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Unbounded Orbits for Semicircular Outer Billiard

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Abstract. We show that the outer billiard around a semicircle has an open ball escaping to infinity.

1. Introduction

An *outer billard map* F is defined outside a closed convex curve Γ in the following way. Let z be a point on the plane. Consider the supporting line $L(z)$ from z to Γ such that Γ lies on the right of L. $F(z)$ lies on $L(z)$ so that the point of contact divides the segment $[z, F(z)]$ in half. If Γ contains segments then $F(z)$ is not defined if $L(z)$ contains a segment. In this case $F(z)$ is defined almost everywhere but it is discontinuous.

In [14] Moser outlined the proof of the fact that if Γ is smooth and strictly convex then all trajectories are bounded (the complete proof (for C^6-curves) was later given in [3]). Moser also asked [14, 15] what happens if Γ is only piecewise smooth, mentioning in particular that even in the case where Γ is quadrangle, the question of boundedness of the orbits was open.

The majority of the subsequent papers on this subject dealt with the most degenerate case when Γ is a polygon. In this case $F(z)$ is obtained from z by reflection around a vertex. In [11,12,23] boundedness of the trajectories was proved for the so called *quasi-rational* polygons, a class including rational polygons as well as regular n-gones. Since affine equivalent curves have conjugated outer billiards all triangular outer billiards have bounded (in fact, periodic) orbits. It was proved in [5] that if Γ is a trapezoid then all trajectories are bounded. Schwartz [18, 19] considers kites – quadrangles with vertices $(-1, 0), (0, 1), (A, 0)$ and $(0, -1)$, and proves that for all irrational A, there exists an unbounded orbit (if A is rational then the orbits are bounded since the kite is rational). It is believed that unbounded orbits exist for almost all N-gonal outer billiards for $N \geq 4$ but this question is far from settled. In particular it is unknown if there exists a polygon with an infinite

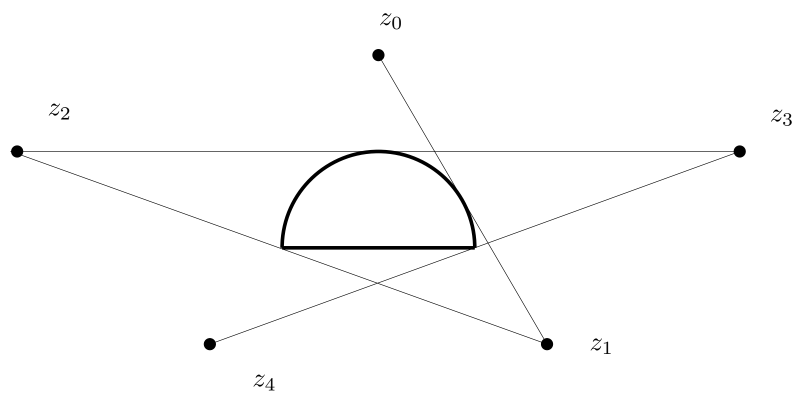

Figure 1. Four iterations of semicircular outer billiard.

measure of unbounded orbits (it seems that for kites almost all orbits are periodic but that case is quite special because a foliation by lines parallel to the x axis is preserved by F^2).

An intermediate case that consists of curves containing both segments and strictly convex pieces received much less attention. Numerical simulations reported in [21] show that for the outer billiard around a semicircle there is a large set of unbounded trajectories. In this paper we show that this numerical conclusion is indeed correct.

Theorem 1. *The outer billiard around a semicircle has an open ball escaping to infinity.*

2. Main ingredients

In all what follows, the billiard curve Γ will be fixed for definiteness to be the semicircle given by the upper part of the unit circle of $\mathbb{R}^2$, and F will denote the outer billiard map around Γ. Let ℓ_1 denote the halfline $\{x \geq 1, y = 0\}$. Let $\mathcal{D}$ be the infinite region bounded by ℓ_1, $F^2\ell_1$ and $\{x = x_0\}$ where x_0 is a large constant. We shall show in the appendix that $F^2\ell$ is a graph of a function $y = h(x)$ where $h(x) = 2 + O(1/x^2)$. Thus

$$\mathcal{D} = \left\{(x, y) : x \geq x_0, 0 \leq y \leq h(x)\right\}.$$

Denote by $\mathcal{F}$ the first return map to $\mathcal{D}$ under F^2. Theorem 1 will of course follow if we show the existence of balls that escape to infinity under the iteration by $\mathcal{F}$. The proof of this fact consists of two parts.

We can rescale the coordinates in $\mathcal{D}$ and think of it as a cylinder (where the boundaries are identified by F^2). A further change of coordinates allows to derive a normal form expression for the return map $\mathcal{F}$ that consists of a periodic (piecewise) linear part $\bar{L} : \mathcal{D} \to \mathcal{D}$, and an asymptotic expansion in powers of

$1/R$, where R denotes the radial coordinate in the fundamental domain (we call a map of $\mathcal{D}$ periodic if it commutes with integer translations of R). The normal form will have singularity lines corresponding to the discontinuities of F^2 and of its derivative that result from the flat piece and the corners in Γ. This normal form is presented at the end of this section while its proof, based on lengthy but straightforward computations, is deferred to Appendix A.

Since $\bar{L}$ is periodic there is a map $\hat{L} : \mathbb{T}^2 \to \mathbb{T}^2$ such that $\pi\bar{L} = \hat{L}\pi$ where π is the projection of the cylinder $\mathcal{D}$ to the torus. Hence, we can reduce the dynamics of $\bar{L}$ to a dynamics on $\mathbb{T}^2$ and look for an escaping orbit on $\mathcal{D}$ which projects to the simplest orbit on $\mathbb{T}^2$, namely a periodic orbit. In our case, we will exhibit a fixed point of $\hat{L}$ that moves up in $\mathcal{D}$ by two units in the R direction under each iteration of $\bar{L}$. The question is hence that of the stability of this orbit when the additional terms of the normal form are considered, which constitutes the second part of the proof. The first crucial observation is that the linear part $\hat{L}$ happens to be elliptic so that the escaping orbit of $\bar{L}$ is actually accompanied by a small ball around it.

Under $\bar{L}$, the radial coordinate R of the points in the escaping ball goes to infinity linearly with time, and viewed on $\mathbb{T}^2$, our problem becomes similar to that of establishing the stability of a periodic point under a time dependent perturbation. If the perturbation was independent of time, it would be possible to derive the result from Moser's theorem on the stability of elliptic fixed points. Time dependent perturbations were studied in [4, 16] but there it was assumed that the perturbation vanishes at the fixed point. This is not true in our case, however there are two features which considerably simplify the problem.

(1) The unperturbed map is globally linear so that the approximation by the linear map does not become worse as we move away from the fixed point.
(2) Rather than a general smooth function, the perturbation has a special form. Namely, if we use a coordinate system centered around the escaping orbit of $\bar{L}$, the main term of our return map will be the linear elliptic map $\bar{L}$, the term of order $\mathcal{O}(\frac{1}{\bar{R}_n})$, where $\bar{R}_n \sim 2n$ is the radial coordinate at time n, is quadratic, the term of order $\mathcal{O}(\frac{1}{\bar{R}_n^2}) = \mathcal{O}(\frac{1}{n^2})$ is cubic and so on (see Lemma 2). The $\mathcal{O}(\frac{1}{n^2})$-terms clearly do not alter the stability displayed in the linear picture, and the $\mathcal{O}(\frac{1}{n})$-perturbation contains only one resonant term which may cause divergence (rather than infinitely many resonant terms which might have appeared for a general smooth perturbation). In our case, the only resonance is related to the fact that the small perturbation of an area preserving map can be area contracting or area expanding (and in the latter case stability is clearly impossible). Stability is thus insured by the nullity of the resonant term, which we obtain *a posteriori* due to the area preservation property of the outer billiard map.

Finally, a particular attention has also to be given to the verification that the candidate escaping ball stays away from the singularity lines of the map.

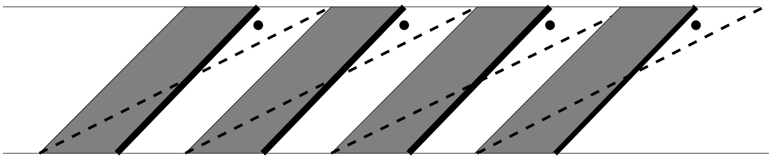

Figure 2. Dynamics of $\mathbf{L}$. R decreases on the shaded parallelograms and increases on white parallelograms. The thick solid lines are discontinuities of $\mathbf{L}$. The thick dashed lines are singularities of higher order terms which are invisible in the linear approximation. Also shown is an escaping orbit which projects to a constant orbit of $\mathbb{T}^2$.

We now specify with two statements the two principal moments of the proof discussed above.

Define on the cylinder $\mathbf{C} = [R_0, \infty) \times \mathbb{T}$ the map

$$\mathbf{L}(R,\phi) = \left(R - \frac{4}{3} + \frac{8}{3}\{\phi - R\}, \{\phi - R\} \right).$$

We will use the notation $[x]$ for the integer part of x and $\{x\}$ for its fractional part $x - [x]$.

Lemma 2. *There exists a smooth change of coordinates $G : (x,y) \in \mathcal{D} \mapsto (R,\phi) \in \mathbf{C} = [R_0,\infty) \times \mathbb{T}$ of the form*

$$R = \frac{2}{3}x - \frac{1}{6} + \mathcal{O}\left(\frac{1}{x}\right), \quad \phi = \frac{y}{2} + \mathcal{O}\left(\frac{1}{x}\right) \tag{1}$$

so that the following holds.[1]

(a) *The singularities of $\mathcal{F}$ are $\mathcal{O}(1/R)$ close to one of the following curves*
- *the singularities of $\mathbf{L}^2$*
- $\{\phi - \frac{R}{2}\} = \frac{1}{4}$ *and* $\{\phi - \frac{R}{2}\} = \frac{3}{4}$
- $\{\tilde{\phi} - \frac{\tilde{R}}{2}\} = \frac{1}{4}$ *and* $\{\tilde{\phi} - \frac{\tilde{R}}{2}\} = \frac{3}{4}$ *where* $(\tilde{R},\tilde{\phi}) = \mathbf{L}(R,\theta)$.

(b) *If (R,ϕ) is $\mathcal{O}(1/R)$ far from the singularities then*

$$\mathcal{F}(R,\phi) = \mathbf{L}^2(R,\phi) + \Big(P\big(\{R\},\phi\big)/[R], Q\big(\{R\},\phi\big)/[R]\Big) + \mathcal{O}(1/R^2)$$

where $P, Q : [0,1] \times \mathbb{T} \to \mathbb{R}$ are piecewise polynomials of degree 2

(c) *The map $\mathcal{F}$ preserves a measure with density*

$$1 + W(\phi)/R + \mathcal{O}(R^{-2})$$

where W is an affine function.

Lemma 3. *Any map satisfying the conclusions of Lemma* 2 *has an open ball of points escaping to infinity.*

[1]We use the same notation for $\mathcal{F}$ and $G \circ \mathcal{F} \circ G^{-1}$.

Remark. A more explicit expression for G in Lemma 2 is given by formulas (14), (15), (18), (19), (21) and (23) of Appendix A. The reader will notice that those formulas are rather cumbersome however for our proof (that is for Lemma 3) we only need the properties listed in Lemma 2.

Lemma 3 is proven in Section 3. We follow [16] but our case is simpler since we deal with a perturbation of the linear system and the first order perturbative terms are polynomials while [16] considers an arbitrary perturbation. The proof of Lemma 2 is given in Appendix A.

3. Construction of unbounded orbits

Following [16] we first observe that the limit map $\mathbf{L}^2$ has an escaping orbit given by $R_n = 2n$ and $\phi_n = \phi_0 = 7/8$, namely $\mathbf{L}^2(R_n, \phi_0) = (R_{n+1}, \phi_0)$. Notice that the latter escaping orbit remains away from the singularity lines. Define

$$\mathbb{L} = d\mathbf{L} = \begin{pmatrix} -\frac{5}{3} & \frac{8}{3} \\ -1 & 1 \end{pmatrix}.$$

Notice that the trace of $\mathbb{L}$ is equal to $-2/3$ which implies that it is elliptic, and so is $\mathbb{L}^2$. Hence a full ball will accompany the escaping point to infinity. To deal with the higher order perturbative terms, a first observation is that $\mathcal{F}$ has no singularities in balls of sufficiently small but fixed radius around the escaping points $(2n, \phi_0)$.

We will therefore consider a point $\{R_N, \phi_N\}$ in a small neighborhood of $\{2N, \phi_0\}$ and study its dynamics. For $n \geq N$, we will denote $\{R_n, \phi_n\}$ the $n - N$ iterate of $\{R_N, \phi_N\}$, and introduce $U_n = R_n - 2n$, $\upsilon_n = \phi_n - \phi_0$. Let s be such that $\cos(2\pi s) = -1/3$. We can introduce a suitable complex coordinate $z_n = U_n + i(aU_n + b\upsilon_n)$ such that $D\mathcal{F}$ becomes a rotation by angle $2\pi s$ near the origin. In these coordinates $\mathcal{F}$ takes the following form in a small neighborhood of $(0, 0)$

$$z_{n+1} = e^{i2\pi s} z_n + \frac{A(z_n)}{N} + \mathcal{O}(N^{-2}) \tag{2}$$

where

$$A(z) = w_1 + w_2 z + w_3 \bar{z} + w_4 z^2 + w_5 z\bar{z} + w_6 \bar{z}^2 .$$

Lemma 4.

(a) *We have that* $\mathcal{R}e(e^{-i2\pi s} w_2) = 0$.

(b) *There exists* $\epsilon > 0$ *and a constant* C *such that if* $|z_N| \leq \epsilon$*, then for every* $n \in [N, N + \sqrt{N}]$

$$|z_n| \leq |z_N| + CN^{-1} .$$

Part (b) is the main result of the lemma. Part (a) is an auxiliary statement needed in the proof of (b). Namely, part (a) says that the resonant coefficient mentioned in Section 2 vanishes.

Before we prove this lemma, let us observe that it implies that for sufficiently large N, all the points $|z_N| \leq \epsilon/2$ are escaping orbits. Indeed by $[\sqrt{N}]$ applications of lemma 4 there is a constant C such that

$$|z_l| \leq \frac{\epsilon}{2} + CN^{-\frac{1}{2}}$$

for every $l \in [N, 2N]$. It now follows by induction on k that if $l \in [2^k N, 2^{k+1} N]$ then

$$|z_l| \leq \epsilon_k$$

where

$$\epsilon_k = \frac{\epsilon}{2} + \frac{C}{\sqrt{N}} \sum_{j=0}^{k} \left(\frac{1}{\sqrt{2}}\right)^j$$

(N has to be chosen large so that $\epsilon_k \leq \epsilon$ for all k). This proves Lemma 3.

Proof of Lemma 4. Let $\bar{n} = n - N$. For $\bar{n} \leq \sqrt{N}$ equation (2) gives

$$z_n = e^{i2\pi\bar{n}s} z_N + \frac{1}{N} \sum_{m=0}^{\bar{n}-1} e^{i2\pi ms} A(e^{i2\pi(\bar{n}-m-1)s} z_{N+\bar{n}-m-1}) + \mathcal{O}(N^{-\frac{3}{2}}) . \tag{3}$$

In particular for these values of n we have

$$z_n = e^{i2\pi s(n-N)} z_N + \mathcal{O}\left(\frac{1}{\sqrt{N}}\right).$$

Substituting this into (3) gives

$$z_n = e^{i2\pi\bar{n}s} z_N + \frac{1}{N} \sum_{m=0}^{\bar{n}-1} e^{i2\pi ms} A(e^{i2\pi(\bar{n}-m-1)s} z_N) + \mathcal{O}\left(\frac{1}{N}\right).$$

To compute the sum above expand A as a sum of monomials and observe that

$$\sum_{m=0}^{\bar{n}-1} e^{i2\pi ms} \left(e^{i2\pi(\bar{n}-m-1)s} z_N\right)^\alpha \left(e^{-i2\pi(\bar{n}-m-1)s} \bar{z}_N\right)^\beta$$

is bounded for $\alpha + \beta \leq 2$ unless $\alpha = \beta + 1$ (that is $\alpha = 1$, $\beta = 0$). Therefore

$$z_n = e^{i2\pi\bar{n}s} z_N \left(1 + \tilde{w}_2 \frac{\bar{n}}{N}\right) + \mathcal{O}\left(N^{-1}\right) \tag{4}$$

where $\tilde{w}_2 = e^{-i2\pi s} w_2$.

Consider now the disc D_N around 0 of radius $N^{-0.4}$. Then by (4)

$$\frac{\text{Area}(\mathcal{F}^{\bar{n}} D_N)}{\text{Area}(D_N)} = \left(1 + 2\mathcal{R}e(\tilde{w}_2)\frac{\bar{n}}{N}\right) + \mathcal{O}\left(N^{-0.6}\right).$$

On the other hand there exists $z \in D_N$ such that denoting $z' = \mathcal{F}^{\bar{n}} z$ we have

$$\frac{\text{Area}(\mathcal{F}^{\bar{n}} D_N)}{\text{Area}(D_N)} = \frac{1 + W(z)/N}{1 + W(z')/(\bar{n} + N)} + \mathcal{O}\left(N^{-2}\right) = 1 + \mathcal{O}\left(N^{-1.4}\right)$$

since $W(z) - W(z') = \mathcal{O}\left(N^{-0.4}\right)$. Comparing those two expressions for the ratio of areas we obtain that $\mathcal{R}e(\tilde{w}_2) = 0$.

This proves part (a) of Lemma 4. Part (b) now follows from (4). □

4. Open questions

In this paper we consider a very simple piecewise smooth curve – the semicircle and prove that there is a positive measure set of escaping orbits. This is just the first step in the study of outer billiards around piecewise smooth curves. In the current section we discuss some of the immediate questions raised by this work.

Question 1. Prove that unbounded orbits exist generically for the following classes of curves

(a) circular caps;
(b) curves consisting of finitely many strictly convex pieces and finitely many segments;
(c) unions of two circular arcs.

We observe that since our proof depends on the existence of an elliptic fixed point for a certain auxiliary map which is an open condition we also obtain the existence of unbounded orbits for caps close to the semicircle. However the limiting cases of caps close to the full circle appear to be much more difficult. It is not difficult to extend Lemma 2 to small piecewise smooth perturbations of the identity in the plane corresponding to curves (a)–(c) above (see the Appendix), however proving that the limiting map has unbounded orbits is more complicated.

Question 2. Let P be the set of maps of the cylinder $\mathbb{R} \times \mathbb{T}$ of the form

$$\mathcal{F}(R, \phi) = \mathrm{id} + L\big(\{R\}, \phi\big)$$

where L is piecewise linear which are invertible and area preserving. Is it true that a generic element of P has unbounded orbits?

We observe that the affirmative answer to Question 2 would be a significant step in answering cases (a) and (b) of Question 1, however it would give little for case (c) (because in cases (a) and (b) the outer billiard map is discontinuous while in case (c) it is continuous but not smooth).

Another interesting direction of research is to describe different possible types of behavior for outer billiards. In particular we say that the orbit $\{z_n\}$ is *oscillatory* if

$$\limsup |z_n| = +\infty\,, \quad \liminf |z_n| < \infty\,.$$

We say that an oscillatory orbit is *erratic* if in addition

$$\liminf d(z_n, \Gamma) = 0\,.$$

We observe that unbounded orbits constructed in [18, 19] are erratic and it is conjectured there that for outer billiard around kites every orbit is either periodic or erratic.

Question 3. Does generic outer billiard for classes (a)–(c) of Question 1 have

(a) oscillatory (in particular erratic) orbits;
(b) infinite measure of bounded (in particular quasiperiodic) orbits;
(c) bounded non-quasiperiodic orbits?

Concerning parts (b) and (c) of Question 3 we observe that the semicircular outer billiard has infinitely many elliptic periodic orbits close to $R \in \mathbb{N}, \phi = 1/2$ (that are periodic points for the linear part) however verifying their KAM stability requires checking a nonzero twist condition, and thus computing an asymptotic of the orbits with a higher precision than what it is done in the Appendix.

As it is the case for Question 1, the natural first step in investigating Question 3 is to study the limiting maps of Question 2. In case the limiting map is elliptic the dynamics is piecewise isometric for a suitable metric. We refer the reader to [7] for a survey of general properties of piecewise isometries and to [8] and references wherein for an interesting case study with an emphasis on existence of unbounded orbits.

Question 4. Does there exist a curve Γ such that the limiting map of Lemma 2 has hyperbolic linear part?

Thus Question 4 raises the problem of existence of a curve such that the outer billiard dynamics is chaotic near infinity. We note that [6] gives an example of a curve such that the outer billiard is chaotic near the curve itself. See also [2, 10] for the discussion of unstable orbits near the boundary of piecewise smooth outer billiards.

In conclusion we mention that there are several mechanical systems (with collisions) of which the dynamics for large energies is given by a small piecewise smooth perturbation of an integrable map (see [2, 9, 20, 24], for example) and all the questions discussed here are interesting for these systems as well.

Appendix A. Normal form

Here we prove Lemma 2. The proof consists of three parts.

In Subsection A.1 we derive the formulas for F^2 in polar coordinates.

In Subsection A.2 we make a coordinate change to simplify the expression of F^2. Our approach follows closely the computations of the normal form for small perturbations of the id (cf [1, 13]), however the resulting normal form is different due to the presence of singularities.

In Subsection A.3 we use the coordinates of Subsection A.2 to compute the first return map of F^2 inside $\mathcal{D}$.

A.1. Semicircular outer billiard.

Here we obtain the asymptotic expansion of F near infinity. Consider coordinates in which the semidisc is given by

$$\{x^2 + y^2 \leq 1, y \geq 0\}.$$

F is piecewise smooth with discontinuities at the following halflines

$$\ell_1 = \{x \geq 1, y = 0\}, \quad \ell_2 = \{x = 1, y \geq 0\}, \quad \ell_3 = \{x = -1, y \leq 0\}.$$

Inside its continuity domains F can be describes as follows:

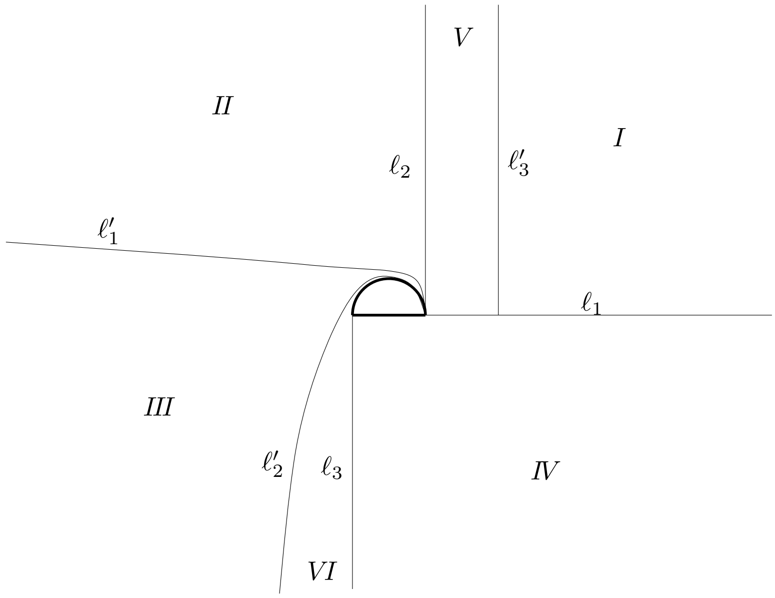

FIGURE 3. Continuity regions for F^2.

between ℓ_1 and ℓ_2-reflection about $O_1 = (1, 0)$;
between ℓ_2 and ℓ_3-reflection about a tangency point to the circular part;
between ℓ_3 and ℓ_1-reflection about $O_2 = (-1, 0)$.

Let $\ell_j' = F^{-1}\ell_j$. Denote by R_j the reflection about O_j. Observe that far from the origin F looks like the reflection about the origin. Therefore we are interested in F^2 which is close to id. F^2 has six continuity domains.

Region I : between ℓ_1 and ℓ_3' we have $F^2 = TR_1$;
Region V : between ℓ_3' and ℓ_2 we have $F^2 = R_2R_1$;
Region II : between ℓ_2 and ℓ_1' we have $F^2 = R_2T$;
Region III : between ℓ_1' and ℓ_2' we have $F^2 = R_1T$;
Region VI : between ℓ_2' and ℓ_3 we have $F^2 = T^2$;
Region IV : between ℓ_3 and ℓ_1 we have $F^2 = TR_2$.

Thus regions I–IV look like the four coordinate quadrants while regions V and VI are small buffers between them (it is easy to see that when the orbit of F^2 visits the last two regions it leaves them immediately). We call the union of regions I, V and II the upper region and the union of regions III, IV and VI the lower region. We consider coordinates (r, θ) which are polar coordinates in the upper region and polar coordinate shifted by π in the lower region. Thus both in upper and the lower region $0 \leq \theta \leq \pi$. Our choice is motivated by the wish to make F^2 in the upper and the lower region look similar. We shall need the formulas for ℓ_j and ℓ_j' in polar coordinates.

Proposition 5. *The discontinuity lines of F^2 are given by the following equations*

$$\ell_1 \subset \{\theta=0\}, \quad \ell_3' \subset \left\{\theta=\frac{\pi}{2}-\frac{3}{r}+\mathcal{O}\left(\frac{1}{r^2}\right)\right\}, \quad \ell_2 \subset \left\{\theta=\frac{\pi}{2}-\frac{1}{r}+\mathcal{O}\left(\frac{1}{r^2}\right)\right\},$$
$$\ell_1' \subset \left\{\theta = \pi - \frac{2}{r}+\mathcal{O}\left(\frac{1}{r^2}\right)\right\}, \quad \ell_2' \subset \left\{\theta = \frac{\pi}{2} - \frac{3}{r}+\mathcal{O}\left(\frac{1}{r^2}\right)\right\},$$
$$\ell_3 \subset \left\{\theta = \frac{\pi}{2} - \frac{1}{r}+\mathcal{O}\left(\frac{1}{r^2}\right)\right\}.$$

Proof. The result for ℓ_j follows by direct computation. To obtain the result for ℓ_j' observe that the preimage of a halfline $\ell(t) = \mathbf{u} + \mathbf{v}t$ satisfies

$$F^{-1}\ell(t) = \mathbf{w} - 2\mathbf{u} - \mathbf{v}t + \mathcal{O}(1/t)$$

where $\mathbf{w}$ is the vector where the supporting line of the semidisc has slope $\mathbf{v}$ (because the midpoint of the segment $[\ell(t), F^{-1}\ell(t)]$ is $\mathcal{O}(1/t)$ close to $\mathbf{w}$).

Alternatively one can compute F^{-1} explicitly and obtain the following parametric equations for ℓ_j' :

$$\ell_1' = \left\{\left(\frac{2}{t} - t, 2\sqrt{1-\frac{1}{t}}\right), \quad t \geq 1\right\},$$
$$\ell_2' = \left\{\left(\frac{1-3t^2}{1+t^2}, \frac{4t}{1+t^2} - t\right), \quad t > 0\right\}, \quad \ell_3' = \{x = 3, y \geq 0\}. \qquad \square$$

Proposition 6. *In our coordinates R_1 and R_2 have the same form given by*

$$R(r,\theta) = \left(r - 2\cos\theta + \frac{2\sin^2\theta}{r} + \mathcal{O}\left(\frac{1}{r^2}\right), \theta + \frac{2\sin\theta}{r} + \frac{2\sin 2\theta}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right)\right)$$

and

$$T(r,\theta) = \left(r, \theta + \frac{2}{r} + \mathcal{O}\left(\frac{1}{r^3}\right)\right).$$

Proof. The proof is based on elementary computations that we describe for R_1 in the region I, the other cases being similar. Let A be a point in region I and let (r,θ) and (x,y) be its polar and cartesian coordinates respectively. Denote $\tilde{A} = F(A)$. Then $\tilde{x} = 2 - x$ and $\tilde{y} = -y$. Hence $\tilde{r} = (4 + x^2 - 4x + y^2)^{1/2} = (r^2 + 4 - 4r\cos\theta)^{1/2} = r - 2\cos\theta + \frac{2-2\cos^2\theta}{r} + \mathcal{O}(1/r^2)$. $\square$

As a direct consequence of Proposition 6 we get

Proposition 7. *In the regions I–IV, F^2 takes the following form*

$$F^2(r,\theta) = \left(r + a(\theta) + \frac{a_1(\theta)}{r} + \mathcal{O}\left(\frac{1}{r^2}\right), \theta + \frac{b(\theta)}{r} + \frac{b_1(\theta)}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right)\right)$$

where

$$a(\theta) = -2\cos\theta\,, \quad b(\theta) = 2(1+\sin\theta)\,, \tag{5}$$
$$b_1(\theta) = 4\cos\theta(1+\sin\theta)$$
$$\text{and}\quad a_1(\theta) = 2\sin^2\theta \text{ in regions } I \text{ and } IV,$$
$$a_1(\theta) = 2\sin^2\theta + 4\sin\theta \text{ in regions } II \text{ and } III\,.$$

In the region VI F^2 is given by

$$F^2(r,\theta) = \left(r, \theta + \frac{4}{r} + \mathcal{O}\left(\frac{1}{r^3}\right)\right) \tag{6}$$

while in the region V it is given by[2]

$$F^2(r,\theta) = \left(r + \frac{8}{r} - 4\left(\frac{\pi}{2} - \theta\right) + \mathcal{O}\left(\frac{1}{r^2}\right), \theta + \frac{4}{r} + \mathcal{O}\left(\frac{1}{r^3}\right)\right). \tag{7}$$

Proof. This follows from the previous proposition by a direct computation, since $F^2 = TR$ in the regions I and IV and $F^2 = RT$ in the regions II and III. The fact that $F^2 = R^2$ and $F^2 = T^2$ in the regions V and VI respectively directly yields (6) and (7). □

Remark. (5) can also be obtained by a simpler computation as follows. Let A_0 be a point, say, in region I, with polar coordinates (r,θ). Denote $A_1 = FA_0$, $A_2 = F^2A_0$ and let B_1 and B_2 be the midpoints of A_0A_1 and A_1A_2 respectively. Then the triangles $\triangle A_0A_1A_2$ and $\triangle B_1A_1B_2$ are similar so that $\overrightarrow{A_0A_2} = 2\overrightarrow{B_1B_2}$. If r is large then B_1 and B_2 are close to the points where the lines with slope $\overrightarrow{OA_0}$ touch the semicircle, that is $B_1 = O_1$, $B_2 \approx (-\sin\theta, \cos\theta)$. Therefore

$$\overrightarrow{A_0A_2} \approx 2\big(-(1+\sin\theta), \cos\theta\big)\,.$$

a and b are the radial and the angular components of $\overrightarrow{A_0A_2}$ giving (5).

Computing how far is $\overrightarrow{B_1B_2}$ from the limiting vector allows to express the higher order terms in terms of the curvature of Γ and its derivatives, however for the semicircle it seems simpler to use the explicit formulas of Proposition 6.

A.2. Normal form coordinates

In this section we make a coordinate change to simplify the outer billiard map near infinity. In particular we refine the result of [22] about the asymptotics of the outer billiard orbits.

Proposition 8. *There exists a piecewise smooth change of coordinates $G : (r,\theta) \mapsto (\rho,\psi)$, with discontinuity lines ℓ_1 and ℓ_1' and the lines $\theta = \pi/2$ in the upper and lower regions, such that F^2 takes the following* normal form *in the variables (ρ,ψ).*

---

[2]Notice that in region V we have $\theta = \frac{\pi}{2} + \mathcal{O}\left(\frac{1}{r}\right)$ so the second and the third terms in (7) are of the same order.

In the regions I–IV we have

$$\rho_{n+1} = \rho_n + \mathcal{O}(1/\rho_n^3)\,, \tag{8}$$

$$\psi_{n+1} = \psi_n + \frac{1}{\rho_n} + \frac{c}{\rho_n^2} + \frac{u}{\rho_n^3} + \mathcal{O}\left(\frac{1}{\rho_n^4}\right), \tag{9}$$

where $c = c^{upper} = \frac{1}{4}$ inside the upper region and $c = c^{lower} = -\frac{1}{4}$ in the lower region.

In the regions V and VI, we have

$$\psi_{n+1} = \psi_n + \frac{1}{\rho_n} + \frac{u_2}{\rho_n^2} + \mathcal{O}\left(\frac{1}{\rho_n^3}\right), \tag{10}$$

$$\rho_{n+1} = \rho_n + \frac{u_1}{\rho_n} + 64(\psi_n - \frac{1}{3}) + \mathcal{O}\left(1/\rho_n^2\right), \quad \text{in region} \quad V \tag{11}$$

$$\rho_{n+1} = \rho_n + \frac{v_1}{\rho_n} + \mathcal{O}\left(1/\rho_n^2\right), \quad \text{in region} \quad VI \tag{12}$$

where u_1, v_1, u_2 are constants that may be different in the upper and lower regions.[3]

Moreover, for $(r, \theta) \in \mathcal{D}$, we have the following expression for the Jacobian of G

$$\mathrm{Jac}(G) = \frac{1}{2}(1-\theta) + \frac{c^{upper}}{2r} + \mathcal{O}\left(\frac{1}{r^2}\right). \tag{13}$$

Proof. We will look for a coordinate change of the form

$$\rho = r\Phi_1(\theta) + \Phi_2(\theta) + \frac{\Phi_3(\theta)}{r} \tag{14}$$

$$\psi = \Psi(\theta) + \frac{\Psi_1(\theta)}{r} + \frac{\Psi_2(\theta)}{r^2}. \tag{15}$$

This is a usual expression of a change of coordinates in view of a normal form for a twist map and the order of the $1/r$ powers is determined by the order of the perturbative terms required in (8) and (9) that in term correspond to what is needed to obtain in the sequel an expression of the return map to $\mathcal{D}$ as in Lemma 2. Notice that (8) and (9) should be viewed as functions of the lower and upper coordinates (r, θ) rather than the original polar coordinates.

Remark. We will observe that ρ has a discontinuity when the orbit crosses from the upper to the lower region, a fact that translates into the discontinuity of the derivative of our coordinate change Φ_1 at the same crossing. This fact is crucial since we shall see in the next subsection that the main change to ρ will come from crossing between the regions.

Given the expression of F^2 in Proposition 7, observe that iterating by F^2 in the regions *I–IV* yields

$$\rho_{n+1} - \rho_n = \Phi_1' b + a\Phi_1 + \left(\Phi_1' b_1 + \frac{\Phi_1'' b^2}{2} + a\Phi_1' b + a_1\Phi_1 + \Phi_2' b\right)\frac{1}{r_n} + \mathcal{O}\left(\frac{1}{r_n^2}\right)$$

---

[3]The explicit values of these constants is not necessary for the sequel.

so that if we require that

$$\frac{\Phi_1'}{\Phi_1} = -\frac{a}{b}\,, \tag{16}$$

$$\Phi_2' = -\frac{1}{b}\left(a_1\Phi_1 + \Phi_1'(ab+b_1) + \frac{\Phi_1''b^2}{2}\right) \tag{17}$$

then $\rho_{n+1} - \rho_n = \mathcal{O}(\frac{1}{r_n^2})$, and since $1 \leq \Phi_1 \leq 2$ and $-1 \leq \Phi_2 \leq 1$ it follows that $\rho_{n+1} - \rho_n = \mathcal{O}(\frac{1}{\rho_n^2})$. Given the expressions of a, b, a_1, b_1 in the different regions, we can choose

$$\Phi_1(\theta) = 1 + \sin\theta\,, \tag{18}$$

$$\Phi_2^{upper} = 1 - |\cos\theta|\,, \quad \Phi_2^{lower} = |\cos\theta| - 1\,. \tag{19}$$

Likewise, expanding $\psi_{n+1} - \psi_n$ we get

$$\psi_{n+1} - \psi_n = \Psi'\frac{b}{r_n} + \Psi'\frac{b_1}{r_n^2} + \frac{\Psi''}{2}\left(\frac{b}{r_n}\right)^2 + \frac{\Psi_1'b}{r_n^2} - \frac{a\Psi_1}{r_n^2} + \mathcal{O}\left(\frac{1}{r_n^3}\right).$$

So if we require that

$$\Psi' = \frac{1}{b\Phi_1} \tag{20}$$

then, using the computational fact $\frac{\Phi_1 b_1}{b} = \frac{(\Phi_1 b)'}{2}$, we obtain

$$\psi_{n+1} - \psi_n = \frac{1}{\rho_n} + \frac{\Phi_2 + b\Phi_1^2\Psi_1' + b\Phi_1\Phi_1'\Psi_1}{\rho_n^2} + \mathcal{O}\left(\frac{1}{\rho_n^3}\right).$$

Observe that (20) is satisfied by

$$\Psi(\theta) = \frac{2}{3} - \frac{1}{3}\left(\frac{1}{(1+t)^3} + \frac{t^2}{(1+t)^3} + \frac{t}{(1+t)^2} + \frac{1}{(1+t)}\right), \tag{21}$$

where $t = \tan\frac{\theta}{2}$. (To obtain (21) observe that the change of variables $t = \tan\frac{s}{2}$ transforms

$$\Psi(\theta) = \int_0^\theta \frac{ds}{2(1+\sin s)^2} = \int_0^{2\tan^{-1}\theta} \frac{1+t^2}{(1+t)^4}dt \tag{22}$$

since $\sin s = \frac{2t}{1+t^2}$, $\frac{ds}{dt} = \frac{2}{1+t^2}$. See also [22].)

Next, Ψ_1 is defined by $\Psi_1(0) = 0$ and

$$(\Psi_1\Phi_1)' = \frac{c}{b\Phi_1} - \frac{\Phi_2}{b\Phi_1} \tag{23}$$

and c^{upper} and c^{lower} are chosen so that $\Psi_1(\pi) = \Psi_1(0) = 0$ both in the upper and lower regions. The values $c^{upper} = -c^{lower} = \frac{1}{4}$ are then obtained from the computation of $\int_0^\pi \frac{1}{b\Phi_1}$ and $\int_0^\pi \frac{\Phi_2}{b\Phi_1}$ using the same change of variables as in (22).

Finally, Φ_3 and Ψ_2 are chosen in a similar fashion as $\Phi_1, \Phi_2, \Psi, \Psi_1$ to guarantee that

$$\rho_{n+1} - \rho_n = \mathcal{O}\left(\frac{1}{\rho_n^3}\right) \quad \text{and} \quad \psi_{n+1} - \psi_n = \frac{1}{\rho_n} + \frac{c}{\rho_n^2} + \frac{u}{\rho_n^3} + \mathcal{O}\left(\frac{1}{\rho_n^3}\right).$$

We do not have to explicit the functions Φ_3 and Ψ_2 since we do not need to know the constant u for the sequel.

Observe that our coordinate change is designed to simplify the map in the regions I–IV so they do not bring much simplification in the buffer regions V and VI. However in those regions the angular coordinate equals to $\pi/2 + \mathcal{O}(1/\rho)$, so we can use the Taylor expansion around $\pi/2$. Namely equations (6)–(7) and the facts that $\Phi_1(\pi/2) = 2$ and $\Psi'(\pi/2) = 1/8$ imply (10)–(11) for some constants u_1 and u_2 that may be different in the lower and upper regions and that we will not need to know explicitly for the rest of the proof.

To obtain the Jacobian estimate (13), recall that in $\mathcal{D}$ we have $\theta = \mathcal{O}(1/r)$. Thus

$$\mathrm{Jac}(G) = |\Phi_1\Psi'(\theta) + (\Psi_1\Phi_1)'(0)/r| + \mathcal{O}(1/r^2).$$

Next, due to (20) $\Phi_1\Psi' = \frac{1}{b} = \frac{1}{2}(1-\theta) + \mathcal{O}(\theta^2)$ while $(\Psi_1\Phi_1)'(0) = \frac{c^{upper}}{2}$. □

Remark. The explicit expression for Ψ is quite complicated, however in the computations of Section A.3 we shall only use (20) and the fact that

$$\Psi(\pi) = \frac{2}{3}, \quad \Psi\left(\frac{\pi}{2}\right) = \frac{1}{3}. \tag{24}$$

Remark. A similar argument shows that a change of variables of the type

$$\rho = \sum_{j=0}^{k-1} r^{1-j}\Phi_j(\theta), \quad \psi = \sum_{j=0}^{k-1} r^{-j}\Psi_j(\theta)$$

brings F^2 to the forms

$$\rho_{n+1} = \rho_n + \mathcal{O}\left(\rho^{-k}\right), \quad \psi_{n+1} = \psi_n + \sum_{j=0}^{k} c_j\rho^{-j} + \mathcal{O}\left(\rho^{-(k+1)}\right).$$

A.3. Proof of Lemma 2

According to our division of the plane into upper and lower regions, we will represent $\mathcal{F}$ as a composition of two maps. Namely let $\tilde{\mathcal{D}}$ be the region bounded by the line $y = 0$, $x \le -\tilde{x}_0$ (where $\tilde{x}_0 \gg x_0$ used in the definition of $\mathcal{D}$) and its image by F^2.

We represent $\mathcal{F} = \mathcal{F}_2\mathcal{F}_1$ where $\mathcal{F}_1$ corresponds to the passage from $\mathcal{D}$ to $\tilde{\mathcal{D}}$ and $\mathcal{F}_2$ corresponds to the passage from $\tilde{\mathcal{D}}$ to $\mathcal{D}$.

We introduce changes of coordinates in $\mathcal{D}$ and $\tilde{\mathcal{D}}$ that make them, up to identification of their boundary lines, diffeomorphic to half cylinders of the form

$\phi, \tilde{\phi} \in \mathbb{T}$, $\rho, \tilde{\rho} \geq \rho_0 + \mathcal{O}(1)$. Equation (9) shows that these changes of coordinates are given by $(\rho, \psi) \mapsto (\rho, \phi)$ and $(\tilde{\rho}, \tilde{\psi}) \mapsto (\tilde{\rho}, \tilde{\phi})$ where

$$\phi = \rho\psi - c^{upper}\psi + \mathcal{O}\left(\frac{1}{\rho^2}\right), \quad \tilde{\phi} = \tilde{\rho}\tilde{\psi} - c^{lower}\tilde{\psi} + \mathcal{O}\left(\frac{1}{\tilde{\rho}^2}\right)$$

where we have used the bounds $\psi = \mathcal{O}(\frac{1}{\rho})$ in $\mathcal{D}$ and $\tilde{\psi} = \mathcal{O}(\frac{1}{\tilde{\rho}})$ in $\tilde{\mathcal{D}}$. Conversely, observe that

$$\psi = \frac{\phi}{\rho} + \frac{c^{upper}\phi}{\rho^2} + \mathcal{O}\left(\frac{1}{\rho^3}\right), \quad \tilde{\psi} = \frac{\tilde{\phi}}{\tilde{\rho}} + \frac{c^{lower}\tilde{\phi}}{\tilde{\rho}^2} + \mathcal{O}\left(\frac{1}{\tilde{\rho}^3}\right).$$

We will now study the iteration by F^2 from $\mathcal{D}$ to $\tilde{\mathcal{D}}$ and give an expression for $\mathcal{F}_1$. We first introduce $\tilde{\mathcal{D}}' = F^{-2}(\tilde{\mathcal{D}})$ and observe that until entering $\tilde{\mathcal{D}}'$ the normal form of F^2 in the (ρ, ψ) coordinates can be used, albeit a special care must be given to an eventual passage in region V. The discontinuity lines of the differential of F^2 that limit the region V are l_2 and l_3'. Their equations in the (ρ, ψ) coordinates become

$$\begin{aligned} \ell_2 &\subset \left\{\psi = \frac{1}{3} - \frac{1}{4\rho} + \mathcal{O}\left(\frac{1}{\rho^2}\right)\right\}, \\ \ell_3' &\subset \left\{\psi = \frac{1}{3} - \frac{3}{4\rho} + \mathcal{O}\left(\frac{1}{\rho^2}\right)\right\}. \end{aligned} \tag{25}$$

This is due to the fact that $1/(2r) = 1/\rho + \mathcal{O}(1/\rho^2)$ in a $\mathcal{O}(1/r)$-neighborhood of the vertical axis (that contains ℓ_2 and ℓ_3'), and the equalities $\Psi(\pi/2) = 1/3$ and $\Psi'(\pi/2) = 1/8$.

Until entering region V we have

$$\begin{aligned} \rho_k &= \rho + \mathcal{O}\left(\frac{1}{\rho^2}\right), \\ \psi_k &= \psi + \frac{k}{\rho} + \frac{kc^{upper}}{\rho^2} + \frac{ku}{\rho^3} + \mathcal{O}\left(\frac{1}{\rho^3}\right). \end{aligned}$$

The value $c^{upper} = 1/4$ allows to compute the entrance times $\overline{n}$ and n to the region V and to $\tilde{\mathcal{D}}'$ respectively. Namely

$$\overline{n} = \left[\Psi\left(\frac{\pi}{2}\right)\rho - \Psi\left(\frac{\pi}{2}\right)c^{upper} - \phi\right] = \left[\frac{\rho}{3} - \phi - \frac{1}{12}\right], \tag{26}$$

$$n = \left[\Psi(\pi)\rho - \Psi(\pi)c^{upper} - \phi\right] = \left[\frac{2\rho}{3} - \phi - \frac{1}{6}\right] \tag{27}$$

unless the orbit comes close to a discontinuity. (This precaution is discussed at the end of the section).

We let $\overline{v} = u_1 - 64\left\{\frac{\rho}{3} - \phi - \frac{1}{12}\right\}$ if $\left\{\frac{\rho}{3} - \phi - \frac{1}{12}\right\} \in [1/4, 3/4]$ and $\overline{v} = 0$ otherwise, which corresponds to points that visit (respectively do not visit) the region V, since $\psi_{\overline{n}} = 1/3 - \left\{\frac{\rho}{3} - \phi - \frac{1}{12}\right\}/\rho + \mathcal{O}(1/\rho^2)$. It then follows from (11) that

$$\rho_{\overline{n}+1} = \rho + \frac{\overline{\upsilon}}{\rho} + \mathcal{O}\left(\frac{1}{\rho^2}\right).$$

Define now $\upsilon = \left\{\frac{2\rho}{3} - \phi - \frac{1}{6}\right\}$. We get

$$\rho_n = \rho + \frac{\overline{\upsilon}}{\rho} + \mathcal{O}\left(\frac{1}{\rho^2}\right),$$
$$\psi_n = \psi + \frac{n}{\rho} + \frac{(n-1)c^{upper}}{\rho^2} + \frac{u_2}{\rho^2} + \frac{nu}{\rho^3} - \frac{\tilde{n}\overline{\upsilon}}{\rho^3} + \mathcal{O}\left(\frac{1}{\rho^3}\right)$$

where $\tilde{n} = n - \overline{n}$ is the time remaining after going through the region V. Hence

$$\psi_n = \frac{2}{3} - \frac{\upsilon}{\rho} + \frac{Z(\rho,\phi)}{\rho^2} + \mathcal{O}\left(\frac{1}{\rho^3}\right)$$

where $Z(\rho,\phi) = (-\phi - \frac{1}{6} - \upsilon)c^{upper} + u_2 + \frac{2}{3}u - \frac{\overline{\upsilon}}{3}$. Going back to the upper region polar coordinates and using that $\Psi(\pi) = 2/3, \Psi'(\pi) = 1/2,\ \Psi''(\pi) = 1, \Phi_1(\pi) = 1, \Phi_1'(\pi) = -1, \Phi_1''(\pi) = 0, \Phi_2(\pi) = 0, \Phi_2'(\pi) = 0$, we get

$$r_n = \rho - 2\upsilon + \frac{P_1}{\rho} + \mathcal{O}\left(\frac{1}{\rho^2}\right), \tag{28}$$
$$\theta_n = \pi - \frac{2\upsilon}{\rho} + \frac{Q_1}{\rho^2} + \mathcal{O}\left(\frac{1}{\rho^3}\right). \tag{29}$$

Here and in the sequel P_i and Q_i, $i = 1, 2, \ldots$ will denote piecewise polynomials of degree 2 in the variables υ, ϕ that are not necessary to explicit for the rest of the proof. Switching to the lower region coordinates and iterating once more by F^2 we get

$$\tilde{r}_{n+1} = \rho - 2 - 2\upsilon + \frac{P_2}{\rho} + \mathcal{O}\left(\frac{1}{\rho^2}\right), \tag{30}$$
$$\tilde{\theta}_{n+1} = \frac{2}{\rho} - \frac{2\upsilon}{\rho} + \frac{Q_2}{\rho^2} + \mathcal{O}\left(\frac{1}{\rho^3}\right). \tag{31}$$

Consequently (14) and (15) yield

$$\tilde{\rho}_{n+1} = \rho - 4\upsilon + \frac{P_3}{\rho} + \mathcal{O}\left(\frac{1}{\rho^2}\right),$$
$$\tilde{\psi}_{n+1} = \frac{1}{\rho} - \frac{\upsilon}{\rho} + \frac{Q_3}{\rho^2} + \mathcal{O}\left(\frac{1}{\rho^3}\right).$$

This finally gives the $(\tilde{\rho}, \tilde{\phi})$ coordinates of the iterate of (ρ, ϕ) inside $\tilde{\mathcal{D}}$ as

$$\tilde{\rho} = \rho - 4 + 4\tilde{\phi} + \frac{P_4}{\rho} + \mathcal{O}\left(\frac{1}{\rho^2}\right),$$
$$\tilde{\phi} = \left\{\phi - \frac{2\rho}{3} + \frac{1}{6}\right\} + \frac{Q_4}{\rho} + \mathcal{O}\left(\frac{1}{\rho^2}\right)$$

where we used that $1 - \upsilon = \left\{\phi - \frac{2\rho}{3} + \frac{1}{6}\right\}$.

Repeating similar computations in the lower region, with this difference that in the last iteration before entering $\mathcal{D}$, the -2 discontinuity term of equation (30) is replaced by the $+2$ term (see (5)) and c^{upper} is replaced by c^{lower} in (27) we get, denoting by $(\hat{\rho}, \hat{\phi})$ the image in $\mathcal{D}$ of $(\tilde{\rho}, \tilde{\phi}) \in \tilde{\mathcal{D}}$

$$\hat{\rho} = \tilde{\rho} + 4\hat{\phi} + \frac{P_5}{\tilde{\rho}} + \mathcal{O}\left(\frac{1}{\tilde{\rho}^2}\right),$$
$$\hat{\phi} = \left\{\tilde{\phi} - \frac{2\tilde{\rho}}{3} - \frac{1}{6}\right\} + \frac{Q_5}{\tilde{\rho}} + \mathcal{O}\left(\frac{1}{\tilde{\rho}^2}\right).$$

Introducing

$$R = \frac{2\rho}{3} - \frac{1}{6}, \quad \tilde{R} = \frac{2\tilde{\rho}}{3} + \frac{7}{6}, \quad \hat{R} = \frac{2\hat{\rho}}{3} - \frac{1}{6}$$

we get up to $\mathcal{O}\left(\frac{1}{\rho}\right)$-terms

$$\tilde{\phi} \sim \{\phi - R\}, \quad \tilde{R} \sim R - \frac{4}{3} + \frac{8\tilde{\phi}}{3},$$
$$\hat{\phi} \sim \{\tilde{\phi} - \tilde{R}\}, \quad \hat{R} \sim \tilde{R} - \frac{4}{3} + \frac{8\hat{\phi}}{3}.$$

This shows that the linear part of $\mathcal{F}$ has the required form.

To prove the statement about the singularities in the upper region, we must find the equation of the preimage of ℓ_2, so we use the normal form of F^2 and (25) to find that a point (ρ, ϕ) would hit ℓ_2 after $n = \rho/3 + \mathcal{O}(1)$ iterations if and only if

$$\frac{1}{3} - \frac{1}{4\rho} = \frac{1}{\rho}\left(\phi + n + \frac{1}{12}\right) + \mathcal{O}\left(\frac{1}{\rho^2}\right).$$

Thus

$$n = \frac{\rho}{3} - \frac{1}{12} - \left(\phi + \frac{1}{4}\right) + \mathcal{O}\left(\frac{1}{\rho}\right).$$

Since $\frac{\rho}{3} - \frac{1}{12} = R/2$ we have that the preimage of ℓ_2 is $\mathcal{O}(\frac{1}{R})$-close to $\{\phi - R/2\} = 3/4$. Likewise the preimage of ℓ_3' is $\mathcal{O}(\frac{1}{R})$-close to $\{\phi - R/2\} = 1/4$.

The computations in the lower regions are similar.

As for the density of the invariant measure of the return map, start by denoting $\mathbf{F}$ the original map in the coordinate system (r, θ). Let $h_1 : (\rho, \psi) \mapsto (r, \theta)$ denote the inverse map of the conjugacy obtained in subsection A.2, and let $h_2 : (\rho, \phi) \mapsto (\rho, \psi)$ denote the inverse of the rescaling used in Subsection A.3. We are interested in $\mathcal{F} = h_2^{-1} \circ h_1^{-1} \circ \mathbf{F}^{N+1} \circ h_1 \circ h_2$. Since $\mathbf{F}$ preserves the area element, we get that $\mathcal{F}$ preserves the density $\mathrm{Jac}(h_1 \circ h_2)(\rho - 2\phi + \mathcal{O}(1/\rho))d\rho d\phi$ (this is because due to (14) and (15) we have that the r coordinate of $h_1 \circ h_2(\rho, \phi)$ is $\rho - 2\phi + \mathcal{O}(1/\rho)$). Next, (13) implies that $\mathrm{Jac}(h_1)$ in $\mathcal{D}$ equals

$$2 + 2\theta - c/\rho + \mathcal{O}\left(\frac{1}{\rho^2}\right) = 2 + \frac{4\phi - c}{\rho} + \mathcal{O}\left(\frac{1}{\rho^2}\right)$$

with $c = c^{upper}$. Finally, since $\psi = \phi/\rho + c\phi/\rho^2 + \mathcal{O}(1/\rho^3)$ we have that $\mathrm{Jac}(h_2) = \frac{1}{\rho}(1 - \frac{2c}{\rho}) + \mathcal{O}(\frac{1}{\rho^3})$. As a consequence the density preserved by $\mathcal{F}$ is of the form $2 - 5c/\rho + \mathcal{O}(1/\rho^2)$. This ends the proof of Lemma 2. □

Remark. A similar argument shows that in Lemma 2 $\mathcal{O}(1/\rho^2)$-terms are piecewise polynomials of degree 3, $\mathcal{O}(1/\rho^3)$-terms are piecewise polynomials of degree 4 etc. We shall not use this fact here, but it could be useful in other problems, for example, for verification of KAM stability of bounded periodic orbits.

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