Ramsey Theory on Trees and Applications

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Modern Ramsey Theory on infinite structures began with the following seminal result of Ramsey.

Theorem 1 (Ramsey, [14]). *For each positive integer k and each finite coloring of all k-sized subsets of the natural numbers, $\mathbb{N}$, there is an infinite set M of natural numbers such that each k-sized subset of M has the same color.*

This result was motivated by and applied to solve a problem in logic regarding canonical k-ary relations on the natural numbers. Ramsey's Theorem has been extended in a myriad of directions, for instance, varying sizes of sets colored, varying the number of colors allowed, including infinitely many colors, and coloring more complex structures. Progress in Ramsey theory has led to progress in a wide array of mathematical areas, such as model theory, set theory, and logic in general, as well as algebra, analysis, topology and dynamics. In this talk, we concentrate on Ramsey theory on trees and applications to homogeneous structures.

A key result en route to the proof that the Boolean Prime Ideal Theorem is strictly weaker than the Axiom of Choice (see [9]) is the Ramsey-type theorem of Halpern and Läuchli on trees. There are many variations of the Halpern-Läuchli Theorem (see [18]); here we shall state the strong tree version. Let T be a finitely branching tree of height ω with no terminal nodes, and let $T(n)$ denote the nodes on the n-th level of T. A subtree $S \subseteq T$ is called a *strong subtree* of T if for each level of n of S at which some node in S branches, every node in $S(n)$ branches maximally in T. The following is the Strong Tree Version of the Halpern-Läuchli Theorem, proved in another form in [8].

Theorem 2. *Let $d \geq 1$ and let T_i, $i < d$, be finitely branching trees of height ω. Given any finite coloring of $\bigcup_{n<\omega} \prod_{i<d} T_i(n)$, there are strong subtrees $S_i \subseteq T_i$, all with the same infinite set L of branching levels, such that, for all $n \in L$, all members of $\prod_{i<d} S_i(n)$ have the same color.*

For one tree, Milliken strengthened the Halpern-Läuchli Theorem by showing that for any given any finitely branching strong tree T of height ω, given any finite strong tree U and a coloring of all copies of U in T by finitely many colors, there is a strong subtree $S \subseteq T$ of infinite height in which all copies of U have the same color. (See [13].) In the terminology of [18], the collection of strong subtrees of T forms a topological Ramsey space.

Milliken's Theorem has found numerous applications to homogeneous relational structures including the following. In [16], Sauer applied Milliken's Theorem in his proof that the Rado graph $\mathcal{R}$, also known as the infinite random graph,

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and other homogeneous universal binary structures, have finite Ramsey degrees. This means that for each finite graph G, there is a finite number t_G such that given any finite coloring of all copies of G in $\mathcal{R}$, there is a copy $\mathcal{R}'$ of $\mathcal{R}$ in which all copies of G take on at most t_G colors. Moreover, for all graphs G with two or more vertices, the Ramsey number t_G is greater than one. Avilés and Todorcevic applied Milliken's Theorem in [1] to find a finite basis for analytic strong n-gaps. More recently, they developed a new type of Miiliken's Theorem in [2] in order to classify minimal analytic gaps. A dual version of the Halpern-Läuchli Theorem was established by Todorcevic and Tyros in [19].

Building on Sauer's techniques, Dobrinen, Laflamme and Sauer employed Milliken's Theorem to prove in [5] that the Rado graph, and more generally simple binary relational structures, have the rainbow Ramsey property, even though they do not have the Ramsey property. The rainbow Ramsey property states that for each finite k, each finite graph G, and each coloring of the copies of G in $\mathcal{R}$ by ω many colors, where each color appears at most k times, there is a copy $\mathcal{R}'$ inside $\mathcal{R}$ where each color appears at most once.

Extending Sauer's result in another direction, Laflamme, Sauer and Vuksanovic used Milliken's Theorem to obtain canonical partitions for finitary as well as countable colorings of n-tuples in countable homogeneous binary relational structures in [12]. More recently, Vlitas has established a Ramsey-classification theorem for equivalence relations on sets of finite strong subtrees of finitely many countably infinite strong trees in [20].

Turning now to trees on uncountable cardinals, Shelah proved in [17] that it is consistent with ZFC (the standard axioms of set theory) that a version of Milliken's Theorem holds for one strong tree on a measurable cardinal κ. In that theorem, the tree has height κ and less than κ-sized branching on each level, and the coloring is on m-sized subsets of levels of the tree, where m is some fixed positive integer. This result was augmented by Džamonja, Larson and Mitchell in [6] to prove homogeneity for colorings of m-sized antichains in a strong tree on a measurable cardinal. They then applied that result to obtain canonical partitions of m-sized subsets of the κ-rationals in [6] and canonical partitions for colorings of finite subgraphs of the universal graph on κ vertices in [7]. Recently, Dobrinen and Hathaway in [4] proved consistency of the strong subtree version of Milliken's Theorem for finitely many trees on a measurable cardinal, also establishing results for trees on weakly compact cardinals.

All of the proofs of the results mentioned in this paragraph use the set-theoretic method of forcing, using ideas from an unpublished proof of Harrington of the strong tree version of the Halpern-Läuchli Theorem for finitely many finitely branching strong trees of countable height. Recently, Dobrinen has built on these ideas to prove a version of Milliken's Theorem relevant to the universal homogeneous triangle-free graph.

A *triangle-free graph* is a graph which omits triangles. The universal homogeneous triangle-free graph is the Fraïssé limit of the Fraïssé class of all finite triangle-free graphs, which we shall denote by $\mathcal{H}_3$. Each countable triangle-free graph embeds into $\mathcal{H}_3$. A construction of $\mathcal{H}_3$ was given by Henson in [10], where among other things, he proved that for any coloring of the vertices of $\mathcal{H}_3$ into two colors, there is either a copy of $\mathcal{H}_3$ in the first color, or else there are copies of each finite triangle-free graph in the second color. Later, it was proved by Komjáth and Rödl in [11] that for any coloring of the vertices in $\mathcal{H}_3$ into two colors, there is a copy of $\mathcal{H}_3$ with all vertices having the same color.

The question of colorings of vertices being resolved, interest turned to colorings of copies of finite triangle-free graphs G in $\mathcal{H}_3$. The *Ramsey degree* of a finite triangle-free graph G is the smallest number t_G such that for any coloring of the copies of G in $\mathcal{H}_3$ into finitely many colors, there is always a copy $\mathcal{H}'$ of $\mathcal{H}_3$ in which all copies of G take on at most t_G colors. If there is no such bound, then we write $t_G = \infty$. The *big Ramsey numbers problem* for $\mathcal{H}_3$ is the problem of finding out whether or not each finite triangle-free graph G has $t_G < \infty$.

Sauer proved in [15] that for G being an edge, that is a graph with two vertices with one edge between them, $t_G = 2$. In recent work, the Dobrinen has developed a notion of strong tree coding triangle-free graphs. Using ideas from Harrington's forcing proof of the Halpern-Läuchli Theorem, the author has proved an analogue of Milliken's Theorem for these *strong triangle-free trees*, from which it follows that the spaces of strong triangle-free trees are almost topological Ramsey spaces. Using this plus a new type of so-called subtree envelope, the Dobrinen has recovered the results in [11] for vertices and [15] for edges, as well as other finite graphs. At the time of writing this abstract, it looks like all finite triangle-free graphs have finite Ramsey degrees, though the paper [3] is not yet in final form.

This talk will provide an overview of the various versions of the Halpern-Läuchli Theorem and Milliken Theorem and the applications mentioned in this abstract. The author aims to convey the fascinating confluence of ideas from logic, Ramsey theory and set theory leading to applications to solving problems in model theory/universal relational structures.

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