda_1 A}{b-a}\right)\boldsymbol{w}_b.$$

b) Show that the covariance is given as follows:

$$\mathrm{Cov}(R_P(\boldsymbol{w}_a), R_P(\boldsymbol{w}_b)) = \frac{1}{A} + ab\,\frac{AC - B^2}{AB^2},$$

where $A = \boldsymbol{e}^T\boldsymbol{V}^{-1}\boldsymbol{e}$, $B = \boldsymbol{\mu}^T\boldsymbol{V}^{-1}\boldsymbol{e}$, and $C = \boldsymbol{\mu}^T\boldsymbol{V}^{-1}\boldsymbol{\mu}$.

Solution 3.31.

a) *Hint*: express $\boldsymbol{w}_G$ and $\boldsymbol{w}_D$ in terms of $\boldsymbol{w}_a$ and $\boldsymbol{w}_b$, and then insert the expressions into

$$\boldsymbol{w} = (\lambda_1 A)\boldsymbol{w}_G + (\lambda_2 B)\boldsymbol{w}_D.$$

b) *Hint*: insert the matrix expressions of $\boldsymbol{w}_G$ and $\boldsymbol{w}_D$ into

$$\mathrm{Cov}\left(\mathrm{R}_P(\boldsymbol{w}_a), \mathrm{R}_P(\boldsymbol{w}_b)\right) = (1-a){\boldsymbol{w}_G}^T\boldsymbol{V}\boldsymbol{w}_b + a{\boldsymbol{w}_D}^T\boldsymbol{V}\boldsymbol{w}_b.$$

3.32. Let $a > 0$ and $b \neq 0$. Show that the utility functions $u(x) = ax + b$ and $u(x) = ax - \frac{b}{2}x^2$, where $x < \frac{a}{b}$, obey

$$\mathbb{E}\big(u(X)\big) = u(\mathbb{E}(X)) \; + \; \frac{1}{2}u''(\mathbb{E}(X))\,\mathrm{Var}(X).$$

Solution 3.32. *Hint*: for the quadratic utility, use $\mathrm{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2$.

3.33. A power utility function refers to one of the form $u(x) = x^a$. When is u a risk-averse utility function?

Solution 3.33. *Hint*: argue that $a \in (0,1)$.

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Chapter 4
Selected Solutions for Chapter 4

4.1 Exercises

4.1.1 Conceptual Exercises

4.1. State the definition of the *market portfolio*.

Solution 4.1.

4.2. Explain how the sign of the beta of a stock indicates the direction of the movement of the stock price with respect to that of the market portfolio.

Solution 4.2. *Hint*: "$\beta > 0$" indicates that on average the stock (excessive) returns move in the same direction as the market (excessive returns).

4.3. Let A be a class of stocks with beta between 0.5 and 2. What is the main property that all stocks in A have in terms of the market returns?

Solution 4.3.

4.4. Give a financial interpretation of the mathematical expression: $\frac{\mu_M - r_f}{\sigma_M}$.

Solution 4.4. $\frac{\mu_M - r_f}{\sigma_M}$ represents the market risk premium per unit of risk taken by the investor.

4.5. Can you find an example of a company in the U.S. with a negative beta? If so, do you think that they are as abundant as companies with a positive beta? Explain.

Solution 4.5.

4.6. Consider random variable $X = 1,2,3$ satisfying $\mathbb{P}(X = i) = \frac{1}{3}$, $i = 1,2,3$. Find the quantile function of X.

Solution 4.6.

$$Q(p) = i \quad \text{if } p \in (\frac{i-1}{3}, \frac{i}{3}], \;\; i = 1,2,3.$$

4.7. Consider a random variable $X = 1,2,3,4,5,6,7,8,9,10$ with p.d.f. given by $\mathbb{P}(X = i) = \frac{1}{11}$ for $i \neq 7$ and $\mathbb{P}(i = 7) = \frac{2}{11}$. Find the quantile function of X.

Solution 4.7.

4.8. Use the table given in Example 4.17 on page 183 to determine $\text{Var}_p(X)$ for arbitrary p.

Solution 4.8.

$$\text{VaR}_p(X) = Q(p) = \begin{cases} -70 & \text{if } p \in (0, 0.1] \\ -30 & \text{if } p \in (0.1, 0.3] \\ 0 & \text{if } p \in (0.3, 0.7] \\ 30 & \text{if } p \in (0.7, 0.9] \\ 70 & \text{if } p \in (0.9, 1]. \end{cases}$$

4.9. Use the table given in Example 4.17 on page 183 to determine $C\ \text{Var}_p(X)$ for $p = 30\%$ and $p = 50\%$.

Solution 4.9.

4.10. What does a negative VaR imply?

Solution 4.10. The portfolio has a high probability to make a profit over the time horizon under the consideration. For instance, a one-day 1% VaR = -$18.58 means that the portfolio has a 99% chance to make more than $18.58 over the next day.

4.11. Returns and risks are two aspects involved in every investment. Identify each statement below as true or false or identify scenarios when it is true and when it is false. Justify your answer.

a) The quantity α relates to factors affecting the performance of an individual stock or the fund manager's skill in selecting the stocks.
b) The factor β relates an individual stock-to-market risks.
c) A higher α stock and a lower β stock would be preferred choices.
d) The quantity $\alpha = 0$ if a stock market is efficient.

Solution 4.11.

a) True, from the definition of $\mathbb{E}(\varepsilon)$, which is the expectation of idiosyncratic return of the asset.
b) True, from the definition of β, i.e. $\mathbb{E}(r) = \alpha + \beta\mathbb{E}(r_M)$.
c) True for those investors who prefer individual stocks. It is false or less true for investors who prefer market indices or ETF funds.
d) False. A vanishing stock α and market efficiency do not impact each other.

4.12. Briefly justify your answer to each of the following.

a) Suppose that empirical evidence were sufficient to confirm the CAPM. What would be an investment implication?
b) Suppose that extraordinary premiums from the (Fama-French) three-factor investing during the period of 1926-1996 were to repeat today, what would be an investment implication?

Solution 4.12.

4.13. All investments carry some form of risk. Major risks include, but are not limited to, the following:

a) Systematic risk
b) Interest rate risk
c) Liquidity risk[30]
d) Regulatory/political risk[31]
e) Leverage risk[32]
f) Credit risk
g) Currency risk
h) Counterparty risk[33]

Find an example for each type of risk listed above.

Solution 4.13.

4.1.2 Application Exercises

4.14. Use the data in Table 4.1 below to compute the Sharpe ratio of the S&P 500 for the period 1986 to 1999. Note that the risk-free rate is not constant.

Solution 4.14. Calculations of this type should be done using a spreadsheet. We provide some of the mathematical details manually. The ith excess return is

$$y_i = r_i - r_{f,i},$$

where r_i is the ith return data point, counting from top to bottom, of the S&P 500 column and $r_{f,i}$ is the risk-free rate for the i year. For example,

[30] Liquidity risk is the risk that an investor cannot execute a buy/sell order in the market due to the lack of anticipated/reasonable bid/ask spread or sufficient volume.

[31] Regulatory changes or governmental policy changes may have significant impact on asset values. Such risk can be either systemic risk or market risk.

[32] Such risk is often associated with unexpected and unfavorable volatility.

[33] Counterparty here means the other party in a financial transaction.

Table 4.1 Annual return rate data from 1986 to 1999 for the S&P 500 and 1-year Treasury bills. Data Source: istockanalyst.com

Year	S&P500 annual return	1-year T-bill rate
1986	18.82%	7.21%
1987	5.40%	5.46%
1988	15.99%	6.52%
1989	31.56%	8.37%
1990	-2.97%	7.38%
1991	30.51%	6.25%
1992	7.45%	3.95%
1993	10.09%	3.35%
1994	1.33%	3.39%
1995	37.28%	6.59%
1996	22.69%	4.82%
1997	33.60%	5.30%
1998	30.73%	4.98%
1999	21.10%	4.31%

$$y_1 = 18.82\% - 7.11\% = -3.57\% = \frac{11.61}{100}.$$

The Sharpe ratio

$$\mathbb{S}_P = \frac{\mathbb{E}[r_P - r_f]}{\sigma(r_P - r_f)}$$

is estimated in terms of the data by

$$\widehat{\mathbb{S}}_P = \frac{\overline{y}}{\widehat{\sigma}(y)},$$

where $\overline{y}$ is the sample mean of the excess return $r_P - r_f$, i.e.

$$\overline{y} = \frac{1}{n}\sum_{i=1}^{n} y_i = \frac{1}{14}\,\frac{185.7}{100} = \frac{13.2642}{100}$$

and the sample variance of the excess return is

$$\begin{aligned}\widehat{\sigma}^2(y) &= \frac{1}{n-1}\left[\sum_{i=1}^{n} y_i^2 - n\,\overline{y}^2\right] \\ &= \frac{158.7489}{100^2}.\end{aligned}$$

Hence:

$$\widehat{\mathbb{S}}_P = \frac{13.2642}{12.5994} = 1.0528.$$

4.15. Assume a risk-free rate of 1.5%. Answer the questions below using the information in the following table:

Portfolio	A	B	C	D	E	F
Expected Return	3.2%	8.1%	9.8%	5.1%	10.7%	4.8%
Standard Deviation	2.7%	9.9%	13.7%	6.2%	17%	6.1%

a) Among the portfolios in the table, which one is closest to the market portfolio? Justify your answer.
b) Plot the capital market line (CML) based on your answer in part (a).
c) For portfolio C, what is the portfolio risk premium per unit of portfolio risk?
d) Suppose we are willing to make an investment only with $\sigma = 6.2\%$. Is a return of 6.5% a realistic expectation for us?

Solution 4.15.

4.16.

a) Given the information in the table below,

Stock	**Beta**	**Expected Return**
A	1.25	
B	0.7	
C	-0.4	

and assuming that r_f is 3% and that the market return is 5%, find the expected returns for each stock listed in the table and plot them on an SML graph.

b) Suppose the table below provides further information about the stocks in part a):

Stock	**Current Price**	**Expected Price**	**Expected Dividend**
A	29.5	21.5	0.71
B	47	49.75	1.85
C	35.4	38.7	1.05

Indicate your estimated returns on each stock on the graph from part a), and decide your buy/sell/hold rating on each stock based on your graph. Justify your decisions.

Solution 4.16. (for part a only)

a) Use $\mu_P = r_f + \beta(\mu_M - r_f) = 0.03 + 0.02\beta$ to compute: $\mu_A = 0.055$, $\mu_B = 0.044$ and $\mu_C = 0.0292$.

4.17. Suppose that we borrow an amount equal to 25% of our original wealth at the risk free-rate 4.125%. Use the CML to find μ_P and σ_P.

Solution 4.17.

4.18. Assume the risk-free rate is 1.5% and consider the information in the table below:

Portfolio	Expected Return	Standard Deviation
A	3.2%	2.7%
B	8.1%	9.9%
C	9.8%	13.7%
D	5.1%	6.2%
E	10.7%	17%
F	4.8%	6.1%

a) Which of these six portfolios offers investors the best combination of risk and return? Justify your answer from a capital market perspective.

b) Use the formula

$$\mu_P = w_0 r_f + (1 - w_0)\mu_M$$

to determine your investment asset allocation.

c) If you plan to invest $100,000, what is your investment strategy based on the information given in this exercise?

Solution 4.18. (for part a only)

a) B, since the Sharpe ratio, $\frac{\mu_P - 0.015}{\sigma_P}$, is max.

4.19. Given the following information, find the security's beta and expected return:

a) The risk-free rate is 1.72%. The market portfolio has its standard deviation as 15.92% and its expected return as 5%. The covariance of the security with the market is 0.04.
b) The risk-free rate is 2%. The market portfolio has its standard deviation as 14% and its expected return as 11%. The security is uncorrelated with the market and has a standard deviation of 39.7%.
c) The risk-free rate is 1.72%. The market portfolio has its standard deviation as 15.92% and its expected return as 5%. The covariance of the security with the market is -0.04.

Solution 4.19.

4.20. The ticker symbol for the Goldman Sachs Group is GS. The table below provides the daily closing prices for GS and S&P 500 index on the trading days during the period between June 30 and July 14 of 2011. Let r and r_M be the daily log returns of GS and S&P 500 index respectively. Using the model $r = \alpha + \beta\, r_M$, determine GS's α and β for the period.

Date	**GS**	**S&P 500**
June 30	$133.09	$1320.64
July 1	$136.65	$1339.67
July 5	$134.5	$1337.88
July 6	$133.89	$1339.22
July 7	$135.01	$1353.22
July 8	$134.08	$1343.8
July 11	$132.02	$1319.49
July 12	$130.31	$1313.64
July 13	$129.7	$1317.72
July 14	$129.89	$1308.87

Solution 4.20. *Hint*:

Date	**GS**	**S&P 500**	y	x
June 30	\$133.09	1320.64	(let $y_i = r_i$)	(let $x_i = r_{M_i}$)
July 1	\$136.65	1339.67	0.026397321	0.014306847
July 5	\$134.5	1337.88	-0.015858713	-0.001337043
July 6	\$133.89	1339.22	-0.004545632	0.001001083
July 7	\$135.01	1353.22	0.008330282	0.010399583
July 8	\$134.08	1343.8	-0.006912213	-0.006985516
July 11	\$132.02	1319.49	-0.01548321	-0.018256123
July 12	\$130.31	1313.64	-0.013037199	-0.004443388
July 13	\$129.7	1317.72	-0.004692136	0.00310106
July 14	\$129.89	1308.87	0.001463847	-0.006738801
sample mean			-0.002704184	-0.0009947

4.21. The ticker symbols (stock symbols) for the Goldman Sachs Group and SPDR S&P 500 ETF are GS and SPY[34] respectively. The table below provides the daily closing prices for GS and SPY on the trading days during the period between June 30 and July 14 of 2011. Let P be a portfolio consisting of longing 200 shares of SPY and shorting 100 shares of GS. Suppose that $r_f = 0$. Find the maximum drawdown and the Sharpe ratio for the portfolio in the time period indicated in the table.

Date	**GS**	**SPY**
June 30	\$133.09	\$131.97
July 1	\$136.65	\$133.92
July 5	\$134.5	\$133.81
July 6	\$133.89	\$133.97
July 7	\$135.01	\$135.36
July 8	\$134.08	\$134.4
July 11	\$132.02	\$131.97
July 12	\$130.31	\$131.4
July 13	\$129.7	\$131.84
July 14	\$129.89	\$130.93

[34] SPDR (Spiders) is a short form of Standard & Poor's depositary receipt, an exchange-traded fund (ETF) that tracks the Standard & Poor's 500 Index (S&P 500). Each share of SPY contains one-tenth of the S&P index and trades at approximately one-tenth of the dollar value of the S&P 500. Thus, the rate of daily returns of SPY and S&P 500 index are basically the same.

Solution 4.21.

4.22. Although most stocks' α and β can be found online, the actual values of α and β for the same stock may be different at different sites. Besides, how the actual values are calculated might be considered as proprietary information. Thus, it is critical to understand the factors that affect the calculations.

Use Yahoo Finance as a data source to complete each of the following problems:

a) The stock symbol of Apple Inc. is AAPL. Estimate α and β for AAPL by using the weekly adjusted closing prices over the last two years and the S&P 500 index as the market portfolio.
b) Estimate α and β for AAPL by using the weekly adjusted closing prices over the last four years and the S&P 500 index as the market portfolio.
c) Estimate α and β for AAPL by using the daily adjusted closing prices over the last two years and the S&P 500 index as the market portfolio.
d) Estimate α and β for AAPL by using the weekly adjusted closing prices over the last two years and the NASDAQ-100 index as the market portfolio.
e) Observe the results above and give the factors that the actual calculated value of β depends on.

Solution 4.22.

4.1.3 Theoretical Exercises

4.23. Establish (4.5) on page 158, i.e., show

$$\mu_M = \frac{C - Br_f}{B - Ar_f}, \quad \sigma_M^2 = \frac{Ar_f^2 - 2Br_f + C}{(B - Ar_f)^2}, \quad \boldsymbol{w}_M = \frac{\boldsymbol{V}^{-1}(\boldsymbol{\mu}_M - r_f\boldsymbol{e})}{B - Ar_f}.$$

Solution 4.23.

4.24. Show that under linear factor model framework, portfolio variance can be decomposed into common factor variance and idiosyncratic variance. (*Hint*: Apply relation (4.25) on page 190.)

Solution 4.24.
Proof: Let $R_P = \sum_{i=1}^{n} w_i R_i$ be a portfolio return, where $\mathbf{w} = [w_1, w_2, \cdots, w_n]^\top$ is the weighting of the assets. Then $\mathbf{w}^\top \Sigma \mathbf{w}$ represents the total variance of the portfolio. The result is readily seen as $\Sigma = \beta F \beta^\top + \Psi$.

4.25. Use the single factor model (4.26) on page 192, namely,

$$R = \alpha + \beta R_M + \varepsilon,$$

to express β in terms of the variance of the total market return and the covariance of the market return with an individual security's return.

Solution 4.25. Since

$$\begin{aligned}\mathrm{Cov}(R, R_M) &= \mathrm{Cov}(\alpha + \beta R_M + \varepsilon, R_M) \\ &= \mathrm{Cov}(\alpha, R_M) + \beta\, \mathrm{Cov}(R_M, R_M) + \mathrm{Cov}(\varepsilon, R_M) \\ &= \beta\, \mathrm{Cov}(R_M, R_M) = \beta\, \mathrm{Var}(R_M).\end{aligned}$$

Hence:

$$\beta = \frac{\mathrm{Cov}(R, R_M)}{\mathrm{Var}(R_M)}.$$

4.26. Use the single factor model (4.26) on page 192 namely,

$$R = \alpha + \beta R_M + \varepsilon,$$

to obtain a corresponding asset pricing formula.

Solution 4.26.

4.27. A self-financing or dollar-neutral portfolio is established by using the proceeds of the short sales to finance the long purchases. In other words, under the assumption of *Frictionless Trading*, a self-financing portfolio is a zero cost portfolio. For example, the excess return $R_M - r_f$ can be viewed as the return of a portfolio that is formed by using the borrowed amount at interest rate r_f to purchase shares of SPY.

A traditionally common representation of many asset pricing models is in the linear factor form:

$$E(R_i - r_f) = \alpha + \sum_{k=1}^{m} \beta_k \lambda_k,$$

where λ_k with $k = 1, 2, \ldots, m$ are the values of the corresponding risk factors.

Given two risk factors—one is the excess return on the market portfolio and the other is an economic recession factor—use a self-financing portfolio to establish a two-factor linear model for the excess return of a security.

Solution 4.27.

4.28. Let

$$A = \frac{\frac{1}{n}\sum_{i=1}^{n} X_i Y_i - \overline{XY}}{\sqrt{\frac{1}{n^2}\sum_{i=1}^{n}(X_i - \overline{X})^2 \sum_{i=1}^{n}(Y_i - \overline{Y})^2}},$$

$$B = \frac{n\sum_{i=1}^{n} X_i Y_i - \left(\sum_{i=1}^{n} X_i\right)\left(\sum_{i=1}^{n} Y_i\right)}{\sqrt{n\sum_{i=1}^{n} X_i^2 - \left(\sum_{i=1}^{n} X_i\right)^2}\sqrt{n\sum_{i=1}^{n} Y_i^2 - \left(\sum_{i=1}^{n} Y_i\right)^2}}.$$

Prove that $A = B$.

Solution 4.28.
Proof: Note that

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2 &= \frac{1}{n}\sum_{i=1}^{n}(X_i^2 - 2X_i\overline{X} + \overline{X}^2) \\
&= \frac{1}{n}\sum_{i=1}^{n} X_i^2 - 2\frac{1}{n}\sum_{i=1}^{n} X_i\overline{X} + \frac{1}{n}\sum_{i=1}^{n}\overline{X}^2 \\
&= \frac{1}{n}\sum_{i=1}^{n} X_i^2 - 2\overline{X}^2 + \overline{X}^2 \\
&= \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \overline{X}^2 \\
&= \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} X_i\right)^2.
\end{aligned}$$

Similarly,

$$\frac{1}{n}\sum_{i=1}^{n}(Y_i - \overline{Y})^2 = \frac{1}{n}\sum_{i=1}^{n} Y_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} Y_i\right)^2$$

Substituting the corresponding expressions in A, the desired result will be followed by straightforward algebraic manipulations.

4.29. Given a portfolio P, show that its Sortino ratio is no less than its Sharpe ratio.

Solution 4.29.

4.30. Let $\boldsymbol{A}$ be an $n \times m$ matrix, and $\boldsymbol{A}^T$ be the transpose of $\boldsymbol{A}$. Prove the following property:

$$\operatorname{rank}(\boldsymbol{A}^T\boldsymbol{A}) = \operatorname{rank}(\boldsymbol{A}).$$

Solution 4.30.
Proof: Since

$$\operatorname{rank}(\boldsymbol{A}) = n - \operatorname{nullity}(\boldsymbol{A})$$

where nullity$(\boldsymbol{A})$ is the dimension of the kernel of $\boldsymbol{A}$. (which is also known as the null space of $\boldsymbol{A}$), it is sufficient to show that $\ker(\boldsymbol{A}^T\boldsymbol{A}) = \ker(\boldsymbol{A})$, i.e., show

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0} \iff \boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}.$$

We have:
($\Longrightarrow$): Straightforward.
($\Longleftarrow$): If $\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$, then $\boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = 0$, which implies $\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$.

4.31. We continue from the last exercise. Let

$$\boldsymbol{A} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \quad \text{with } x_1 \neq x_2 \quad \text{and} \quad \boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

be given. Find the best fit to the system

$$\boldsymbol{A} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \boldsymbol{y}$$

such that the norm $\left| \boldsymbol{A} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} - \boldsymbol{y} \right|$ is minimized.

Hint: Let $\boldsymbol{\gamma} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ and $L = |\boldsymbol{A}\boldsymbol{\gamma} - \boldsymbol{y}|^2$, then

$$L = (\boldsymbol{A}\boldsymbol{\gamma} - \boldsymbol{y})^T(\boldsymbol{A}\boldsymbol{\gamma} - \boldsymbol{y}) = \boldsymbol{\gamma}^T\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{\gamma} - \boldsymbol{\gamma}^T\boldsymbol{A}^T\boldsymbol{y} - \boldsymbol{y}^T\boldsymbol{A}\boldsymbol{\gamma} + \boldsymbol{y}^T\boldsymbol{y}.$$

Solution 4.31.

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Chapter 5
Selected Solutions for Chapter 5

5.1 Exercises

5.1.1 Conceptual Exercises

5.1. Suppose that a binomial tree has an initial price of \$80. If the tree has 21 periods, then is \$80 one of the possible prices at time t_{21}? If the tree has 300 periods, then is \$80 one of the possible prices at time t_{300}? Justify your answers.

Solution 5.1. *Hint*: the price occurs if the period n is a even number.

5.2. For an n-period binomial tree, give a financial interpretation of each of the following: $\mathrm{u}_n\,\mathrm{d}_n = 1$, $\mathrm{u}_n - 1$, and $\mathrm{d}_n - 1$.

Solution 5.2.

a) $\mathrm{u}_n\,\mathrm{d}_n = 1$: for any node at a given time step, an upward price movement followed by a downward movement does not change the security's price.
b)
c)

5.3. Explain why the condition, $\mathrm{d}_n < \mathrm{e}^{(m-q)h_n} < \mathrm{u}_n$, holds for n-period CRR trees with n sufficiently large.

Solution 5.3.

5.4. How many 1-period subtrees are in an n-period binomial tree?

Solution 5.4. $\frac{n(n+1)}{2}$.

5.5. For an n-period binomial tree, let N_U be the number of security price upticks from time t_0 to t_n. Explain why N_U is a binomial random variable. What are its expected value and variance if the tree has 40 steps and the uptick probability is 60%?

Solution 5.5. $\mathbb{E}(N_U) = 24$.

5.6. An n-period CRR tree has reflection symmetry about the horizontal line $S(t) = S_0$ since the tree recombines. Agree or disagree? Justify your answer.

Solution 5.6. Disagree.

5.7. For an n-period risk-neutral binomial tree, show that if $\mathrm{u}_n < \mathrm{e}^{(\mathrm{r}-q)\,h_n}$, then there is an arbitrage.

Solution 5.7. *Hint*: at t_0, short sell $\mathrm{e}^{-q\,h_n}$ units of the security and use the proceeds to long (buy) a bond paying the risk-free rate r. At time t_1, liquidating the bond brings in $S_0\,\mathrm{e}^{(\mathrm{r}-q)\,h_n}$.

5.8. The risk-neutral uptick probability p_n^* is related to the real-world probability p_n by $p_n^* = p_n - \eta_n\,\sqrt{p_n(1-p_n)}$, where $\eta_n = \frac{\mathbb{E}(R(t_0,t_1)) - \mathrm{r}\,h_n}{\sqrt{\mathrm{Var}(R(t_0,t_1))}}$. Interpret η_n.

Solution 5.8. *Hint*: slope of the capital market line.

5.9. If in an n-period real world CRR tree, the real world probability p_n is replaced by the risk-neutral uptick probability p_n^*, then the expected annualized return rate m is unchanged, but the annualized variance σ^2 changes. Agree or disagree? Justify your answer.

Solution 5.9. Disagree.

5.10. For an n-period binomial tree, the collection of all paths has 2^{2^n} subcollections of paths, where the empty subcollection is included. Agree or disagree? Justify your answer.

Solution 5.10. Agree.

5.1.2 Application Exercises

5.11. A trader believes that a certain stock currently at \$51.25 per share has by the end of the trading day a 70% chance of increasing by 50¢ and a 30% chance of decreasing by 25¢. Using a 1-step binomial tree with this information, what is the expected price of the stock at the end of the day?

Solution 5.11. \$51.53.

5.12. Assume that the current share price of a stock is \$100 with a volatility of 10%. Using a CRR tree model over a year with each being one trading day, predict the maximum spread in the stock's possible prices a trading day from now. Is your prediction impacted if you employ a Taylor approximation to $\mathrm{e}^{\pm\sigma\sqrt{h_{252}}}$ using the CRR assumptions?

Solution 5.12.

5.13. Suppose that a nondividend-paying stock with current price of \$45 has an instantaneous annual expected return of 8% and annual volatility of 15%.

a) Using a 100-period CRR tree, forecast the price of the stock 3 months from now, i.e., find the expected price of the stock 3 months from now.
b) What is your forecast if you use an 80-period CRR tree?
c) Using an 80-period CRR tree, determine the probability of your forecasted price occurring.

Solution 5.13. Due to the large number of periods is large, we employ the CRR tree to determine u_{100}, d_{100}, and p_{100} in terms of h_{100}, σ, and m:

$$\mathrm{u}_{100} \approx \mathrm{e}^{\sigma\sqrt{h_{100}}}, \qquad \mathrm{d}_{100} \approx \mathrm{e}^{-\sigma\sqrt{h_{100}}}, \qquad p_{100} \approx \frac{\mathrm{e}^{(m-q)\,h_{100}} - \mathrm{d}_{100}}{\mathrm{u}_{100} - \mathrm{d}_{100}}.$$

We have:

$$\begin{aligned} &n = 100, \quad h_{100} = \frac{0.25}{100} = 0.0025, \quad \sqrt{h_{100}} = 0.05, \\ &m = 0.08, \quad q = 0, \quad \sigma = 0.15, \quad S_0 = \$45, \\ &\mathrm{u}_{100} = 1.00753, \quad \mathrm{d}_{100} = 0.992528, \quad p_{100} = 0.51146. \end{aligned}$$

a)
b) \$45.91.
c) *Hint*: for the forecasted price \$45.91, check whether k is an integer.

5.14. Consider a nondividend-paying stock with a current price of \$45, an instantaneous annual expected return of 8%, and annual volatility of 15%. Assume an 80-period CRR tree.

a) Can the stock price be at the same value 3 months from now? If so, how many times would the price have to increase and decrease for this to happen?
b) What is the probability that the stock price will be at the same value 3 months?
c) What is the probability that in 3 months the stock price is greater than its current price?

Solution 5.14. We have

$$\begin{aligned} &n = 100, \quad h_{100} = \frac{0.25}{100} = 0.0025, \quad \sqrt{h_{100}} = 0.05, \\ &m = 0.08, \quad q = 0, \quad \sigma = 0.15, \quad S_0 = \$45. \end{aligned}$$

The 80-period CRR tree yields:

$$\mathrm{u}_{80} = 1.00842, \quad \mathrm{d}_{80} = 0.99165, \quad p_{80} = 0.512813.$$

a) 40

b)
c)

5.15. Suppose that the current date is January 21, 2016. Estimate the volatility σ and instantaneous drift μ_{RW} for Google (ticker symbol GOOGL) using its adjusted closing prices from Yahoo! Finance for the period from January 20, 2016 to September 15, 2015. The data will consist of 90 daily log returns over the past 91 trading days. Carry out similar estimates with the past 60 daily log returns and then the past 30 daily log returns. Annualize your results using 252 trading days in a year. Discuss your findings.

Solution 5.15. For the past 90 daily log returns, the historical σ is 0.236 or 23.6% and $\mu_{\text{RW}} = 0.276$ or 27.6%.

For the past 60 daily log returns, the historical σ is 0.243 or 24.3% and $\mu_{\text{RW}} = 0.225$ or 22.5%.

5.16. Assume that a nondividend-paying security has a current price of \$50, instantaneous expected return of 8%, and volatility of 15%. Estimate the probability, as a fraction (not percentage) to two decimal places, that 3 months from now the price of the real-world, continuous-time price of the security is greater than \$50. What fractional probability to two decimal places does a 100-period real-world CRR tree predict? How about a 1,000-period real-world CRR tree? Find a value of n for which an n-period real-world CRR tree gives the same fractional probability to two, three, and four decimal places as that obtained from the continuous-time security price model. Use a software for this problem.

Solution 5.16.

5.1.3 *Theoretical Exercises*

5.17. For an n-period binomial tree, show that

$$\mathbb{P}\left(S(t_n) = S_0 \mathrm{u}_n^i \mathrm{d}_n^{n-i} \,|\, S(t_0) = S_0\right) = \binom{n}{i} p_n^i (1 - p_n)^{n-i}, \qquad i = 0, 1, 2, \ldots, n.$$

Solution 5.17. *Hint*: the probability of the specific price, $S_0 \mathrm{u}_n^i \mathrm{d}_n^{n-i}$, is the sum of the probabilities of all paths to that price.

5.18. Determine the number of elements in the sample space Ω_n of price paths of an n-period binomial tree.

Solution 5.18.

5.19. Show that $\sigma_n^2 h_n = p_n(1 - p_n)\left[\ln\left(\frac{\mathrm{u}_n}{\mathrm{d}_n}\right)\right]^2$.

Solution 5.19. *Hint*:

$$\sigma_n^2 h_n = \mathbb{E}\left(\ln\left(\frac{S(t_n)}{S(t_0)}\right)^2\right) - \left[\mathbb{E}\left(\ln\left(\frac{S(t_n)}{S(t_0)}\right)\right)\right]^2$$
$$= p_n\,(\ln \mathrm{u}_n)^2 \;+\; (1-p_n)\,(\ln \mathrm{d}_n)^2 - (p_n \ln \mathrm{u}_n \;+\; (1-p_n)\ln \mathrm{d}_n)^2.$$

5.20. Show that for a CRR tree, the uptick probability $p_n \approx \frac{\mathrm{e}^{(m-q)\,h_n} - \mathrm{d}_n}{\mathrm{u}_n - \mathrm{d}_n}$ satisfies $p_n \approx \frac{1}{2}\left(1 + \frac{\mu_{\mathrm{RW}}}{\sigma}\sqrt{h_n}\right)$ for n sufficiently large.

Solution 5.20. *Hint*:

$$p_n \approx \frac{\mathrm{e}^{(m-q)\,h_n} - \mathrm{d}_n}{\mathrm{u}_n - \mathrm{d}_n} \approx \frac{1 + (m-q)\,h_n - 1 + \sigma\sqrt{h_n} - \frac{1}{2}\sigma^2 h_n}{1 + \sigma\sqrt{h_n} + \frac{1}{2}\sigma^2 h_n - 1 + \sigma\sqrt{h_n} - \frac{1}{2}\sigma^2 h_n}.$$

5.21. For a CRR tree, show $X_{n,j} = \begin{cases} \frac{1-p_n}{\sqrt{p_n(1-p_n)}} & \text{with probability } p_n \\[2ex] \frac{-p_n}{\sqrt{p_n(1-p_n)}} & \text{with probability } 1-p_n. \end{cases}$

Solution 5.21. *Hint*: the CRR formulas can be expressed as:

$$\ln \mathrm{u}_n = -\ln \mathrm{d}_n, \qquad \mu_n\, h_n = (2p_n - 1)\ln \mathrm{d}_n, \qquad \sigma_n^2\, h_n = p_n(1-p_n)\,(2\ln \mathrm{u}_n)^2.$$

then use Equation (5.43) on page 230.

5.22. A *Jarrow-Rudd (JR) tree* for the price of a security paying a continuous dividend yield rate q is a binomial tree where $p_n = \frac{1}{2}$ and the per-period expectation μ_n^* and variance σ_n^2 defined by

$$\mu_n^* = \frac{1}{h_n}\,\mathbb{E}\left(\ln\left(\frac{S(t_j)}{S(t_{j-1})}\right)\right), \qquad (\sigma_n^*)^2 = \frac{1}{h_n}\;\mathrm{Var}\left(\ln\left(\frac{S(t_j)}{S(t_{j-1})}\right)\right),$$

satisfy $\mu_n^* \to \mu_*$ and $\sigma_n^* \to \sigma$ as $n \to \infty$, where $\mu_* = \mathrm{r} - q - \frac{\sigma^2}{2}$. Then for n sufficiently large, the constraint equations for $\mathrm{u}_n, \mathrm{d}_n, p_n$ in a JR tree are

$$p_n = \frac{1}{2}, \quad \mu_*\, h_n \approx p_n \ln \mathrm{u}_n \;+\; (1-p_n)\ln \mathrm{d}_n, \quad \sigma^2 h_n \approx p_n(1-p_n)\left[\ln\left(\frac{\mathrm{u}_n}{\mathrm{d}_n}\right)\right]^2.$$

a) Show that $\mathrm{u}_n \approx \mathrm{e}^{\mu_*\, h_n + \sigma\sqrt{h_n}}$ and $\mathrm{d}_n \approx \mathrm{e}^{\mu_*\, h_n - \sigma\sqrt{h_n}}$ for n sufficiently large.

b) Show that the security price formula for a JR tree has the following form for n sufficiently large: $S^{\mathrm{JR}}(t_n) \approx S_0\,\mathrm{e}^{\mu_*\,\tau \,+\, \sigma\sqrt{\tau}\,\mathbb{Z}_n^{\mathrm{JR}}}$, where $\mathbb{Z}_n^{\mathrm{JR}} = \frac{1}{\sqrt{n}}\sum_{j=1}^n X_{n,j}^{\mathrm{JR}}$ and $X_{n,j}^{\mathrm{JR}} \approx \pm 1$ with probability $1/2$ for each possibility.

c) Verify that the triangular array $\mathfrak{X}_{n,j}^{\mathrm{JR}} = \frac{X_{n,j}^{\mathrm{JR}}}{\sqrt{n}}$, where $n \geq 1$ and $j = 1, \ldots, n$, satisfies the hypotheses of the Lindeberg CLT.

d) Using the Lindeberg CLT, determine the continuous-time security price formula to which the JR tree security price converges in distribution. Is this the same formula obtained by using a risk-neutral CRR tree?
e) **(Risk-Neutral JR tree)** Is the JR tree risk neutral? If not, then how would you make it risk neutral? For a sufficiently large n, what would be the approximate governing equations of a risk-neutral JR tree, i.e., the equations expressing $\mathrm{u}_n, \mathrm{d}_n, p_n$ in terms of the inputs $\mathrm{r} - q, \sigma, h_n$?

Solution 5.22.

a)
b)
c)
d)
e) *Hint*: consider Equation (5.48).

5.23. For $t > 0$ and $K > 0$, show that $\mathbb{E}\left(S(t) \mid S(t) > K\right) = \mathbb{E}(S(t)) \frac{\mathsf{N}(\mathrm{d}_{+,\mathrm{RW}}(t))}{\mathsf{N}(\mathrm{d}_{-,\mathrm{RW}}(t))}$ and $\mathbb{E}\left(S(t) \mid S(t) < K\right) = \mathbb{E}(S(t)) \frac{\mathsf{N}(-\mathrm{d}_{+,\mathrm{RW}}(t))}{\mathsf{N}(-\mathrm{d}_{-,\mathrm{RW}}(t))}$.

Solution 5.23. Hints: the conditional expectation of an absolutely continuous random variable Y given an event A with probability $P(A) > 0$ is defined by:

$$\mathbb{E}(Y|A) = \frac{\mathbb{E}(Y\mathbf{1}_A)}{\mathbb{P}(A)} = \frac{1}{\mathbb{P}(A)} \int_A^\infty y f_Y(y) dy,$$

where $\mathbf{1}_A$ is the indicator function on A and f_Y is the p.d.f of Y. In the case of a security, we consider:

$$\mathbb{E}(S(t)|S(t) > K) = \frac{\mathbb{E}(S(t)\mathbf{1}_{\{S(t)>K\}})}{\mathbb{P}(S(t) > K)}, \quad \mathbb{E}(S(t)|S(t) < K) = \frac{\mathbb{E}(S(t)\mathbf{1}_{\{S(t)<K\}})}{\mathbb{P}(S(t) < K)}.$$

Since we know the expressions of the denominators (Section 5.4), only the numerators need to be calculated. Employing the lognormal density, we have

$$\mathbb{E}(S(t)\mathbf{1}_{\{S(t)>K\}}) = \int_K^\infty x \, \frac{1}{\sigma\sqrt{2\pi t}\, x} \, \exp\left(-\frac{\left(\ln\left(x/S(0)\right) - \mu_{\mathrm{RW}} t\right)^2}{2\sigma^2 t} \right) dx.$$

Change to a new variable: $v = \ln\left(\frac{x}{S(0)}\right) - \mu_{\mathrm{RW}} t$. Show that

$$\mathbb{E}(S(t)\mathbf{1}_{\{S(t)>K\}}) = S(0)\mathrm{e}^{\mu_{\mathrm{RW}} t \, + \, \frac{\sigma^2 t}{2}} \int_{\ln\left(\frac{K}{S(0)}\right) - \mu_{\mathrm{RW}} t}^\infty \frac{\exp\left(-\frac{(v-\sigma^2 t)^2}{2\sigma^2 t}\right)}{\sigma\sqrt{2\pi t}} \, dv.$$

Simplify the integral futher by letting

$$w = \frac{v - \sigma^2 t}{\sigma\sqrt{t}}.$$

Show that

$$\mathbb{E}(S(t)\mathbf{1}_{\{S(t)>K\}}) = S(0)\mathrm{e}^{(m-q)\,t} \int_{-\mathsf{d}_{+,\mathrm{RW}}(t)}^{\infty} \frac{\mathrm{e}^{-\frac{w^2}{2}}}{\sqrt{2\pi}} = \mathbb{E}(S(t))\ \mathsf{N}(\mathsf{d}_{+,\mathrm{RW}}(t)).$$

A similar analysis yields

$$\mathbb{E}(S(t)\mathbf{1}_{\{S(t)<K\}}) = \mathbb{E}(S(t))\mathsf{N}(-\mathsf{d}_{+,\mathrm{RW}}(t)).$$

Conclude that

$$\mathbb{E}(S(t)|S(t)>K) = \mathbb{E}(S(t))\frac{\mathsf{N}(\mathsf{d}_{+,\mathrm{RW}}(t))}{\mathsf{N}(\mathsf{d}_{-,\mathrm{RW}}(t))},$$
$$\mathbb{E}(S(t)|S(t)<K) = \mathbb{E}(S(t))\frac{\mathsf{N}(-\mathsf{d}_{+,\mathrm{RW}}(t))}{\mathsf{N}(-\mathsf{d}_{-,\mathrm{RW}}(t))}.$$

5.24. For $t > 0$, $K > 0$, and $K_2 > K_1 > 0$, show that $\mathbb{P}\big(S(t) < K\big) = \mathsf{N}(-\mathsf{d}_{-,\mathrm{RW}}(t))$ and $\mathbb{P}\big(K_1 < S(t) < K_2\big) = \mathsf{N}(\mathsf{d}^{K_1}_{-,\mathrm{RW}}(t)) - \mathsf{N}(\mathsf{d}^{K_2}_{-,\mathrm{RW}}(t))$.

Solution 5.24. Hints: using $S(t) = S(0)\,\mathrm{e}^{\mu_{RW}\,t\,+\,\sigma\sqrt{t}\,Z(t)}$, where $Z \sim N(0,1)$, establish $\mathbb{P}(S(t) < K) = \mathsf{N}(-\mathsf{d}_{-,\mathrm{RW}}(t))$, where $\mathsf{d}_{-,\mathrm{RW}}(t) = \frac{\ln(\frac{S(0)}{K})+\mu_{\mathrm{RW}}t}{\sigma\sqrt{t}}$. For $K_2 > K_1 > 0$, note that

$$\mathbb{P}(K_1 < S(t) < K_2) = \mathbb{P}(S(t) < K_2) - \mathbb{P}(S(t) \leq K_1).$$

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Chapter 6
Selected Solutions for Chapter 6

6.1 Exercises

6.1.1 Conceptual Exercises

6.1. Consider an oversimplified stock price behavior as described by a two-period binomial tree. In each period the stock price either goes up by a factor u with probability p or goes down by a factor d with probability $1 - p$. Identify the corresponding probability space $(\Omega, \mathfrak{F}, \mathbb{P})$.

Solution 6.1. *Hint*: See Example 6.4 on page 256.

6.2. Continue from Exercise 6.1.

Let $\omega_1 = UU$ (i.e. the event that the stock price goes up in both periods). Construct a sub-σ-algebra $\mathcal{F}$ such that $\{\omega_1\} \in \mathcal{F} \subsetneq 2^{\Omega}$.

Solution 6.2. *Hint*: See Example 6.4.

6.3. Continue from Exercise 6.1.

Construct a filtration $\{\mathcal{F}_t \subseteq \mathfrak{F},\ t = t_0, t_1, t_2\}$ such that as time t increases, $\mathcal{F}_t$ reveals more information about the evolution of the stock price.

Solution 6.3.

6.4. Consider an oversimplified stock price behavior as described by a three-period binomial tree. In each period the stock price either goes up by a factor u with probability p or goes down by a factor d with probability $1 - p$. Identify the corresponding sample space Ω (in the notation for the probability space $(\Omega, \mathfrak{F}, \mathbb{P})$). How many elements are there in the σ-algebra $\mathfrak{F} = 2^{\Omega}$? (*Hint*: Each simple event corresponds to a path.)

Solution 6.4.

6.5. Continue from Exercise 6.4.

Given the corresponding sample space

$$\begin{aligned}\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7, \omega_8\}, \text{ where} \\ \omega_1 = UUU,\ \omega_2 = UUD,\ \omega_3 = UDU,\ \omega_4 = UDD, \\ \omega_5 = DUU,\ \omega_6 = DUD,\ \omega_7 = DDU,\ \omega_8 = DDD,\end{aligned}$$

Construct a sub-σ-algebra $\mathcal{F}$ such that

$$\{\omega_1, \omega_2, \omega_3, \omega_4\} \in \mathcal{F} \subsetneq 2^{\Omega}.$$

Solution 6.5.

$$\mathcal{F} = \{\emptyset, \{\omega_1, \omega_2, \omega_3, \omega_4\}, \{\omega_5, \omega_6, \omega_7, \omega_8\}, \Omega\}.$$

6.6. Continue from Exercise 6.4.

Construct a filtration $\{\mathcal{F}_t \subseteq \mathfrak{F},\ t = t_0, t_1, t_2, t_3\}$ such that as time t increases, $\mathcal{F}_t$ reveals more information about the evolution of the stock price.

Solution 6.6. *Hint*: Understand Example 6.17 on page 269 first.

6.7. Let $\mathfrak{B} = \{\mathfrak{B}(t)\}$ be standard Brownian motion. What is the probability that $\mathfrak{B}(1)$ lies between -1 and 1?

Solution 6.7. *Hint*: $\mathfrak{B}(1)$ is a standard normal random variable.

6.8. Let $X = \{X_t\}$ and $Y = \{Y_t\}$ be two processes. Verify the following identity:

$$\Delta(X_t Y_t) = X_t\,\Delta Y_t + Y_t\,\Delta X_t + \Delta X_t\,\Delta Y_t$$

where ΔX_t, ΔY_t and $\Delta(X_t Y_t)$ are defined as the corresponding increments over time interval $[t, t + \Delta t]$. (*Hint*: see (6.38)on page 292.)

Solution 6.8. From page 292 we have:

$$\begin{aligned}\text{LHS} &= X_{t+\Delta t} Y_{t+\Delta t} - X_t Y_t \\ \text{RHS} &= X_t\,(Y_{t+\Delta t} - Y_t) + Y_t\,(X_{t+\Delta t} - X_t) + (X_{t+\Delta t} - X_t)(Y_{t+\Delta t} - Y_t).\end{aligned}$$

Simplify the RHS to obtain the LHS.

6.9. Let $\mathfrak{B} = \{\mathfrak{B}(t)\}$ be standard Brownian motion. Describe your visualization of $-\mathfrak{B}$. Is it different from that of $\mathfrak{B}$?

Solution 6.9. No.

6.10. Use the definition of covariation to derive

$$d\mathfrak{B}(t)\,dt = 0.$$

Solution 6.10.
Proof: Let $\mathbf{P}: 0 = t_0 < t_1 < t_2 < \cdots < t_n = t$ be a partition on the interval $[0,t]$ with $t_k = \frac{kt}{n}$, $k = 0,1,\ldots,n-1$, $n > 0$.

$$\begin{aligned}[\mathfrak{B}, X]_t &= \lim_{|\mathbf{P}|\to 0} \sum_{k=0}^{n-1} (\mathfrak{B}(t_{k+1}) - \mathfrak{B}(t_k))(t_{k+1} - t_k) \\ &= \lim_{n\to\infty} \frac{t}{n} \sum_{k=0}^{n-1} (\mathfrak{B}(t_{k+1}) - \mathfrak{B}(t_k)) = t \lim_{n\to\infty} \frac{1}{n} \int_0^t d\mathfrak{B}(u) = 0,\end{aligned}$$

which implies

$$d\mathfrak{B}(t)\, dt = d[\mathfrak{B}, X]_t = 0.$$

6.11. Suppose that a stock price is modeled by a process $S = \{S(t)\}$ where

$$S(t) = \mathrm{e}^{0.7t^2 + 2.3\mathfrak{B}(t)}.$$

What is the expected growth rate of the stock at any given time t?

Solution 6.11. *Hint*: See page 307 for computational steps.

6.12. Let $\{S(t)\}$ be governed by the s.d.e.

$$dS(t) = \mu S(t)dt + \sigma S(t) d\mathfrak{B}(t),$$

where μ and $\sigma > 0$ are constants. Is $\{S(t)\}$ an Itô diffusion process?

Solution 6.12.

6.13. Recall s.d.e. (6.58) on page 304:

$$dY = \left(f_t + \mu f_x + \frac{1}{2}\sigma^2 f_{xx} \right) dt + \sigma f_x\, d\mathfrak{B}.$$

Compute $\mathbb{E}(dY|\mathcal{F}_t)$ and $\mathrm{Var}(dY|\mathcal{F}_t)$, where $\{\mathcal{F}_t\}$ is the Brownian filtration described in Definition 6.30. Indicate properties of conditional expectation that you applied.

Solution 6.13.

6.14. Let $\{S(t)\}$ be governed by the s.d.e.

$$dS(t) = \mu S(t)dt + \sigma S(t) d\mathfrak{B}(t),$$

where μ and $\sigma > 0$ are constants. Let $Y(t) = \mathrm{e}^{-rt} S(t)$. Use the Itô product rule (see (6.40) on page 292) to compute $dY(t)$.

Solution 6.14. *Hint*: Let $f(t) = \mathrm{e}^{-rt}$. Use the Itô product rule to obtain

$$dY(t) = d(f(t)S(t)) = f(t)dS(t) + d(f(t))S(t) + d[f, S]_t.$$

6.15. Continue from the last exercise. Is $\{Y(t)\}$ defined in Exercise 6.14 a martingale under a risk-neutral probability measure?

Solution 6.15.

6.16. Compute $\int_0^t \mathfrak{B}(s)\, d\mathfrak{B}(s)$ first, then express the s.i.e. in the form of s.d.e.

Solution 6.16. *Hint*: See that famous example in Section 6.7.3 on page 301.

6.17. Is it true that $X = \{X(t), t \geq 0\}$ is a Brownian motion process if and only if $Y = \{Y(t), t \geq 0\}$, where $Y(t) = \mathrm{e}^{X(t)}$, is a geometric Brownian motion process?

Solution 6.17.

6.18. In continuous-time financial mathematics, in what situations is geometric Brwonian motion useful?

Solution 6.18.

6.1.2 *Application Exercises*

6.19. Rewrite s.d.e. (6.27) on page 279 into

$$dS(t) = \mu\, S(t)\, dt + \sigma\, S(t)\, d\mathfrak{B}_t.$$

Consider a time period of length Δt. Compute the ratio of the per-period standard deviation to the per-period drift, i.e. $\frac{\sqrt{\mathrm{Var}(\Delta S(t)}}{\mathbb{E}(\Delta S(t))}$, and interpret your result.

Solution 6.19. *Hint*: Notice that

$$\frac{\sqrt{\mathrm{Var}(\Delta S(t)}}{\mathbb{E}(\Delta S(t))} = \frac{\sigma}{\mu\sqrt{\Delta t}} \to \infty \quad \text{as} \quad \Delta t \to 0.$$

6.20. Find a solution to the s.d.e. $dX(t) = \mu X(t) dt + \sigma X(t) d\mathfrak{B}(t)$ by applying Itô's lemma with $f(x) = \ln x$.

Solution 6.20.

6.21. Verify that p, u and d given by (6.88) on page 319

$$\mathrm{p} = \frac{\mathrm{e}^{(\mathrm{r}-q)\frac{t}{n}} - \mathrm{d}}{\mathrm{u} - \mathrm{d}}, \quad \mathrm{u} = \mathrm{e}^{\sigma\sqrt{\frac{t}{n}}} \quad \mathrm{d} = \mathrm{e}^{-\sigma\sqrt{\frac{t}{n}}}$$

are an approximation to the solution of system (6.86):

$$\begin{cases} \mathrm{pu} + (1-\mathrm{p})\mathrm{d} = \mathrm{e}^{(\mu+\frac{1}{2}\sigma^2)\frac{t}{n}} \\ \mathrm{pu}^2 + (1-\mathrm{p})\mathrm{d}^2 = \mathrm{e}^{2(\mu+\sigma^2)\frac{t}{n}} \\ \mathrm{ud} = 1 \end{cases}$$

Solution 6.21. *Hint*: The notation may be eased by letting $\xi = \mu + \frac{1}{2}\sigma^2$, and by rewriting the system (6.86):

$$\begin{cases} \mathrm{pud} + (1-\mathrm{p})\mathrm{d}^2 = \mathrm{d}e^{\xi\frac{t}{n}} \\ \mathrm{pu}^2 + (1-\mathrm{p})\mathrm{d}^2 = e^{(2\xi+\sigma^2)\frac{t}{n}} \\ \mathrm{ud} = 1 \end{cases}$$

which is equivalent to

$$\begin{cases} \mathrm{p} + (1-\mathrm{p})\mathrm{d}^2 = \mathrm{d}e^{\xi\frac{t}{n}} & (1) \\ \mathrm{pu}^2 + (1-\mathrm{p})\mathrm{d}^2 = e^{(2\xi+\sigma^2)\frac{t}{n}} & (2) \\ \mathrm{ud} = 1 & (3) \end{cases}$$

Note that (1) is equivalent to

$$(1-\mathrm{p})\mathrm{d}^2 = \mathrm{d}e^{\xi\frac{t}{n}} - \mathrm{p}. \qquad (1').$$

6.1.3 Theoretical Exercises

6.22. Given a standard Brownian motion $\mathfrak{B}(t)$, show that each of following stochastic processes is also a standard Brownian motion:

a) $X(t) = \frac{1}{\sqrt{c}}\mathfrak{B}(c\,t)$ for all constants $c > 0$.
b) $Y(t) = \mathfrak{B}(t+c) - \mathfrak{B}(t)$ for all constants $c > 0$.

Solution 6.22.

6.23. Let $\{\mathfrak{B}(t)\}$ be a standard Brownian motion. Let $t_1, t_2, \ldots, t_n \in (0, \infty)$ with $0 < t_1 < t_2 < \cdots < t_n$. Show that the random vector (or multivariate random variable) $(\mathfrak{B}(t_1), \mathfrak{B}(t_2), \ldots, \mathfrak{B}(t_n))$ has a multivariate normal distribution for any fixed choice of n time points $0 < t_1 < t_2 < \cdots < t_n$, $n \geq 1$.

Solution 6.23.

6.24. Continue from the last exercise. Show that the joint probability density function of $(\mathfrak{B}(t_1), \mathfrak{B}(t_2), \ldots, \mathfrak{B}(t_n))$ is:

$$f(x_1, \ldots, x_n) = \frac{\exp\left(-\frac{1}{2}\left(\frac{x_1^2}{t_1} + \sum_{j=1}^{n-1} \frac{(x_{j+1}-x_j)^2}{t_{j+1}-t_j}\right)\right)}{\sqrt{(2\pi)^k\, t_1\,(t_2-t_1)\cdots(t_n-t_{n-1})}},$$

where $-\infty < x_j < \infty$ for $j = 1, \ldots, n$.

Solution 6.24.

6.25. Let $\{\mathfrak{B}(t)\}$ be a standard Brownian motion. Compute $\mathbb{E}(\mathfrak{B}(t))$ for $t \geq 0$ and $\operatorname{Cov}(\mathfrak{B}(s), \mathfrak{B}(t))$ for $s, t \geq 0$. Is Brownian motion a white noise?

Solution 6.25. *Hint*: Compute $\operatorname{Cov}(\mathfrak{B}(s), \mathfrak{B}(t))$.

6.26. Let $\{Z_n : n \geq 1\}$ be a sequence of independent identically distributed random variables with mean $\mu = 0$ and finite variance σ^2. Let

$$S_n = Z_1 + Z_2 + \cdots + Z_n \quad \text{and} \quad X_n = S_n^2 - n\sigma^2.$$

Show that $\{X_n\}$ is a martingale with respect to the natural filtration of the sequence $\sigma(S_n : n \geq 1)$.

Solution 6.26. *Hint*: Show that $\mathbb{E}(X_{n+1}|S_1, S_2, \ldots, S_n) = X_n$.

6.27. Let $W = \{W_i\}$ be a random walk defined in Example 6.27. Show that W is not a white noise.

Solution 6.27. *Hint*: Compute $\operatorname{Cov}(W_i, W_{i-1})$.

6.28. Prove (6.51) on page 299:

$$[X]_t = \int_0^t \sigma^2(s)\, ds,$$

where X satisfies (6.50) on page 298 and $\mu(t)$ is continuous on $[0, \infty)$.

Solution 6.28.
Proof:

$$\begin{aligned} d[X]_t = dX_t dX_t &= \mu^2(t) dt\, dt + 2\,\mu(t)\,\sigma(t)\,dt\,d\mathfrak{B}(t) + (\sigma(t))^2\, d\mathfrak{B}_t\, d\mathfrak{B}_t \\ &= (\sigma(t))^2\, dt. \end{aligned}$$

6.29. Show that $\mathfrak{B}_t^2 - [\mathfrak{B}]_t$ is a martingale with respect to the filtration generated by the Brownian motion itself. That is, $\mathfrak{B}$ is adapted to its natural filtration $\{\mathcal{F}_t\}$, where $\mathcal{F}_t = \sigma(\{\mathfrak{B}_s,\, s \leq t\})$.

Solution 6.29. *Hint*: For $0 \leq s \leq t$, show that $\mathbb{E}(\mathfrak{B}_t^2 - [\mathfrak{B}]_t | \mathcal{F}_s) = \mathfrak{B}_s^2 - [\mathfrak{B}]_s$.

6.30. Show that as positive integer $n \to \infty$ the sequence of random variables $\{W_t^{(n)}\}$, defined by (6.89) on page 321, converges in distribution to a normal random variable in $\mathcal{N}(0, t)$, where $t > 0$ is an integer.

Solution 6.30.

6.31. Let $X = \{X(t)\}$ be an Itô process represented by s.d.e. (6.56)

$$dX(t) = \mu(X(t), t)\, dt \; + \; \sigma(X(t), t)\, d\mathfrak{B}(t), \quad 0 \leq t \leq T.$$

Show that if we define a process $Y = \{Y(t)\}$ by $Y(t) = f(X(t))$, $0 \leq t \leq T$, then Itô's formula (see (6.57) on page 304) has a convenient form:

$$dY = f'(X)dX + \frac{1}{2}f''(X)(dX)^2,$$

where $X = X(t)$. Clearly, this form is easier to remember than the form in (6.57) on page 304 because it bears greater similarity to the Taylor expansion.

Solution 6.31.

6.32. Let $X = (X_1, X_2)$, where X_1 and X_2 are Itô processes governed respectively by the s.d.e.'s

$$dX_i = \mu_i dt + \sigma_i d\mathfrak{B}_i, \quad i = 1,2.$$

Two-dimensional Itô's lemma states that if $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are standard Brownian motion processes with correlation ρ_{12}, then $Y = f(X)$, where $f : \mathbb{R}^2 \to \mathbb{R}$ is twice continuously differentiable, is an Itô process satisfying the s.d.e.

$$\begin{aligned} df(X) &= f_1(X)dX_1 + f_2(X)dX_2 \\ &+ \frac{1}{2}\left(f_{11}(X)dX_1^2 + 2f_{12}(X)dX_1dX_2 + f_{22}(X)dX_2^2\right), \end{aligned} \tag{6.90}$$

where $f_i = \frac{\partial f}{\partial X_i}$ and $f_{ij} = \frac{\partial^2 f}{\partial X_i \partial X_i}$, $i, j = 1, 2.$[29]

Show that (6.90) is equivalent to (6.57) if $X_2 = t$ (i.e. $\mu_2 = 1$ and $\sigma_2 = 0$).

Solution 6.32.
Proof: With $X_1 = X_1$ and $X_2 = t$, we have

$$\begin{aligned} f(X) &= f(X_1, t), \quad dX_2 = dt, \\ (dX_1)^2 &= (\mu_1 dt + \sigma_1 d\mathfrak{B}_1)^2 = \sigma_1^2 dt, \\ (dX_2)^2 &= 0, \\ dX_1 dX_2 &= 0, \end{aligned}$$

(6.1) becomes

$$\begin{aligned} df(X) &= f_1(X)dX_1 + f_t\,dt + \frac{1}{2}f_{11}(X)dX_1^2 \\ &= f_1(\mu_1 dt + \sigma_1 d\mathfrak{B}_1) + f_t\,dt + \frac{1}{2}\sigma^2 f_{11}\,dt. \end{aligned}$$

6.33. Prove the Itô product rule. (*Hint*: prove (6.39) on page 292 by applying two dimensional Itô's lemma given in Exercise 6.32).

Solution 6.33.

6.34. An investment in a foreign asset carries exchange risk. The model under our consideration is introduced by Briys and Solnik [4] in study of hedging such risk.

[29] More precisely speaking, we assume that $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are defined on the same filtered probability space $\{\Omega, \mathfrak{F}, \{\mathcal{F}_t\}, \mathbb{P}\}$ and adapted to the filtration $\{\mathcal{F}_t\}$.

Let $V(t)$ be the local (domestic) currency value of a foreign asset at time t. Let $S(t)$ be the exchange rate at time t expressed as the local currency value of one unit of foreign currency (e.g. 1.11 USD/Euro). The model assumes that both $\{V(t)\}$ and $\{S(t)\}$ are geometric Brownian motion processes:

$$\frac{dV}{V} = \mu_V dt + \sigma_V d\mathfrak{B}_V$$
$$\frac{dS}{S} = \mu_S dt + \sigma_S d\mathfrak{B}_S,$$

where two standard Brownian motion processes $\mathfrak{B}_V$ and $\mathfrak{B}_S$ have correlation ρ_{VS}.

Let $V^* = VS$, the value of the foreign investment expressed in domestic currency. Compute $\frac{dV^*}{V^*}$ and interpret your answer. (*Hint*: apply Itô product rule.)

Solution 6.34. *Hint*: Notice that both V and S are Itô processes.

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Chapter 7
Selected Solutions for Chapter 7

7.1 Exercises

Unless stated otherwise, an option is either American or European.

7.1.1 Conceptual Exercises

7.1. Is a forward a contingent claim? Is an option a contingent claim?

Solution 7.1.

7.2. There are mainly three types of derivative traders: hedgers, speculators and arbitrageurs. What are their definitions?

Solution 7.2. *Hint*: For hedgers and speculators, see page 331.
For arbitrageurs, see page 334.

7.3. Sketch the terminal payoff diagram of a forward with expiration T and forward price K. If you short this forward, what is the terminal payoff diagram?

Solution 7.3.

7.4. What are the key features of futures that differ from forwards?

Solution 7.4. Standardization (exchanged-traded), margin account requirements and the mark-to-market daily account settlement.

7.5. If you believe that the market price of a stock will stay at approximately the same price for a period of time, can you still make money from the stock if your hunch is correct? Explain your answer.

Solution 7.5.

7.6. What is the possible maximum gain or loss if you sell a call?

Solution 7.6.

The maximum gain equals premium.
The maximum loss is unlimited.

7.7. What is the possible maximum gain or loss if you buy a call?

Solution 7.7.

7.8. Consider the call options given in Example 7.45 (see page 372). Are they all in the money? Identify their intrinsic value(s).

Solution 7.8. Yes.

7.9. If a call is in the money sufficiently close to the expiration date, then the call price will rise dollar for dollar with the stock price. Agree or disagree? Explain.

Solution 7.9.

7.10. An at-the-money American call with a strike price of \$80 is being sold for \$200. Assume that the stock goes up to \$84 per share on the day of expiration.

a) If you bought the option, what is your return from exercising the call and liquidating your stock position? If you did not buy the option, but had bought 100 shares of the stock in the market at \$80 per share and then sold them on the option's expiration date at \$84 per share, what would be your return? Do the two scenarios have equivalent gain/loss?
b) If you do not exercise the option, what is your approximate return from selling the call right before expiration?
c) Which would you then prefer? Exercise the call or sell the call?

Solution 7.10.

7.11. You sell an American call on 1 round lot of a stock at \$40 per share. A month later, the market value of that stock is \$46 per share. If the buyer exercises the option, you will be obligated to deliver 100 shares at \$6 below current market value.

a) If you own those shares, what is your gain/loss from settling the position?
b) If you had naked short sold the American call, what is your gain/loss from settling the position?

Solution 7.11.

a) You must sell 100 shares to the holder at the strike price \$40 to receive \$4,000. You lose \$600 since you could have received \$4,600 from selling your shares in the market at \$46 had you not sold the call.
b) Since you sold the call naked, you have to buy 100 shares of the stock at \$46 and then sell them at \$40, which is a loss of \$600.

7.12. You paid \$300 for an American call on a stock several months ago. It will expire next month and is now worth only \$100. What are the feasible actions that you can take? What are the consequences of your actions?

Solution 7.12. You can sell the call and take a loss of \$200 or hold until just before expiration, hoping that the stock's price will move sufficiently upward and sell the call then to break even or make a profit. If you do not sell the call by the expiration date, the option will likely become worthless. The reason is that to make a profit (excluding taxes and any transaction costs) you need the option's value to more than triple on the sale date, which is only one month away.

7.13. Argue how American put buying/selling works.

a) How American put buying works.
   Buyers of a put expect the underlying stock to fall in value. In each of the following cases, what are the feasible actions of a buyer and their outcomes in terms of monetary gain or loss?
   Case 1: The price of the stock increases after the buyer purchased the put.
   Case 2: The price of the stock almost does not change.
   Case 3: The price of the stock decreases and the exercise price of the put is higher than the price of the stock at the expiration date.
b) How American put selling works.
   Under the plan of selling puts, you grant someone else the right to sell 100 shares to you at the exercise price. At the time you sell, you receive a premium. Like the call seller, you do not have much control over the outcome of your investment since the buyer will decide whether to exercise the put you sold him. In each of the following cases, what are the feasible actions of a put seller and their monetary results?
   Case 1: The price of the stock increases.
   Case 2: The price of the stock remains stable till the expiration date.
   Case 3: The price of the stock decreases and the put is in the money at the expiration.

Solution 7.13.

7.14. Hedge/hedging is a strategy used to offset investment risk. A perfect hedge is one eliminating the possibility of future gain or loss.

A stockholder worried about declining stock prices, for instance, can hedge his or her holdings by buying a put on the stock or by selling a call.

a) How does each case work?
b) Which type of hedging is preferable?

Solution 7.14. For part a) only:

a) Let S=stock price/share at time 0 and S_t =stock price at time t.
Case 1: To hedge by buying a put P=P(S,K,t) (price of put at time 0) . The possible maximum gain of buying put is K-P. The possible maximum loss of buying put is P. Therefore, the possible maximum gain of hedged holdings is infinite (∞). The possible maximum loss of hedged holdings is S-(K-P), assuming S$\geq$K at time 0.

Case 2: To hedge by selling a call at unit price C=C(S,K,t) at time 0, the possible maximum gain of selling a call is C. The possible maximum loss of naked call if infinite (∞). Therefore, the possible maximum gain of hedged holdings is C, assuming S$\geq$K at time 0, while the possible maximum loss of hedged holdings is S-C.

7.15. (Call Time Spread Bearish) Recall Example 1. Given XYZ 40 call price table:

Expiration	*Nov*	*Dec*	*Jan*
Premium	2	3	5

If one expected XYZ stock to decline, one might establish a bear spread by taking a position opposite of a bullish one.

Make two transactions to establish a spread in the hope of making a profit if XYZ stock's price declines and of limiting the loss if the expectation turns out to be wrong.

Solution 7.15.
(**Correction:** "Recall Example 1." in the beginning of the statement of this exercise should be removed.)

7.16. (Price Put Spread Bearish) Open a bear spread by using the following puts:

XYZ Dec 40 Put at 3
XYZ Dec. 45 Put at 7

in the hope of making a profit if XYZ stock declines in price. What is the possible maximum gain or loss? Justify your answers.

Solution 7.16.

7.1.2 *Application Exercises*

7.17. (Forward price and arbitrage) Suppose that the current spot price of a continually paying dividend asset is \$222, the interest rate is $r = 3\%$ and the dividend yield is $q = 2\%$.

a) What are the one-month and eight-month forward prices for the asset in an arbitrage-free market?
b) Let Π be a portfolio on time interval $[0, T]$ consisting of three positions starting from time 0: borrow \$222 at the rate 3%, long 1 unit of the asset and short the three-month forward at $F_T(0) = \$222.56$. Is Π an arbitrage portfolio? If your answer is no, show a proof. If your answer is yes, explain how you can make a profit by taking the arbitrage opportunity.

Solution 7.17.

7.18. (Forward value) Given $K_0 = \$222$, $F_T(t) = \$252$, $T - t = 6$ months, and $r = 3\%$, determine the value of the forward in an arbitrage-free market and interpret the result.

Solution 7.18. The value of the forward = 25.82 (the amount that the seller would compensate the buyer).

7.19. (Swaps) Assume that the terms of the swap contract include the following:

a) the notional principal is one million dollars,
b) the life of the contract is 2 years,
c) A pays B three-month $LIBOR + 0.2\%$,
d) B pays A 1.5% fixed,
e) there is an exchange of payments every 3 months from the initialization.

Given the $LIBOR$ rates in the table below, calculate both the floating cash flow and fixed cash flow of the swap.

Period	*LIBOR*	*Payment from A to B*	*Payment from B to A*
0	1%		
1	0.8%		
2	1%		
3	1.2%		
4	1.06%		
5	1.1%		
6	1.2%		
7	1.4%		
8			

Solution 7.19. *Hint*:

Period	$LIBOR$	*Payment from A to B*	*Payment from B to A*
0	1%		
1	0.8%	\$3,000	\$3,750
2	1%	\$2,500	\$3,750
3	1.2%		
4	1.06%		
5	1.1%		
6	1.2%		
7	1.4%		
8			

7.20. (Swaps) Suppose that both companies X and Y need to borrow US dollars and that company X would like to borrow at a fixed rate, whereas company Y would like to borrow at a floating rate. If X can borrow at 6.00% fixed and $LIBOR + 0.60\%$ floating, and Y can borrow at 5.00% fixed and $LIBOR + 0.20\%$ floating, what is the range of possible cost savings that company X can realize through an interest rate swap with company Y? Use an example of swap mechanics to demonstrate how a cost saving to be done for either company (ignoring credit risk differences).

Solution 7.20.

7.21. Identify the range of profit, loss and break-even point as outcomes of the corresponding strangle strategy given in Example 7.39 (page 366).

Solution 7.21. Range of profit and loss are $(0,\, K_1 - a) \cup (K_1 + a,\, \infty)$ and $(K_1 - a,\, K_1 + a)$ respectively.

Breakeven points are $K_1 - a$ or $K_1 + a$.

7.22. Use a formula to express the terminal payoff of each spread strategy given in Example 7.40 (page 366).

Solution 7.22.

7.23. Use a formula to express the terminal payoff of the butterfly spread given in Example 7.42 (page 367).

Solution 7.23.

7.1.3 Theoretical Exercises

7.24. (Arbitrage)

a) Suppose that the price of a stock at time t, denoted by $S(t)$, is modeled by a one-step binomial tree over the time period $[0, T]$ with

$$S(T) = \begin{cases} S_b & \text{with probability } p \\ S_a & \text{with probability } 1 - p, \end{cases}$$

where $S_b > S_a$.
Show that $S_b > S(0) > S_a$ is a necessary condition for a non-arbitrage opportunity for any investor (assuming $r_f = 0$).

b) Show that there exists a (risk-neutral) probability $p > 0$ holding the equation

$$S(0) = pS_b + (1 - p)S_a.$$

Solution 7.24.
Proof (for part a only):
(a) If $S_b > S_a > S(0)$, then borrow money to buy stock.
If $S(0) > S_b > S_a$, then short the stock.

7.25. (Forward value) Prove (7.5) (page 344).

Solution 7.25.

7.26. Show that if the price of the underlier of a forward contract follows a geometric Brownian motion, so does the forward price process (see Example 7.15).

Solution 7.26. *Hint*: Apply Itô's lemma and see Example 7.15 on page 343.

7.27. (General property of options) Let $C_A(0, K, T_1)$ and $C_A(0, K, T_2)$ be two American call options on the same terms except that they have different expirations with $T_1 < T_2$. Show that $C_A(0, K, T_1) \leq C_A(0, K, T_2)$.

Solution 7.27.
Proof: Because of time value.

7.28. Establish the following bounds for American puts on nondividend-paying underliers:

$$K \geq P^A(0) \geq \max\{K - S(0), 0\}.$$

Solution 7.28.

7.29. Establish the following relation between American and European puts on the same nondividend-paying underlier and with the same expiration T and strike K:

$$P^A(0) \geq P^E(0).$$

Solution 7.29.

7.30. Is it never advantageous to exercise early an American put on a nondividend-paying stock? Justify your answer.

Solution 7.30. One may exercise early if $S(0)$ is very small as the profit is bounded by K.

7.31. Establish the following put-call parity bounds for American options:

$$S(t_0)\,\mathrm{e}^{-q\tau} - K \leq C^A(T) - P^A(t_0) \leq S(t_0) - K\,\mathrm{e}^{-r\tau},$$

where $\tau = T - t_0$.

Solution 7.31.

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Chapter 8

Selected Solutions for Chapter 8

8.1 Exercises

8.1.1 Conceptual Exercises

8.1. A modeler who knows nothing about the BSM model is trying to find a formula for the present value $C(0)$ of a European call option, where the underlying security has current price $S(0)$ and the strike price is K. She proposes the following formula after considerable experimentation:

$$C(0) = w_1 S^n(0) \ + \ w_2 K^m,$$

where the weights w_1 and w_2 are to be determined. Without using any information about the BSM model, give a two-sentence argument that determines the possible values of n and m.

Solution 8.1.

8.2. Give a brief intuitive reason why a European call option is more risky than its underlying security.

Solution 8.2.

8.3. Express the return rate of a European call during an instant dt as a s.d.e.

Solution 8.3. *Hint*: Itô's formula.

8.4. If a stock satisfies the CAPM, then does a European call on the stock also satisfy the CAPM? Justify your answer.

Solution 8.4. Yes. *Hint*: try to obtain

$$\mu_C - r_t = \frac{S}{C^E}\frac{\partial C^E}{\partial S}(\mu_s - r_t).$$

8.5. Traders often abide by simple intuitive rules concerning volatility. Here are some examples you may have heard:

"Sell a stock when its volatility is high."
"Favor puts when volatility is high."
"Buy a stock when volatility is low."

Are these rules of thumb captured by the BSM model for underliers? Justify your answer.

Solution 8.5. Yes.

8.6. Explain why a European call and put have the same implied volatility.

Solution 8.6. *Hint*: put-call parity.

8.7. Briefly critique the MJD model's assumption that the risk of price jumps is diversifiable.

Solution 8.7.

8.1.2 *Application Exercises*

8.8. (Price change in options versus stocks) Traders use options for speculation. To get an intuitive feel for why this is the case, we consider an example of how the price of a European call option changes with variations in the underlying security. A financial company's stock currently has a price of \$40. The risk-free interest rate is 7% per annum and the stock has volatility parameter of 28%. Consider a European call option on the stock for a strike price of \$41 with expiration in 6 months. Let t be the current time and $t + h$ an hour later.

a) From time t to $t + h$, the price of the stock increases by 1%. What is the percentage change in the value of the call? Would the price of the put move by the same percentage?
b) From time t to $t + h$, the price of the stock decreases by 1%. What is the percentage change in the value of the call? Would the price of the put move by the same percentage?

Solution 8.8. By Table 8.1, we find:

a) A 1% increase in the stock's price over the given hour causes a 6.8% increase in the price of the call. For this case, the price of the put will decrease by 5.9%.
b)

Current Prices	Security Price Increases 1%	Security Price Decreases 1%
$S_t = \$40$	$S_{t+h} = \$40.4$	$S_{t+h} = \$39.6$
$C_t = \$3.347$	$C_{t+h} = \$3.575$	$C_{t+h} = \$3.127$
$P_t = \$2.937$	$P_{t+h} = \$2.765$	$P_{t+h} = \$3.117$

Table 8.1 Change in value of the call and put prices with respect to security price change.

8.9. (European calls as insurance) After careful research, a fund manager would like to purchase 100,000 shares of a nondividend-paying stock currently trading at \$60. The fund manager estimates the stock's volatility at 15% and believes the stock will rise over the coming months. The \$6 million needed to buy the shares now will not be available until a month away. He is concerned that if the stock rises over the next month, it will become too expensive to buy. How can the fund manager insure against the risk of the stock price increasing? How much would this insurance cost? Suppose that the risk-free rate is 2%.

Solution 8.9.

8.10. (European puts as insurance) An investor owns 10,000 shares of a stock paying no dividend and currently trading at \$160 per share. The stock has a volatility of 20% and the current risk-free rate is 2.5%. Three months from now she would like to liquidate the shares to purchase an investment property for \$1,500,000. She is concerned that if the stock price falls over the next three months, she would not be able to buy the property. On the other hand, she does not want to sell her shares now since there is also the possibility that the stock price will increase over the next three months and so she would miss out on such gains. How can she mitigate against this risk? Note that since the portfolio has a single stock, Markowitz portfolio theory does not directly apply.

Solution 8.10. *Hint*: she can insure against the risk using puts.

8.11. (Warrants) Assume that the total assets of a company satisfies the BSM model. Suppose that the current total value of the company is \$50 million. Assume that its stock pays no dividend and is presently trading at \$50 per share. The stock's volatility is 25% and the risk-free rate is 6%. The company plans to issue $300,000$ warrants with strike price of \$70 and maturity in 3 years. Each warrant is based on 1 share of the company's stock. Determine how much money the company will raise if it sells all the warrants at a fair price.

Solution 8.11. \$1,304,146.869.

Table 8.2 Delta-hedging table for Exercise 8.12.

Delta Hedging Values Per Share							
t	S_t	$\Delta_C(t)$	$-C_t^E$ short call	I_t investment	$-\mathbb{L}_t$ loan balance	$\Delta_C(t)\,S_t$ long $\Delta_C(t)$ shares	$\mathcal{V}_C(t)$ portf. value
Day 0	$75.000000		-$2.264248				$0.000000
Day 1	$75.472305	0.574123		$2.264373			$0.004812
Day 2	$75.646263						
Day 3	$75.523208						$0.029374
Day 4	$ 76.652920						
Day 5	$77.036379						
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
Day 79	$69.932887	0.000000	$ 0.000000	$ 2.274071	-$2.146712	$0.000000	$ 0.127359
Day 80	$69.904710						

8.12. (Delta hedging European calls that end out-of-the-money at expiration) Assume that a firm sells 1,000 European calls (in round lots) on a nondividend-paying stock with current price $75, strike $75, annual volatility of 20%, and 80 days to expiration. Suppose that the risk-free rate is 2% and assume 365 days in a year. In Table 8.2 some entries are shown for delta hedging based on a MATLAB code that outputs values to at least nine decimal places, but were rounded at the sixth decimal place so the entries appear less congested. If you work to six decimal places only, then there naturally will be rounding errors in the day-to-day delta hedging and not all your numerical values will exactly match those in the table. See Example 8.9 on page 428 and Remark 8.10 in that example.

a) Complete the values for Days 1-5 and Day 80 in Table 8.2. Assume that the firm sold the European calls at the BSM price. Did the firm experience a profit or a loss? Determine how much.
b) If the firm sold the calls at $3.50 per share of the stock, did the firm have a profit or loss? Determine the amount.

Solution 8.12. a) The firm had a profit of $\$12,736.60$.
b) The firm gets a total profit of $\$136,854.61$.

8.13. (Delta hedging European calls that end in-the-money at expiration) Suppose that a firm sells 800 European calls (in round lots) on a nondividend-paying stock with current price $110, strike price $110, annual volatility of 20%, and 90 days to expiration. Suppose that the risk-free rate is 3% and assume 365 days in a year (see Remark 8.10). Table 8.3 shows a portion of delta hedging

Delta Hedging Values Per Share							
t	S_t	$\Delta_C(t)$	$-C_t^E$	$\mathtt{I}_t$	$-\mathbb{L}_t$	$\Delta_C(t)\,S_t$	$\mathcal{V}_C(t)$
			short call	investment	loan balance	long $\Delta_C(t)$ shares	port. value
Day 0	$75.000000	0.538849	-$2.264248	$ 2.264248	-$40.413671	$40.413671	$0.000000
Day 1	$75.472305	0.574123	-$ 2.511847	$2.264373	-$43.078091	$43.330378	$0.004812
Day 2	$75.646263	0.587081	-$2.597501	$ 2.264497	-$44.060672	$44.410471	$ 0.016795
Day 3	$75.523208	0.577927	-$2.510389	$ 2.264621	-$43.371758	$43.646900	$0.029374
Day 4	$ 76.652920	0.660400	-$3.194983	$ 2.264745	-$49.695938	$ 50.621594	-$0.004582
Day 5	$77.036379	0.687378	-$3.438249	$ 2.264869	-$51.776973	$ 52.953142	$0.002789
Day 6	$ 76.986565	0.684698	-$3.388942	$2.264993	-$51.573471	$ 52.712563	$ 0.015143
Day 7	$ 77.171148	0.697974	-$3.501393	$2.265117	-$52.600799	$ 53.863448	$ 0.026373
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
Day 79	$69.932887	0.000000	$ 0.000000	$ 2.274071	-$2.146712	$0.000000	$ 0.127359
Day 80	$69.904710	0.000000	$0.000000	$ 2.274196	-$2.146830	$ 0.000000	$ 0.127366

Table 8.3 Delta-hedging table for Exercise 8.13.

Delta Hedging Values Per Share							
t	S_t	$\Delta_C(t)$	$-C_t^E$	$\mathtt{I}_t$	$-\mathbb{L}_t$	$\Delta_C(t)\,S_t$	$\mathcal{V}_C(t)$
			short call	investment	loan balance	long $\Delta_C(t)$ shares	port. value
Day 0	$ 110.000000			$ 4.757733			
Day 1	$111.276638						-$0.004965
Day 2	$108.680788	0.499957					- $0.103196
Day 3	$110.757606						
Day 4	$110.049030						
Day 5	$111.589771						-$0.161611
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
Day 89	$125.092736	1.000000	-$15.101777	$4.792664	- $ 115.392349	$125.092736	-$ 0.608726
Day 90	$124.937041						

using a MATLAB code; see the comment in Exercise 8.12 about rounding errors.

a) Compute the values for Days 1-5 and Day 90 in Table 8.3 under the assumption that the firm sold the European calls at the BSM price. Did the firm experience a profit or a loss? Determine how much.

b) If the firm sold the calls at \$5.50 per share of the stock, did the firm have a profit or loss? Determine the amount.

Solution 8.13.

8.1.3 *Theoretical Exercises*

8.14. For a cum-dividend security price $S_t^c = e^{qt} S_t$, where S_t follows geometric Brownian motion, show that $dS_t^c = m\,S_t^c\,dt + \sigma\,S_t^c\,d\mathfrak{B}_t$.

Solution 8.14. *Hint*: Itô product rule.

8.15. Show that the self-financing condition $\big(dn_t\big)\,S_{t+dt}^c + \big(db_t\big)\,B_{t+dt} = 0$ is equivalent to $dV_t = n_t\,dS_t^c \;+\; b_t\,dB_t$.

Solution 8.15. *Hint*: use

$$dV_t = d(n_t\,S_t^c) + d(b_t\,B_t).$$

8.16. If the BSM p.d.e. does not hold, then there is an arbitrage. Agree or disagree? Justify your answer.

Solution 8.16.

8.17. If $f(x,t)$ is a solution of the BSM p.d.e., then show that for every positive constant $c > 0$, the function $f_c(x,t) = f(c\,x,t)$ is also a solution.

Solution 8.17.

8.18. Given solutions $f_1(x,t), \ldots, f_n(x,t)$ of the BSM p.d.e., show that all linear combinations $c_1\,f_1(x,t) \;+\; \cdots \;+\; c_n\,f_n(x,t)$ are also solutions.

Solution 8.18.

8.19. If a solution $f(x,t)$ of the BSM p.d.e. has an nth partial derivative with respect to x, then show that $x^n\,\dfrac{\partial^n f}{\partial x^n}(x,t)$ is also a solution.

Solution 8.19. *Hint*: use induction, i.e., show that the result is true for $n = 1$. Then assume that the result is true for $n = k$ (integer greater than 2) and prove that it holds for $n = k+1$.

8.20. Show that, for the price process $\widehat{S}_t = S_t\,e^{-q(T-t)}$, the BSM p.d.e. (8.18) on page 391 transforms to a form without dividend:

$$\frac{1}{2}\sigma^2\bar{x}^2\frac{\partial^2 C^E}{\partial \bar{x}^2}(\bar{x},t) + \mathsf{r}\bar{x}\,\frac{\partial C^E}{\partial \bar{x}}(\bar{x},t) + \frac{\partial C^E}{\partial t}(\bar{x},t) - \mathsf{r}\,C^E(\bar{x},t) = 0,$$

where $\bar{x} = x\,e^{-q(T-t)}$.

Solution 8.20. *Hint*: assume

$$f(x,t) = f(x(\bar{x},\bar{t}), t(\bar{x},\bar{t})).$$

Then the partial derivatives are:

$$\frac{\partial f}{\partial x} = \frac{\partial \bar{x}}{\partial x}\frac{\partial f}{\partial \bar{x}} + \frac{\partial \bar{t}}{\partial x}\frac{\partial f}{\partial \bar{t}},$$
$$\frac{\partial f}{\partial t} = \frac{\partial \bar{x}}{\partial t}\frac{\partial f}{\partial \bar{x}} + \frac{\partial \bar{t}}{\partial t}\frac{\partial f}{\partial \bar{t}}.$$

8.21. Using the variables,

$$\tilde{x} = \ln\left(\frac{\bar{x}}{K}\right), \quad \tilde{\tau} = \frac{\sigma^2}{2}(T-t), \quad \mathsf{v}(\tilde{x},\tilde{\tau}) = \frac{C^E(\bar{x},t)}{K}, \quad \tilde{k} = \frac{\mathsf{r}}{(\sigma^2/2)},$$

show that (8.24) and (8.29) transform, respectively, to

$$\frac{\partial \mathsf{v}}{\partial \tilde{\tau}}(\tilde{x},\tilde{\tau}) = \frac{\partial^2 \mathsf{v}}{\partial \tilde{x}^2}(\tilde{x},\tilde{\tau}) + (\tilde{k}-1)\frac{\partial \mathsf{v}}{\partial \tilde{x}}(\tilde{x},\tilde{\tau}) - \tilde{k}\,\mathsf{v}(\tilde{x},\tilde{\tau})$$

and

$$\mathsf{v}(\tilde{x},0) = \max\{e^{\tilde{x}} - 1,\, 0\}, \quad \lim_{\tilde{x}\to-\infty} \mathsf{v}(\tilde{x},\tilde{\tau}) = 0, \quad \mathsf{v}(\tilde{x},\tilde{\tau}) \to e^{\tilde{x}} \text{ as } \quad \tilde{x}\to\infty.$$

Solution 8.21.

8.22. Using a trial solution $\mathsf{v}(\tilde{x},\tilde{\tau}) = \tilde{u}(\tilde{x},\tilde{\tau})\,e^{a\tilde{x}+b\tilde{\tau}}$, show that for the choices $a = -\frac{1}{2}(\tilde{k}-1)$ and $b = -\frac{1}{4}(\tilde{k}+1)^2$, Equation (8.24) transforms into the heat equation

$$\frac{\partial \tilde{u}}{\partial \tilde{\tau}}(\tilde{x},\tilde{\tau}) = \frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2}(\tilde{x},\tilde{\tau})$$

and (8.25) into

$$\tilde{u}(\tilde{x},0) = \max\left\{e^{\frac{1}{2}(\tilde{k}+1)\tilde{x}} - e^{\frac{1}{2}(\tilde{k}-1)\tilde{x}},\, 0\right\}, \qquad \lim_{|\tilde{x}|\to+\infty} \tilde{u}(\tilde{x},\tilde{\tau})\,e^{-c\tilde{x}^2} = 0,$$

where $c > 0$.

Solution 8.22. Hints:

$$\frac{\partial \mathsf{v}}{\partial \tilde{\tau}}(\tilde{x},\tilde{\tau}) = \frac{\partial^2 \mathsf{v}}{\partial \tilde{x}^2}(\tilde{x},\tilde{\tau}) + (\tilde{k}-1)\frac{\partial V}{\partial \tilde{x}}(\tilde{x},\tilde{\tau}) - \tilde{k}\mathsf{v}(\tilde{x},\tilde{\tau}).$$

We can convert the equation above into a diffusion equation, using a trial solution $\mathsf{v}(\tilde{x},\tilde{\tau}) = \tilde{u}(\tilde{x},\tilde{\tau})e^{a\tilde{x}+b\tilde{\tau}}$:

$$\begin{aligned}
\frac{\partial \mathsf{v}}{\partial \tilde{\tau}}(\tilde{x},\tilde{\tau}) &= \frac{\partial \tilde{u}}{\partial \tilde{\tau}}(\tilde{x},\tilde{\tau})\mathrm{e}^{a\tilde{x}+b\tilde{\tau}} + \tilde{u}(\tilde{x},\tilde{\tau})b\mathrm{e}^{a\tilde{x}+b\tilde{\tau}},\\
\frac{\partial \mathsf{v}}{\partial \tilde{x}}(\tilde{x},\tilde{\tau}) &= \frac{\partial \tilde{u}}{\partial \tilde{x}}(\tilde{x},\tilde{\tau})\mathrm{e}^{a\tilde{x}+b\tilde{\tau}} + \tilde{u}(\tilde{x},\tilde{\tau})a\mathrm{e}^{a\tilde{x}+b\tilde{\tau}},\\
\frac{\partial^2 \mathsf{v}}{\partial \tilde{x}^2}(\tilde{x},\tilde{\tau}) &= \frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2}(\tilde{x},\tilde{\tau})\mathrm{e}^{a\tilde{x}+b\tilde{\tau}} + 2\frac{\partial \tilde{u}}{\partial \tilde{x}}(\tilde{x},\tilde{\tau})a\mathrm{e}^{a\tilde{x}+b\tilde{\tau}} + \tilde{u}(\tilde{x},\tilde{\tau})a^2\mathrm{e}^{a\tilde{x}+b\tilde{\tau}}.
\end{aligned}$$

Plug the above three equations into the foremost equation, we can obtain

$$\frac{\partial \tilde{u}}{\partial \tilde{\tau}}(\tilde{x},\tilde{\tau}) = \frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + (2a+\tilde{k}-1)\frac{\partial \tilde{u}}{\partial \tilde{x}}(\tilde{x},\tilde{\tau}) + (a^2+(\tilde{k}-1)a-\tilde{k}-b)\tilde{u}(\tilde{x},\tilde{\tau}).$$

Now determine constraints on a and b to obtain the heat equation. Determine the initial and boundary conditions.

8.23. Derive equation (8.32) on 394 and show that it equals

$$C^E(x,t) = x\,\mathrm{e}^{-q(T-t)}\,\mathsf{N}\big(d_+(x,T-t)\big) \;-\; K\mathrm{e}^{-\mathsf{r}(T-t)}\,\mathsf{N}\big(d_-(x,T-t)\big).$$

Solution 8.23. *Hint*:

$$\tilde{u}(\tilde{x},\tilde{\tau}) = \frac{1}{2\sqrt{\pi\tilde{\tau}}}\int_{-\infty}^{\infty}\tilde{u}(s,0)\mathrm{e}^{\frac{(\tilde{x}-s)^2}{4\tilde{\tau}}}ds,$$

where

$$\begin{aligned}
\tilde{u}(\tilde{x},0) &= \max\{\mathrm{e}^{\frac{1}{2}(\tilde{k}+1)\tilde{x}} - \mathrm{e}^{\frac{1}{2}(\tilde{k}-1)\tilde{x}},0\}\\
&= \begin{cases}\mathrm{e}^{\frac{1}{2}(\tilde{k}+1)\tilde{x}} - \mathrm{e}^{\frac{1}{2}(\tilde{k}-1)\tilde{x}}, & \text{if} \quad \tilde{x}>0,\\ 0, & \text{if} \quad \tilde{x}\le 0.\end{cases}
\end{aligned}$$

8.24. Consider the discounted underlier price process $\{\widetilde{S}_t\}_{t\ge 0}$, where $\widetilde{S}_t = \mathrm{e}^{-\mathsf{r}t}\,S^c_t$ with $S^c_t = \mathrm{e}^{qt}S_t$ the cum-dividend price process, and a discounted self-financing, replication portfolio value process $\{\widetilde{V}_t\}_{t\ge 0}$, where $\widetilde{V}_t = \mathrm{e}^{-\mathsf{r}t}\,V_t$. Show

a) $d\widetilde{S}_t = \sigma\,d\mathfrak{B}^{\mathbb{Q}}_t$, where $d\mathfrak{B}^{\mathbb{Q}}_t = d\mathfrak{B}_t + \left(\frac{m-\mathsf{r}}{\sigma}\right)dt$.
b) $dV_t = \mathsf{r}\,n_t\,V_t\,dt \;+\; n_t(m-\mathsf{r})\,S^c_t\,dt \;+\; n_t\,\sigma\,S^c_t\,d\mathfrak{B}_t$.
c) $d\widetilde{V}_t = n_t\,d\widetilde{S}_t$.

Solution 8.24.

a) *Hint*: Itô's Product Rule.
b) *Hint*: apply $d\mathfrak{B}_t = r\mathfrak{B}_t dt$ and $dS^c_t = mS^c_t + \sigma S^c_t d\mathfrak{B}_t$ into $dV_t = n_t dS^c_t + b_t d\mathfrak{B}_t$, we can obtain

$$dV_t = n_t[mS^c_t dt + \sigma S^c_t d\mathfrak{B}_t] + b_t v\mathfrak{B}_t dt.$$

c)

8.25. Consider a nonvanishing stochastic process $\{X_t\}_{t\ge 0}$ such that

$$\frac{dX_t}{X_t} = a(X_t, t)\,dt + b(X_t, t)\,d\mathfrak{B}_t,$$

where $a(x,t)$ and $b(x,t)$ are deterministic functions. The coefficients $a(X_t, t)$ and $b(X_t, t)$ are called the *drift* and *volatility*, respectively, of $\{X_t\}_{t\geq 0}$. For example, a security price following geometric Brownian motion has constant volatility $b(X_t, t) = \sigma$. Show that the volatility of a European call is strictly greater than the volatility of its underlying security.

Solution 8.25. *Hint*: by Itô's formula,

$$dC^E = \left(\frac{1}{2}\sigma_s^2 S^2 \frac{\partial^2 C^E}{\partial S^2} + \mu_s S \frac{\partial C^E}{\partial S} + \frac{\partial C^E}{\partial t}\right) dt + \sigma_s S \frac{\partial C^E}{\partial S} d\mathfrak{B}.$$

8.26. Assume that a security satisfies the CAPM. Show that the beta of a European call on the security is strictly greater than the beta of the security.

Solution 8.26.

8.27. Establish the following:

If $S(T) = K$, then $\Delta_C(t) \to 1/2$ and $\Delta_P(t) \to -1/2$ as $\tau \to 0$.
If $S(T) > K$, then $\Delta_C(t) \to 1$ and $\Delta_P(t) \to 0$ as $\tau \to 0$.
If $S(T) < K$, then $\Delta_C(t) \to 0$ and $\Delta_P(t) \to -1$ as $\tau \to 0$.
If $t < T$, then $\Delta_C(t) < 1$ and $\Delta_P(t) > -1$.

Solution 8.27.

8.28. Show that the discounted underlier process $X_t = \mathrm{e}^{-(\mathrm{r}-q)\,t}\, S_t$ and discounted derivative price process $Y_t = \mathrm{e}^{-\mathrm{r}\,t}\, f(S_t, t)$ are martingales relative to the risk-neutral measure $\mathbb{Q}$ of Girsanov's theorem.

Solution 8.28. You need to show that $\mathbb{E}_{\mathbb{Q}}(X_t|\mathcal{F}_u) = X_u$ and $\mathbb{E}_{\mathbb{Q}}(Y_t|\mathcal{F}_u) = Y_u$.

8.29. In a continuous-time approach, we saw that the BSM European option pricing formula can be derived as the solution of the BSM p.d.e. On the other hand, the BSM pricing formula can be determined as the continuum limit of the discrete-time binomial tree model. Is there a discrete-time analog of the BSM p.d.e. in the binomial tree framework? If so, then using appropriate discrete-time interpretations, determine the partial difference equation analog of the BSM p.d.e. directly from the binomial tree.

Solution 8.29. Yes. Assume:

$$\frac{\delta C^E}{\delta S} = \frac{C_u^E - C_d^E}{su - sd}$$

$$\frac{\delta C^E}{\delta t} = \frac{(u-1)(C_d^E - C^E) - (d-1)(C_u^E - C^E)}{(u-1)-(d-1)}.$$

Try to obtain

$$0 = -\tilde{r}C^E + \tilde{\tilde{r}}s\frac{\delta C^E}{\delta S} + \frac{\delta C^E}{\delta t}.$$

8.30. Consider the binomial tree model for option pricing.

a) Give a one-sentence mathematical reason why the constraint $\mathrm{d} < \mathrm{e}^{(\mathrm{r}-q)h} < \mathrm{u}$ holds. Do not use a specific binomial tree model such as a CRR tree, JR tree, etc.
b) Give a financial reason why the condition $\mathrm{d} < \mathrm{e}^{(\mathrm{r}-q)h} < \mathrm{u}$ holds. If this result does not hold, then is any assumption of the BSM model violated? If so, indicate which one.

Solution 8.30.

8.31. Using a 3-period binomial tree, show that a European call price is given by:

$$C(t_0,3) = \mathrm{e}^{-(3h)\mathrm{r}}[p_*^3 C_{\mathrm{u}^3} + 3p_*^2(1-p_*)C_{\mathrm{u}^2\mathrm{d}} + 3p_*(1-p_*)^2 C_{\mathrm{ud}^2} + (1-p_*)^3 C_{\mathrm{d}^3}].$$

Solution 8.31. *Hint*: work backwards through the tree.

8.32. For an n-period binomial tree, show that the price of a European call is given by

$$C(t_0) = \mathrm{e}^{-\mathrm{r}(nh)}\left[\sum_{i=0}^{n}\binom{n}{i}p_*^i(1-p_*)^{n-i}C_{\mathrm{u}^i\mathrm{d}^{n-i}}(t_n)\right],$$

where $C_{\mathrm{u}^i\mathrm{d}^{n-i}}(t_n) = \max\{S(t_0)\mathrm{u}^i\mathrm{d}^{n-i} - K, 0\}$ for $i = 0,1,\ldots,n$.

Solution 8.32. *Hint*: try $n = 1$ and $n = 2$ first, to obtain

$$C(t_0) = \mathrm{e}^{-(2h)\mathrm{r}}\left[p_*^2 C_{\mathrm{u}^2}(t_2) + 2p_*(1-p_*)C_{\mathrm{ud}}(t_2) + (1-p_*)^2 C_{\mathrm{d}^2}(t_2)\right]$$

and

$$\begin{aligned} C(t_0) = \mathrm{e}^{-(3h)\mathrm{r}}\Big[&p_*^3 C_{\mathrm{u}^3}(t_3) + 3p_*^2(1-p_*)C_{\mathrm{u}^2\mathrm{d}}(t_2) \\ &+3p_*(1-p_*)^2 C_{\mathrm{ud}^2}(t_3) + (1-p_*)^3 C_{\mathrm{d}^3}(t_3)\Big]. \end{aligned}$$

Find the pattern and derive the general equation.

More rigorously, use induction to verify your guess, namely, establish the result for $n = 1$ (which is done), assume that it is true for any integer $n = k-1 \geq 1$, and show that it holds for $n = k$. It is helpful to note: At the time t_k in a general binomial tree, there are $k+1$ nodes and the ith node branches out as in Figure 8.1 from t_{k-1} to t_k, where $i = 1,\ldots,k$:

Applying the 1-period case to Fig. 8.1 in your induction argument.

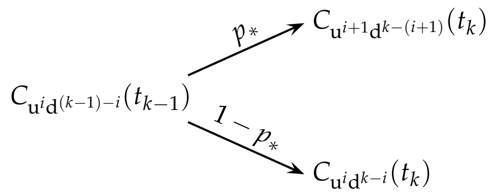

Fig. 8.1 A 1-period binomial subtree at the ith node $C_{\mathrm{u}^i\mathrm{d}^{k-(i+1)}}(t_{k-1})$ branching out from t_{k-1} to t_k.

8.33. Show that the n-period binomial formula for a European call can be expressed as:

$$C(t_0) = S(t_0)\,\mathrm{e}^{-q\tau}\,\mathsf{N}(n,k_*,\hat{p}_*) \;-\; K\mathrm{e}^{-\mathrm{r}\tau}\,\mathsf{N}(n,k_*,p_*),$$

where k_* is the smallest value of i for which $S(t_0)\mathrm{u}^i\mathrm{d}^{n-i} - K > 0$ and

$$\tau = t_n - t_0, \quad \mathsf{N}(n,k_*,p_*) = \sum_{i=k_*}^{n} \binom{n}{i} p_*^i (1-p_*)^{n-i}, \quad \hat{p}_* = \frac{p_*\mathrm{u}}{\mathrm{e}^{(\mathrm{r}-q)h}}.$$

Solution 8.33. *Hint*: try to convert the equation in Exercise 8.32 to

$$\begin{aligned} C(t_0) = S(t_0) &\left(\frac{\sum_{i=k_*}^{n} \binom{n}{i} p_*^i (1-p_*)^{n-i}\mathrm{u}^i\mathrm{d}^{n-i}}{\mathrm{e}^{(nh)\mathrm{r}}} \right) \\ &- \mathrm{e}^{-(nh)\mathrm{r}} K \left(\sum_{i=k_*}^{n} \binom{n}{i} p_*^i (1-p_*)^{n-i} \right) \end{aligned} \tag{8.1}$$

8.34. Find the deltas of a forward and futures on a non-dividend paying stock.

Solution 8.34. Review of forwards and futures:

Assume that the forwards and futures are long and the underlying asset pays no dividend. At the start of a forward or future contract, that is, at time 0, the value of the contract is zero. Let $F_T(t)$ be the forward price at time t, which is the asset's delivery price set at time t. Let $\widehat{F}_{T,n}(t)$ be the futures price (delivery price) set at time t on day n. Forwards and futures have identical prices on contracts with the same maturity date T, same underlying asset, and same number of units of the underlier:

$$F_T(t) = S(t)\mathrm{e}^{\mathrm{r}(T-t)} = \widehat{F}_{T,n}(t).$$

Note that for forwards and futures, the delivery price set at the start of the contract is

$$K_0 = F_T(0) = S(0)\mathrm{e}^{\mathrm{r}T} = \widehat{F}_{T,1}(0).$$

After the start of the contract, the values of the forward and futures deviate. The value of the forward is given by (Chapter 7):

$$\mathfrak{F}_T(t) = S(t) - K_0 e^{\mathsf{r}(T-t)}, \qquad 0 \le t \le T.$$

The value of the futures contract at time t on day n, where t is assumed to be prior to when the new futures contract takes effect at the end of day n, is the spread between the futures price at time t and the previous day's closing price:

$$\widehat{\mathfrak{F}}_{T,n}(t) = \widehat{F}_{T,n}(t) - \widehat{F}_{T,n-1} = S(t)e^{\mathsf{r}(T-t)} - \widehat{F}_{T,n-1}.$$

You now have all the pieces needed to solve the problem.

8.35. Show that delta, gamma, and theta of European calls and puts are:

$$\Delta_C(S_{t_0}, t_0) = e^{-q\tau}\,\mathsf{N}(d_+(S_{t_0}, \tau)), \qquad \Delta_P(S_{t_0}, t_0) = -e^{-q\tau}\,\mathsf{N}(-d_+(S_{t_0}, \tau))$$

$$\Gamma_C(S_{t_0}, t_0) = \frac{e^{-q\tau}\,\mathsf{N}'(d_+(S_{t_0}, \tau))}{S_{t_0}\,\sigma\sqrt{\tau}}, \qquad \Gamma_P(S_{t_0}, t_0) = \Gamma_C(S_{t_0}, t_0)$$

$$\Theta_C(S_{t_0}, t_0) = -\frac{S_{t_0}\,e^{-q\tau}\,\sigma\,\mathsf{N}'(d_+(S_{t_0}, \tau))}{2\sqrt{\tau}} + q\,S_{t_0}\,e^{-q\tau}\,\mathsf{N}(d_+(S_{t_0}, \tau)) - \mathsf{r}\,K e^{-\mathsf{r}\tau}\,\mathsf{N}(d_-(S_{t_0}, \tau))$$

$$\Theta_P(S_{t_0}, t_0) = -\frac{S_{t_0}\,e^{-q\tau}\,\sigma\,\mathsf{N}'(d_+(S_{t_0}, \tau))}{2\sqrt{\tau}} - q\,S_{t_0}\,e^{-q\tau}\,\mathsf{N}(-d_+(S_{t_0}, \tau)) + \mathsf{r}\,K e^{-\mathsf{r}\tau}\,\mathsf{N}(-d_-(S_{t_0}, \tau)).$$

Solution 8.35. *Hint*: use the BSM formulas for European calls and puts as well as put-call parity. The differentiation involves the chain rule.

8.36. Consider a portfolio of derivatives with the same underlying security that pays no dividend. Prove that if the portfolio has zero gamma, then it is theta-market neutral, meaning $\frac{\partial \Theta_P}{\partial S} = \frac{\partial \Theta_P}{\partial x}(S_{t_0}, t_0) = 0$.

Solution 8.36. By (8.109), the portfolio's value satisfies the BSM p.d.e.:

$$\frac{1}{2}\sigma^2 S_{t_0}^2\,\Gamma_P + (\mathsf{r} - q)\,S_{t_0}\,\Delta_P + \Theta_P - \mathsf{r}\,\mathcal{V}_P = 0.$$

If $\Gamma_P = 0$ and the underlier pays no dividend ($q = 0$), then

$$\mathsf{r}S_{t_0}\,\Delta_P + \Theta_P - \mathsf{r}\,\mathcal{V}_P = 0 \qquad \text{or} \qquad \Theta_P = \mathsf{r}\,\mathcal{V}_P - \mathsf{r}S_{t_0}\,\Delta_P.$$

It follows:

$$\frac{\partial \Theta_P}{\partial S} = \mathsf{r}\,\frac{\partial \mathcal{V}_P}{\partial S} - \mathsf{r}S_{t_0}\,\frac{\partial \Delta_P}{\partial S} - \mathsf{r}\,\Delta_P = \mathsf{r}\,\Delta_P - \mathsf{r}S_{t_0}\,\Gamma_P - \mathsf{r}\,\Delta_P = 0.$$

8.37. Consider an MJD security price process, i.e.

$$S_t \stackrel{\mathrm{d}}{=} S_u \, \mathrm{e}^{\mu_0 \,(t-u) + \sigma \, \mathfrak{B}_{t-u}} \prod_{\ell=1}^{\mathfrak{N}_{t-u}} J_\ell,$$

where $0 \le u \le t$ and $\mu_0 = m - q - \lambda \, \kappa - \frac{1}{2}\sigma^2$. Show:

a) $\mathbb{E}\left(\prod_{\ell=1}^{\mathfrak{N}_{t-u}} J_\ell \right) = \mathrm{e}^{\lambda \, \kappa \,(t-u)}$

b) $\mathbb{E}_{\mathbb{P}_{\lambda,\gamma}}\left(S_t \mid \mathcal{F}_u^{\mathrm{MJD}} \right) = S_u \, \mathrm{e}^{(\mu_0 + \frac{1}{2}\sigma^2 + \lambda \, \kappa)\,(t-u)} = S_u \, \mathrm{e}^{(m-q)\,(t-u)}$.

Solution 8.37. a) *Hint*: conditioning relative to $\mathcal{F}_u^{\mathrm{MJD}}$, note that

$$\mathbb{E}\left(\prod_{\ell=1}^{\mathfrak{N}_{t-u}} J_\ell \right) = \sum_{i=0}^{\infty} \mathbb{E}\left(\prod_{\ell=1}^{\mathfrak{N}_{t-u}} J_\ell | \mathfrak{N}_{t-u} = i \right) P(\mathfrak{N}_{t-u} = i).$$

b) *Hint*: use $\mathbb{E}(X_u \, Y_t \mid \mathcal{F}_u) = X_u \, \mathbb{E}(Y_t \mid \mathcal{F}_u)$ and the result from part a).

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