

Chapter 2

Polynomial Solutions of Eigenvalue Problems

2.1 Hahn's q -Operator

Let \mathcal{P} denote the space of polynomials over \mathbb{C} .

In [261] W. Hahn introduced the linear operator $\mathcal{A}_{q,\omega}$ defined by

$$(\mathcal{A}_{q,\omega}p)(x) := \frac{p(qx + \omega) - p(x)}{qx + \omega - x}, \quad p \in \mathcal{P}, \quad x \in \mathbb{R} \setminus \left\{ \frac{\omega}{1-q} \right\} \quad (2.1.1)$$

for all $q \in \mathbb{R} \setminus \{-1, 0\}$ ¹, $\omega \in \mathbb{R}$ and $(q, \omega) \neq (1, 0)$. This class of operators includes the q -derivative operator \mathcal{D}_q ($\omega = 0$), the difference operator Δ ($q = 1$ and $\omega = 1$) and also the differentiation operator D as a limit case ($q \rightarrow 1$ and $\omega = 0$). In order to avoid the latter limiting process, we introduce the operator $\mathcal{A}_{q,\omega}$ in a second way. From (2.1.1) we obtain that $\mathcal{A}_{q,\omega}(1) = 0$, $\mathcal{A}_{q,\omega}(x) = 1$ and the product rule

$$(\mathcal{A}_{q,\omega}(p_1 p_2))(x) = (\mathcal{A}_{q,\omega}p_1)(x)p_2(x) + p_1(qx + \omega)(\mathcal{A}_{q,\omega}p_2)(x) \quad (2.1.2)$$

for $p_1, p_2 \in \mathcal{P}$. Now we have

Theorem 2.1. *For all $q \in \mathbb{R} \setminus \{-1, 0\}$ and $\omega \in \mathbb{R}$ there exists a unique linear operator $\mathcal{A}_{q,\omega}$ on \mathcal{P} satisfying $\mathcal{A}_{q,\omega}(x) = 1$ and the product rule (2.1.2).*

Proof. The product rule (2.1.2) implies that $\mathcal{A}_{q,\omega}(1 \cdot 1) = \mathcal{A}_{q,\omega}(1) \cdot 1 + 1 \cdot \mathcal{A}_{q,\omega}(1)$. Hence we have $\mathcal{A}_{q,\omega}(1) = 0$. So $\mathcal{A}_{q,\omega}$ is uniquely defined on the basis $\{1, x, x^2, \dots\}$ of the space \mathcal{P} by

$$\mathcal{A}_{q,\omega}(x^{n+1}) = \mathcal{A}_{q,\omega}(x)x^n + (qx + \omega)\mathcal{A}_{q,\omega}(x^n), \quad n = 1, 2, 3, \dots$$

and the initial values $\mathcal{A}_{q,\omega}(1) = 0$ and $\mathcal{A}_{q,\omega}(x) = 1$. □

¹ An essential property of the operator $\mathcal{A}_{q,\omega}$ is that its action on a polynomial of degree n leads to a polynomial of degree $n - 1$ for all $n = 1, 2, 3, \dots$. For $q = -1$ this property does not hold (for instance $\mathcal{A}_{-1,\omega}(x^2) = \omega$). That is why the case $q = -1$ is excluded.

As a consequence of this theorem, (2.1.2) together with $\mathcal{A}_{q,\omega}(x) = 1$ can be seen as a definition of $\mathcal{A}_{q,\omega}$. This definition holds for all $q \in \mathbb{R} \setminus \{-1, 0\}$ and $\omega \in \mathbb{R}$, and therefore includes the differentiation operator $D = \mathcal{A}_{1,0}$.

2.2 Eigenvalue Problems

We consider the eigenvalue problem²

$$\varphi(x) (\mathcal{A}_{q,\omega}^2 y_n)(x) + \psi(x) (\mathcal{A}_{q,\omega} y_n)(x) = \lambda_n y_n(qx + \omega) \quad (2.2.1)$$

for polynomials y_n of degree n , where $\mathcal{A}_{q,\omega}^2 y_n = \mathcal{A}_{q,\omega}(\mathcal{A}_{q,\omega} y_n)$ with $\lambda_n \in \mathbb{C}$ and $n \in \{0, 1, 2, \dots\}$. In [106], [372] and [379], for instance, it is shown that polynomial solutions of any degree n , where $n \in \{0, 1, 2, \dots\}$, can only exist if φ is a polynomial of degree at most 2 and ψ is a polynomial of degree 1, say

$$\varphi(x) = ex^2 + 2fx + g, \quad \psi(x) = 2\epsilon x + \gamma, \quad e, f, g, \epsilon, \gamma \in \mathbb{C}, \quad \epsilon \neq 0. \quad (2.2.2)$$

In order to calculate the eigenvalues λ_n , we define (cf. (1.8.2))

$$[-1] = -\frac{1}{q}, \quad [0] = 0, \quad [n] = \sum_{k=0}^{n-1} q^k, \quad n = 1, 2, 3, \dots$$

Note that this definition holds for all $q \in \mathbb{R} \setminus \{-1, 0\}$. Now we obtain

$$[n] - q^{n-k}[k] = [n-k], \quad n \geq k, \quad n, k \in \{0, 1, 2, \dots\} \quad (2.2.3)$$

and

$$[n][n-1] - q^{n-k}[k][k-1] = [n-k][n+k-1], \quad n \geq k, \quad n, k \in \{0, 1, 2, \dots\}. \quad (2.2.4)$$

Further we have

$$\begin{aligned} \mathcal{A}_{q,\omega}(x^n) &= \frac{(qx + \omega)^n - x^n}{qx + \omega - x} = \sum_{k=0}^{n-1} (qx + \omega)^{n-1-k} x^k \\ &= x^{n-1} \sum_{k=0}^{n-1} q^k + r(x) = [n]x^{n-1} + r(x), \quad n = 2, 3, 4, \dots, \end{aligned} \quad (2.2.5)$$

where r is a polynomial of degree at most $n-2$. The eigenvalues λ_n can now be obtained by comparing the coefficients of x^n in (2.2.1):

$$\lambda_n = \frac{[n]}{q^n} (e[n-1] + 2\epsilon), \quad n = 0, 1, 2, \dots \quad (2.2.6)$$

² It will turn out to be convenient to take $y_n(qx + \omega)$ on the right-hand side.

So we have: if (2.2.1) has a polynomial solution of degree n , then λ_n given by (2.2.6) is the corresponding eigenvalue.

Note that this result is also valid for $q = 1$ and $\omega = 0$.

So the eigenvalue problem (2.2.1) can be written in the form

$$\begin{aligned} & (ex^2 + 2fx + g) (\mathcal{A}_{q,\omega}^2 y_n) (x) + (2\epsilon x + \gamma) (\mathcal{A}_{q,\omega} y_n) (x) \\ &= \frac{[n]}{q^n} (e[n-1] + 2\epsilon) y_n(qx + \omega), \quad n = 0, 1, 2, \dots \end{aligned} \quad (2.2.7)$$

In the case of the difference operator Δ (i.e. $q = 1$ and $\omega = 1$) this reads

$$(ex^2 + 2fx + g) (\Delta^2 y_n) (x) + (2\epsilon x + \gamma) (\Delta y_n) (x) = n(e(n-1) + 2\epsilon) y_n(x+1) \quad (2.2.8)$$

for $n = 0, 1, 2, \dots$ and in the case of the differentiation operator D (i.e. $q = 1$ and $\omega = 0$)

$$(ex^2 + 2fx + g) y_n''(x) + (2\epsilon x + \gamma) y_n'(x) = n(e(n-1) + 2\epsilon) y_n(x) \quad (2.2.9)$$

for $n = 0, 1, 2, \dots$. We remark that the operator $\mathcal{A}_{q,\omega}$ is not invariant under translations for $q \neq 1$, unlike the operators Δ and D . This can be seen as follows. If we apply the operator $\mathcal{A}_{q,\omega}$ to the polynomial $p(\cdot + c)$, where $c \in \mathbb{R}$ is a constant, we obtain

$$(\mathcal{A}_{q,\omega} p(\cdot + c)) (x) = \frac{p(qx + \omega + c) - p(x + c)}{qx + \omega - x} \quad (2.2.10)$$

and, if we apply the operator $\mathcal{A}_{q,\omega}$ to the polynomial p first and then replace the argument x by $x + c$, we have

$$\begin{aligned} (\mathcal{A}_{q,\omega} p(\cdot)) (x + c) &= \frac{p(q(x+c) + \omega) - p(x+c)}{q(x+c) + \omega - (x+c)} \\ &= \frac{p(qx + \bar{\omega} + c) - p(x+c)}{qx + \bar{\omega} - x}, \end{aligned} \quad (2.2.11)$$

where $\bar{\omega} = \omega + c(q-1)$. Both results coincide only if $q = 1$.

Now we will give another version of the operator equation (2.2.7) in the case that $(q, \omega) \neq (1, 0)$. In order to do this, we write

$$(\mathcal{A}_{q,\omega} y_n) (x) = \frac{y_n(qx + \omega) - y_n(x)}{qx + \omega - x} =: p_n(x)$$

and

$$\begin{aligned} (\mathcal{A}_{q,\omega}^2 y_n) (x) &= (\mathcal{A}_{q,\omega} p_n) (x) = \frac{p_n(qx + \omega) - p_n(x)}{qx + \omega - x} \\ &= \frac{y_n(q^2x + [2]\omega) - (1+q)y_n(qx + \omega) + qy_n(x)}{q(qx + \omega - x)^2}. \end{aligned}$$

Then the operator equation (2.2.7) can be written in the form

$$C(qx + \omega)y_n(q^2x + [2]\omega) - \{C(qx + \omega) + D(qx + \omega)\}y_n(qx + \omega) + D(qx + \omega)y_n(x) = \lambda_n y_n(qx + \omega),$$

where

$$C(qx + \omega) = \frac{ex^2 + 2fx + g}{q(qx + \omega - x)^2} \quad \text{and} \quad D(qx + \omega) = qC(qx + \omega) - \frac{2\epsilon x + \gamma}{qx + \omega - x}.$$

If we now replace x by $(x - \omega)/q$, we obtain the so-called *symmetric* form

$$C(x)y_n(qx + \omega) - \{C(x) + D(x)\}y_n(x) + D(x)y_n((x - \omega)/q) = \lambda_n y_n(x) \quad (2.2.12)$$

for $n = 0, 1, 2, \dots$ with

$$C(x) = \frac{e(x - \omega)^2 + 2fq(x - \omega) + gq^2}{q(qx + \omega - x)^2} \quad \text{and} \quad D(x) = qC(x) - \frac{2\epsilon(x - \omega) + \gamma q}{qx + \omega - x}. \quad (2.2.13)$$

Finally, we will derive a third version of the operator equation (2.2.7), which involves the operators $\mathcal{A}_{q,\omega}$ and $\mathcal{A}_{1/q,-\omega/q}$. First of all we have

$$\begin{aligned} (\mathcal{A}_{q,\omega}y_n)((x - \omega)/q) &= \frac{y_n(q((x - \omega)/q) + \omega) - y_n((x - \omega)/q)}{q((x - \omega)/q) + \omega - (x - \omega)/q} \\ &= \frac{y_n((x - \omega)/q) - y_n(x)}{(x - \omega)/q - x} = (\mathcal{A}_{1/q,-\omega/q}y_n)(x). \end{aligned}$$

In the case of the q -derivative operator \mathcal{D}_q (i.e. $\omega = 0$), this reads

$$(\mathcal{D}_q y_n)(x/q) = (\mathcal{A}_{q,0}y_n)(x/q) = (\mathcal{A}_{1/q,0}y_n)(x) = \mathcal{D}_{1/q}y_n(x).$$

In the case of the difference operator Δ (i.e. $q = 1$ and $\omega = 1$), this reads

$$(\Delta y_n)(x - 1) = (\mathcal{A}_{1,1}y_n)(x - 1) = (\mathcal{A}_{1,-1}y_n)(x) =: \nabla y_n(x).$$

Now we have

$$\begin{aligned} (\mathcal{A}_{q,\omega}(\mathcal{A}_{1/q,-\omega/q}y_n))(x) &= \frac{(\mathcal{A}_{1/q,-\omega/q}y_n)(qx + \omega) - (\mathcal{A}_{1/q,-\omega/q}y_n)(x)}{qx + \omega - x} \\ &= \frac{y_n(qx + \omega) - (1 + q)y_n(x) + qy_n((x - \omega)/q)}{(qx + \omega - x)^2} \\ &= q^{-1}(\mathcal{A}_{q,\omega}^2 y_n)((x - \omega)/q). \end{aligned}$$

Hence we obtain for $n = 0, 1, 2, \dots$

$$\begin{aligned}
& q\varphi((x-\omega)/q) \left(\mathcal{A}_{q,\omega} \left(\mathcal{A}_{1/q,-\omega/q} y_n \right) \right) (x) \\
& + \psi((x-\omega)/q) \left(\mathcal{A}_{1/q,-\omega/q} y_n \right) (x) = \lambda_n y_n(x). \tag{2.2.14}
\end{aligned}$$

In the case of the q -derivative operator \mathcal{D}_q (i.e. $\omega = 0$), this reads

$$q\varphi(x/q) \left(\mathcal{D}_q \left(\mathcal{D}_{1/q} y_n \right) \right) (x) + \psi(x/q) \left(\mathcal{D}_{1/q} y_n \right) (x) = \lambda_n y_n(x) \tag{2.2.15}$$

for $n = 0, 1, 2, \dots$. In the case of the difference operator Δ (i.e. $q = 1$ and $\omega = 1$), this reads

$$\varphi(x-1) (\Delta (\nabla y_n)) (x) + \psi(x-1) (\nabla y_n) (x) = \lambda_n y_n(x), \quad n = 0, 1, 2, \dots \tag{2.2.16}$$

2.3 The Regularity Condition

In this section we will point out in which cases the eigenvalue problem (2.2.1) has essentially unique polynomial solutions $y_n(x)$ of degrees $n = 0, 1, 2, \dots, N$ for some positive integer N with possibly $N \rightarrow \infty$. Solutions are called essentially unique if they are determined up to a factor independent of x . We have

Theorem 2.2. *Let N denote a positive integer (possibly $N \rightarrow \infty$). Then the following statements are equivalent:*

1. *For each $n = 0, 1, 2, \dots, N$ there exists a solution y_n of the eigenvalue problem (2.2.1) and all eigenspaces are one dimensional.*
2. *For $m, n \in \{0, 1, 2, \dots, N\}$ with $m \neq n$ we have $\lambda_m \neq \lambda_n$.*

Proof. Assume that $\lambda_m = \lambda_n$ for $m \neq n$. Then there is either no polynomial solution for one of the degrees m and n or the solutions y_m and y_n belong to the same eigenspace. This shows that the first statement implies the second.

Now we use induction to show that the second statement implies the first. For $n = 0$ we have $\lambda_0 = 0$ and the one-dimensional eigenspace generated by $y_0(x) = 1$. Now we assume that $n \in \{1, 2, 3, \dots\}$. Suppose that the polynomials $y_\nu(x)$ are solutions of degree ν for $\nu = 0, 1, 2, \dots, n-1$. Then the (monic) polynomial $y_n(x)$ of degree n given by

$$y_n(x) = x^n + \sum_{\nu=0}^{n-1} \alpha_\nu y_\nu(x) \quad \text{with} \quad \alpha_\nu \in \mathbb{C} \tag{2.3.1}$$

is a solution of (2.2.1) if

$$\begin{aligned}
& \varphi(x)\mathcal{A}_{q,\omega}^2(x^n) + \psi(x)\mathcal{A}_{q,\omega}(x^n) \\
& + \varphi(x) \left(\mathcal{A}_{q,\omega}^2 \sum_{v=0}^{n-1} \alpha_v y_v \right) (x) + \psi(x) \left(\mathcal{A}_{q,\omega} \sum_{v=0}^{n-1} \alpha_v y_v \right) (x) \\
& = \lambda_n \left((qx + \omega)^n + \sum_{v=0}^{n-1} \alpha_v y_v (qx + \omega) \right)
\end{aligned}$$

holds. The polynomial $\varphi(x)\mathcal{A}_{q,\omega}^2(x^n) + \psi(x)\mathcal{A}_{q,\omega}(x^n)$ has degree at most n . Hence we may write

$$\varphi(x)\mathcal{A}_{q,\omega}^2(x^n) + \psi(x)\mathcal{A}_{q,\omega}(x^n) = \beta_n(qx + \omega)^n + \sum_{v=0}^{n-1} \beta_v y_v(qx + \omega)$$

with $\beta_n, \beta_v \in \mathbb{C}$. Combining the last two equations, we get

$$\beta_n(qx + \omega)^n + \sum_{v=0}^{n-1} (\beta_v + \lambda_v \alpha_v) y_v(qx + \omega) = \lambda_n \left((qx + \omega)^n + \sum_{v=0}^{n-1} \alpha_v y_v(qx + \omega) \right)$$

and therefore

$$(\beta_n - \lambda_n)(qx + \omega)^n + \sum_{v=0}^{n-1} (\alpha_v(\lambda_v - \lambda_n) + \beta_v) y_v(qx + \omega) = 0.$$

Since $\lambda_v \neq \lambda_n$, this implies that the numbers α_v are uniquely determined by this equation. So in fact this means that the (monic) polynomial solution y_n given by (2.3.1) is uniquely determined, which implies that the corresponding eigenspace is one dimensional. \square

Now we use (2.2.3) and (2.2.4) to find from (2.2.6):

$$\begin{aligned}
q^n(\lambda_n - \lambda_m) &= e([n][n-1] - q^{n-m}[m][m-1]) + 2\mathcal{E}([n] - q^{n-m}[m]) \\
&= e[n-m][n+m-1] + 2\mathcal{E}[n-m] \\
&= [n-m](e[n+m-1] + 2\mathcal{E}), \quad n \geq m, \quad m, n \in \{0, 1, 2, \dots\}.
\end{aligned}$$

Hence we have

$$\lambda_n - \lambda_m = \frac{[n-m]}{q^n} (e[n+m-1] + 2\mathcal{E}), \quad n \geq m, \quad m, n \in \{0, 1, 2, \dots\}. \quad (2.3.2)$$

Since $q \neq -1$, it follows that $[n-m] \neq 0$ for $m \neq n$, so $\lambda_m \neq \lambda_n$ is equivalent to $e[n+m-1] + 2\mathcal{E} \neq 0$. Therefore, theorem 2.2 leads to:

Corollary 2.3. *Let N denote a positive integer (possibly $N \rightarrow \infty$). Then the eigenvalue problem (2.2.1) has polynomial solutions y_n of degree n for all $n = 0, 1, 2, \dots, N$ with one-dimensional eigenspaces if and only if the regularity condition*

$$e[n] + 2\varepsilon \neq 0, \quad n = 0, 1, 2, \dots, 2N - 2 \quad (2.3.3)$$

holds.

2.4 Determination of the Polynomial Solutions

We want to obtain a two-term recurrence relation for the coefficients of the polynomial solutions y_n of (2.2.1). In order to achieve this, we introduce so-called generalized binomial coefficients $\begin{bmatrix} x; c \\ n \end{bmatrix}$ with $x \in \mathbb{R}$, $c \in \mathbb{C}$ and $n \in \{0, 1, 2, \dots\}$, such that

$$\left(\mathcal{A}_{q, \omega} \begin{bmatrix} \cdot; c \\ 0 \end{bmatrix} \right) (x) = 0 \quad \text{and} \quad \left(\mathcal{A}_{q, \omega} \begin{bmatrix} \cdot; c \\ n \end{bmatrix} \right) (x) = \begin{bmatrix} x; c \\ n-1 \end{bmatrix} \quad (2.4.1)$$

for $n = 1, 2, 3, \dots$. This can be done by

$$\begin{bmatrix} x; c \\ 0 \end{bmatrix} := 1 \quad \text{and} \quad \begin{bmatrix} x; c \\ n \end{bmatrix} := \prod_{i=1}^n \frac{x + cq^{i-1} - [i-1]\omega}{[i]}, \quad n = 1, 2, 3, \dots \quad (2.4.2)$$

Note that these generalized binomial coefficients depend on q and ω . However, for ease of expression we omit these in the notation of the symbol $\begin{bmatrix} \cdot \\ \cdot \end{bmatrix}$. In the case of the difference operator (i.e. $q = 1$ and $\omega = 1$) these generalized binomial coefficients reduce to the ordinary binomial coefficients and

$$\Delta \begin{pmatrix} x+c \\ 0 \end{pmatrix} = 0 \quad \text{and} \quad \Delta \begin{pmatrix} x+c \\ n \end{pmatrix} = \begin{pmatrix} x+c \\ n-1 \end{pmatrix}, \quad n = 1, 2, 3, \dots$$

In the case of the differentiation operator D (i.e. $q = 1$ and $\omega = 0$), we have

$$D(1) = 0 \quad \text{and} \quad D \frac{(x+c)^n}{n!} = \frac{(x+c)^{n-1}}{(n-1)!}, \quad n = 1, 2, 3, \dots$$

2.4.1 First Approach

To simplify the expressions, we write $v = \omega + c(1 - q)$. Then we have

$$\begin{bmatrix} x; c \\ n \end{bmatrix} = \prod_{i=1}^n \frac{x + c - [i-1]v}{[i]}, \quad n = 1, 2, 3, \dots$$

Now we write

$$y_n(x) = \sum_{k=0}^n a_{n,k} \begin{bmatrix} x; c \\ k \end{bmatrix}, \quad a_{n,n} \neq 0, \quad n = 0, 1, 2, \dots \quad (2.4.3)$$

For a given $a_{n,n} \neq 0$ we want to determine the other coefficients $a_{n,k}$ in such a way that $y_n(x)$ given by (2.4.3) satisfies (2.2.7). By using (2.2.3), (2.2.4), (2.4.1),

$$x \begin{bmatrix} x; c \\ k-1 \end{bmatrix} = [k] \begin{bmatrix} x; c \\ k \end{bmatrix} + ([k-1]v - c) \begin{bmatrix} x; c \\ k-1 \end{bmatrix}, \quad k = 1, 2, 3, \dots,$$

$$x^2 \begin{bmatrix} x; c \\ k-2 \end{bmatrix} = [k][k-1] \begin{bmatrix} x; c \\ k \end{bmatrix} + [k-1] \{([k-1] + [k-2])v - 2c\} \begin{bmatrix} x; c \\ k-1 \end{bmatrix} \\ + ([k-2]v - c)^2 \begin{bmatrix} x; c \\ k-2 \end{bmatrix}, \quad k = 2, 3, 4, \dots,$$

$$\begin{bmatrix} qx + \omega; c \\ k \end{bmatrix} = q^k \begin{bmatrix} x; c \\ k \end{bmatrix} + vq^{k-1} \begin{bmatrix} x; c \\ k-1 \end{bmatrix}, \quad k = 1, 2, 3, \dots,$$

by substituting (2.4.3) into (2.2.7) and by comparing the coefficients, we find

$$[n-k] (e[n+k-1] + 2\varepsilon) a_{n,k} + \left\{ e[n][n-1]v + 2\varepsilon[n-k]v \right. \\ \left. - e[k] \{([k] + [k-1])v - 2c\} q^{n-k} + (2\varepsilon c - 2f[k] - \gamma) q^{n-k} \right\} a_{n,k+1} \\ - (e([k]v - c)^2 + 2f([k]v - c) + g) q^{n-k} a_{n,k+2} = 0 \quad (2.4.4)$$

for $k = n-1, n-2, n-3, \dots, 0$ with the convention that $a_{n,n+1} := 0$. Hence, if the regularity condition (2.3.3) holds, then (2.4.4) gives us the coefficients $a_{n,k}$ for $k = n-1, n-2, n-3, \dots, 0$ in terms of $a_{n,n} \neq 0$.

The three-term recurrence relation (2.4.4) can be rewritten in a such a way that the possibility of a two-term recurrence relation becomes apparent:

$$[n-k] (e[n+k-1] + 2\varepsilon) a_{n,k} - (e[k-1]v - 2ec + 2f) [k-1] q^{n-k+1} a_{n,k+1} \\ + [n-k-1] (e[n+k] + 2\varepsilon) v a_{n,k+1} - (e[k]v - 2ec + 2f) [k] v q^{n-k} a_{n,k+2} \\ + ((e + 2\varepsilon)v + \{2c(e + \varepsilon) - \gamma - 2f\} q) q^{n-k-1} a_{n,k+1} \\ - (ec^2 - 2fc + g) q^{n-k} a_{n,k+2} = 0.$$

This implies that if

$$(e + 2\varepsilon)v^2 + \{2c(e + \varepsilon) - \gamma - 2f\} vq = -(ec^2 - 2fc + g)q^2 \quad (2.4.5)$$

holds, then the recurrence relation (2.4.4) can be written in the form

$$s(k)a_{n,k} + t(k)a_{n,k+1} + v(s(k+1)a_{n,k+1} + t(k+1)a_{n,k+2}) = 0$$

with

$$s(k) := [n-k] (e[n+k-1] + 2\varepsilon)$$

and

$$t(k) := ((e + 2\varepsilon)v + \{2\varepsilon c - \gamma + 2(ec - f)[k] - e[k - 1]^2 v q\} q) q^{n-k-1}.$$

Now we consider the two-term recurrence relation

$$s(k)a_{n,k} + t(k)a_{n,k+1} = 0, \quad k = n - 1, n - 2, n - 3, \dots, 0. \quad (2.4.6)$$

If there exists a number c satisfying (2.4.5) and if the coefficients $a_{n,k}$ satisfy (2.4.6), then they also satisfy (2.4.4). Therefore, the coefficients $a_{n,k}$ can be determined by (2.4.6) in terms of $a_{n,n} \neq 0$ provided that there exists a number c such that (2.4.5) holds.

In the case that $v \neq 0$, and c satisfies (2.4.5), we can write the two-term recurrence relation for the coefficients $a_{n,k}$ in a different form. If we multiply (2.4.6) by v and use (2.4.5), we obtain

$$\begin{aligned} [n - k](e[n + k - 1] + 2\varepsilon)va_{n,k} \\ - \{e([k - 1]v - c)^2 + 2f([k - 1]v - c) + g\}q^{n-k+1}a_{n,k+1} = 0 \end{aligned} \quad (2.4.7)$$

for $k = n - 1, n - 2, n - 3, \dots, 0$.

With this first approach the case of the differentiation operator D (i.e. $q = 1$ and $\omega = v = 0$) can be treated completely. If there exists a c satisfying (2.4.5), id est $ec^2 - 2fc + g = 0$, then the coefficients of the polynomial solutions

$$y_n(x) = \sum_{k=0}^n a_{n,k} \frac{(x+c)^k}{k!}, \quad a_{n,n} \neq 0, \quad n = 0, 1, 2, \dots \quad (2.4.8)$$

of the second-order differential equation (2.2.9) satisfy the two-term recurrence relation (2.4.6), id est

$$(n - k)(e(n + k - 1) + 2\varepsilon)a_{n,k} + (2(ec - f)k + 2\varepsilon c - \gamma)a_{n,k+1} = 0 \quad (2.4.9)$$

for $k = n - 1, n - 2, n - 3, \dots, 0$. If (2.4.5) has no solution for c , id est $e = f = 0$ and $g \neq 0$, then we find from (2.4.4) with $c = \gamma/2\varepsilon$ the two-term recurrence relation

$$2\varepsilon(n - k)a_{n,k} - ga_{n,k+2} = 0, \quad a_{n,n+1} = 0, \quad k = n - 1, n - 2, n - 3, \dots, 0, \quad (2.4.10)$$

which leads to the (symmetric) Hermite polynomials.

2.4.2 Second Approach

In order to deal with the cases where c cannot be determined by (2.4.5), we use a second approach. For this purpose we will modify (2.4.3) in such a way that in the numerator of the generalized binomial coefficient $x + c$ occurs as the penultimate factor. By using $v = \omega + c(1 - q)$ and the definition (2.4.2), we obtain

$$\begin{aligned}
\begin{bmatrix} x; c + [k-2]vq^{2-k} \\ k \end{bmatrix} &= \prod_{i=1}^k \frac{x + cq^{i-1} + [k-2]vq^{i+1-k} - [i-1]\omega}{[i]} \\
&= \prod_{i=1}^k \frac{x + c + [k-i-1]vq^{i+1-k}}{[i]} \quad (2.4.11)
\end{aligned}$$

for $k = 1, 2, 3, \dots$. Note that

$$\left(\mathcal{A}_{q,\omega} \begin{bmatrix} \cdot; c + [k-2]vq^{2-k} \\ k \end{bmatrix} \right) (x) = \begin{bmatrix} x; c + [k-2]vq^{2-k} \\ k-1 \end{bmatrix}, \quad k = 1, 2, 3, \dots$$

These alternative generalized binomial coefficients can be used to build a second form for the solutions:

$$y_n(x) = \sum_{k=0}^n b_{n,k} \begin{bmatrix} x; c + [k-2]vq^{2-k} \\ k \end{bmatrix}, \quad b_{n,n} \neq 0, \quad n = 0, 1, 2, \dots \quad (2.4.12)$$

For a given $b_{n,n} \neq 0$ we want to determine the other coefficients $b_{n,k}$ in such a way that $y_n(x)$ given by (2.4.12) satisfies (2.2.7). By using

$$x \begin{bmatrix} x; c + [k-2]vq^{2-k} \\ k-1 \end{bmatrix} = [k] \begin{bmatrix} x; c + [k-2]vq^{2-k} \\ k \end{bmatrix} + (v-c) \begin{bmatrix} x; c + [k-2]vq^{2-k} \\ k-1 \end{bmatrix}$$

for $k = 1, 2, 3, \dots$,

$$\begin{aligned}
x^2 \begin{bmatrix} x; c + [k-2]vq^{2-k} \\ k-2 \end{bmatrix} &= [k][k-1] \begin{bmatrix} x; c + [k-2]vq^{2-k} \\ k \end{bmatrix} \\
&\quad + [k-1](v-2c) \begin{bmatrix} x; c + [k-2]vq^{2-k} \\ k-1 \end{bmatrix} + c^2 \begin{bmatrix} x; c + [k-2]vq^{2-k} \\ k-2 \end{bmatrix}
\end{aligned}$$

for $k = 2, 3, 4, \dots$,

$$x \begin{bmatrix} x; c + [k-2]vq^{2-k} \\ k-2 \end{bmatrix} = [k-1] \begin{bmatrix} x; c + [k-2]vq^{2-k} \\ k-1 \end{bmatrix} - c \begin{bmatrix} x; c + [k-2]vq^{2-k} \\ k-2 \end{bmatrix}$$

for $k = 2, 3, 4, \dots$ and

$$\begin{aligned}
\begin{bmatrix} qx + \omega; c + [k-2]vq^{2-k} \\ k \end{bmatrix} &= q^k \begin{bmatrix} x; c + [k-1]vq^{1-k} \\ k \end{bmatrix} \\
&= q^k \begin{bmatrix} x; c + [k-2]vq^{2-k} \\ k \end{bmatrix} + vq \begin{bmatrix} x; c + [k-2]vq^{2-k} \\ k-1 \end{bmatrix}
\end{aligned}$$

for $k = 1, 2, 3, \dots$, we find by substituting (2.4.12) into (2.2.7)

$$\begin{aligned}
& \sum_{k=0}^n [n-k] (e[n+k-1] + 2\varepsilon) q^k \begin{bmatrix} x; c + [k-1]vq^{1-k} \\ k \end{bmatrix} b_{n,k} \\
& + \sum_{k=1}^n \left\{ (e[k-1] + 2\varepsilon) \left([k]q^{1-k} - 1 \right) vq^n + (2[k-1](ec-f) + 2\varepsilon c - \gamma) q^n \right\} \\
& \quad \times \begin{bmatrix} x; c + [k-2]vq^{2-k} \\ k-1 \end{bmatrix} b_{n,k} \\
& - \sum_{k=2}^n (ec^2 - 2fc + g) q^n \begin{bmatrix} x; c + [k-2]vq^{2-k} \\ k-2 \end{bmatrix} b_{n,k} = 0.
\end{aligned}$$

Hence, if $ec^2 - 2fc + g = 0$ can be solved for c , only two sums remain. These lead to a two-term recurrence relation for the coefficients $b_{n,k}$:

$$\begin{aligned}
& [n-k] (e[n+k-1] + 2\varepsilon) b_{n,k} + \left\{ (e[k] + 2\varepsilon) ([k+1]q^{-k} - 1)v \right. \\
& \quad \left. + (2[k](ec-f) + 2\varepsilon c - \gamma) \right\} q^{n-k} b_{n,k+1} = 0
\end{aligned} \tag{2.4.13}$$

for $k = n-1, n-2, n-3, \dots, 0$.

The case of the difference operator Δ (i.e. $q = \omega = v = 1$) can be treated by using both approaches. The first approach leads to solutions of the form

$$y_n(x) = \sum_{k=0}^n a_{n,k} \binom{x+c}{k}, \quad a_{n,n} \neq 0, \quad n = 0, 1, 2, \dots, \tag{2.4.14}$$

where c is a solution of (2.4.5), id est

$$e(c+1)^2 - 2f(c+1) + g = -2\varepsilon(c+1) + \gamma$$

and the coefficients satisfy (2.4.6), which reads

$$\begin{aligned}
& (n-k) (e(n+k-1) + 2\varepsilon) a_{n,k} \\
& - (e(k-1-c)^2 + 2f(k-1-c) + g) a_{n,k+1} = 0
\end{aligned} \tag{2.4.15}$$

for $k = n-1, n-2, n-3, \dots, 0$. The second approach leads to solutions of the form

$$y_n(x) = \sum_{k=0}^n b_{n,k} \binom{x+c+k-2}{k}, \quad b_{n,n} \neq 0, \quad n = 0, 1, 2, \dots \tag{2.4.16}$$

If c satisfies $ec^2 - 2fc + g = 0$, then the coefficients satisfy the two-term recurrence relation (2.4.13), which reads

$$\begin{aligned}
& (n-k) (e(n+k-1) + 2\varepsilon) b_{n,k} \\
& + (ek^2 + 2(ec-f + \varepsilon)k + 2\varepsilon c - \gamma) b_{n,k+1} = 0
\end{aligned} \tag{2.4.17}$$

for $k = n-1, n-2, n-3, \dots, 0$.

2.5 Existence of a Three-Term Recurrence Relation

In this section we will show that the monic polynomial solutions y_n of the operator equation (2.2.1) satisfy a three-term recurrence relation of the form

$$y_{n+1}(x) = (x - c_n)y_n(x) - d_n y_{n-1}(x), \quad c_n, d_n \in \mathbb{C}, \quad n = 1, 2, 3, \dots \quad (2.5.1)$$

For simplification we introduce the operator $\mathcal{S}_{q,\omega}$ on the space \mathcal{P} of polynomials defined by

$$(\mathcal{S}_{q,\omega} p)(x) := p(qx + \omega), \quad p \in \mathcal{P}, \quad x \in \mathbb{R}. \quad (2.5.2)$$

Then the following commutation relations hold:

$$\mathcal{A}_{q,\omega} \mathcal{S}_{q,\omega} = q \mathcal{S}_{q,\omega} \mathcal{A}_{q,\omega} \quad \text{and} \quad \mathcal{S}_{q,\omega}^{-1} \mathcal{A}_{q,\omega} = q \mathcal{A}_{q,\omega} \mathcal{S}_{q,\omega}^{-1}. \quad (2.5.3)$$

Further we will use the notation

$$\widehat{p}(x) := (\mathcal{S}_{q,\omega}^{-1} p)(x) = p((x - \omega)/q), \quad p \in \mathcal{P}, \quad x \in \mathbb{R} \quad (2.5.4)$$

for convenience.

Theorem 2.4. *Let Λ be a linear functional on \mathcal{P} defined by*

$$\Lambda[1] = 1 \quad \text{and} \quad \Lambda[y_n] = 0, \quad n = 1, 2, 3, \dots, \quad (2.5.5)$$

where y_n denotes a polynomial solution of the operator equation (2.2.1). Then the following distributional equation holds for every polynomial $p \in \mathcal{P}$:

$$\Lambda[q\widehat{\varphi}(\mathcal{A}_{q,\omega} p) + \widehat{\psi}p] = 0. \quad (2.5.6)$$

Furthermore, for all $m, n \in \{0, 1, 2, \dots\}$ we have

$$\Lambda[q\widehat{\varphi}(\mathcal{A}_{q,\omega} y_m)(\mathcal{A}_{q,\omega} y_n)] = -\lambda_n \Lambda[y_m y_n]. \quad (2.5.7)$$

Proof. By using (2.5.2), we obtain from (2.2.1) for every polynomial

$$p^*(x) = \sum_{k=0}^n \alpha_k y_k(x), \quad \alpha_k \in \mathbb{C}$$

that

$$\begin{aligned} & \varphi(x) (\mathcal{A}_{q,\omega}^2 p^*)(x) + \psi(x) (\mathcal{A}_{q,\omega} p^*)(x) \\ &= \sum_{k=0}^n \alpha_k \{ \varphi(x) (\mathcal{A}_{q,\omega}^2 y_k)(x) + \psi(x) (\mathcal{A}_{q,\omega} y_k)(x) \} \\ &= \sum_{k=0}^n \alpha_k \lambda_k (\mathcal{S}_{q,\omega} y_k)(x). \end{aligned}$$

Applying $\mathcal{S}_{q,\omega}^{-1}$ to both sides and using the commutation relation (2.5.3), we obtain by using the notation (2.5.4)

$$q\widehat{\varphi}(x) (\mathcal{A}_{q,\omega}\mathcal{S}_{q,\omega}^{-1}\mathcal{A}_{q,\omega}p^*)(x) + \widehat{\psi}(x) (\mathcal{S}_{q,\omega}^{-1}\mathcal{A}_{q,\omega}p^*)(x) = \sum_{k=0}^n \alpha_k \lambda_k y_k(x).$$

From (2.2.5) it can be deduced that for each polynomial $p \in \mathcal{P}$ there exists a polynomial p^* such that

$$p(x) = (\mathcal{S}_{q,\omega}^{-1}\mathcal{A}_{q,\omega}p^*)(x).$$

Now we use the fact that $\lambda_0 = 0$ and $\Lambda[y_k] = 0$ for $k = 1, 2, 3, \dots$ to conclude that

$$\Lambda[q\widehat{\varphi}(\mathcal{A}_{q,\omega}p) + \widehat{\psi}p] = \sum_{k=0}^n \alpha_k \lambda_k \Lambda[y_k] = 0,$$

which proves (2.5.6).

To prove (2.5.7), we apply $\mathcal{S}_{q,\omega}^{-1}$ to the operator equation (2.2.1), use the commutation relation (2.5.3) and multiply the result by y_m :

$$q\widehat{\varphi}(x) (\mathcal{A}_{q,\omega}\mathcal{S}_{q,\omega}^{-1}\mathcal{A}_{q,\omega}y_n)(x)y_m(x) + \widehat{\psi}(x) (\mathcal{S}_{q,\omega}^{-1}\mathcal{A}_{q,\omega}y_n)(x)y_m(x) = \lambda_n y_m(x)y_n(x).$$

If we apply the product rule (2.1.2) to $p_1(x) = (\mathcal{S}_{q,\omega}^{-1}\mathcal{A}_{q,\omega}y_n)(x)$ and $p_2(x) = y_m(x)$, then we find

$$\begin{aligned} & (\mathcal{A}_{q,\omega}((\mathcal{S}_{q,\omega}^{-1}\mathcal{A}_{q,\omega}y_n)y_m))(x) \\ &= (\mathcal{A}_{q,\omega}\mathcal{S}_{q,\omega}^{-1}\mathcal{A}_{q,\omega}y_n)(x)y_m(x) + (\mathcal{A}_{q,\omega}y_n)(x)(\mathcal{A}_{q,\omega}y_m)(x). \end{aligned}$$

Combining the last two results, we find that

$$\begin{aligned} & q\widehat{\varphi}(x) \{ (\mathcal{A}_{q,\omega}((\mathcal{S}_{q,\omega}^{-1}\mathcal{A}_{q,\omega}y_n)y_m))(x) - (\mathcal{A}_{q,\omega}y_n)(x)(\mathcal{A}_{q,\omega}y_m)(x) \} \\ &+ \widehat{\psi}(x) (\mathcal{S}_{q,\omega}^{-1}\mathcal{A}_{q,\omega}y_n)(x)y_m(x) = \lambda_n y_m(x)y_n(x). \end{aligned}$$

Finally, we apply Λ on both sides of this equation to obtain

$$\begin{aligned} & \Lambda[q\widehat{\varphi}(\mathcal{A}_{q,\omega}((\mathcal{S}_{q,\omega}^{-1}\mathcal{A}_{q,\omega}y_n)y_m)) + \widehat{\psi}(\mathcal{S}_{q,\omega}^{-1}\mathcal{A}_{q,\omega}y_n)y_m] \\ & - \Lambda[q\widehat{\varphi}(\mathcal{A}_{q,\omega}y_n)(\mathcal{A}_{q,\omega}y_m)] = \lambda_n \Lambda[y_m y_n]. \end{aligned}$$

The first term equals zero in view of (2.5.6) with $p(x) = (\mathcal{S}_{q,\omega}^{-1}\mathcal{A}_{q,\omega}y_n)(x)y_m(x)$. This completes the proof of (2.5.7). \square

The following theorem states the "weak" orthogonality of the polynomial solutions of the eigenvalue problem (2.2.1).

Theorem 2.5. *Let the regularity condition (2.3.3) hold for the operator equation (2.2.1) with polynomial solutions y_n with $n = 0, 1, 2, \dots$. Then the linear functional Λ given by (2.5.5) satisfies*

$$\Lambda[y_m y_n] = 0 \quad \text{for } m \neq n, \quad m, n \in \{0, 1, 2, \dots\}. \quad (2.5.8)$$

Proof. From (2.5.7) we get for $m, n \in \{0, 1, 2, \dots\}$

$$\begin{cases} \Lambda[q\widehat{\varphi}(\mathcal{A}_{q,\omega}y_n)(\mathcal{A}_{q,\omega}y_m)] = -\lambda_n\Lambda[y_m y_n] \\ \Lambda[q\widehat{\varphi}(\mathcal{A}_{q,\omega}y_m)(\mathcal{A}_{q,\omega}y_n)] = -\lambda_m\Lambda[y_m y_n]. \end{cases}$$

Subtracting these two equations, we find

$$(\lambda_n - \lambda_m)\Lambda[y_m y_n] = 0, \quad m, n \in \{0, 1, 2, \dots\}.$$

The regularity condition (2.3.3) implies that $\lambda_m \neq \lambda_n$ for $m \neq n$, which implies (2.5.8). \square

If the linear functional Λ in the preceding theorem is quasi-definite, it is well known that a recurrence relation of the form (2.5.1) exists for all $n = 0, 1, 2, \dots$. See for instance [146]. Now we will prove:

Theorem 2.6. *With the assumptions of the preceding theorem, assume that there exists a number $N \in \{1, 2, 3, \dots\}$ such that*

$$\Lambda[y_n^2] \neq 0, \quad \text{for } n = 0, 1, 2, \dots, N-1 \quad \text{and} \quad \Lambda[y_N^2] = 0. \quad (2.5.9)$$

Then there exists a three-term recurrence relation of the form

$$y_{n+1}(x) = (x - c_n)y_n(x) - d_n y_{n-1}(x), \quad n = 1, 2, 3, \dots, N+1 \quad (2.5.10)$$

with $d_N = 0$.

Proof. The monic polynomial y_{n+1} can be written as

$$y_{n+1}(x) = xy_n(x) + \sum_{k=0}^n \alpha_k^{(n)} y_k(x), \quad \alpha_k^{(n)} \in \mathbb{C}, \quad n = 0, 1, 2, \dots \quad (2.5.11)$$

Now we want to show that $\alpha_k^{(n)} = 0$ for $k = 0, 1, 2, \dots, n-2$. To do this we multiply this equation by y_ν for $\nu \in \{0, 1, 2, \dots, n-2\}$ and apply the linear functional Λ to both sides, which leads to

$$\alpha_k^{(n)} \Lambda[y_k^2] = 0, \quad k = 0, 1, 2, \dots, n-2.$$

Now we use (2.5.9) to conclude that $\alpha_k^{(n)} = 0$ for $k = 0, 1, 2, \dots, n-2$ for all $n = 2, 3, 4, \dots, N+1$, which proves (2.5.10) with $c_n = -\alpha_n^{(n)}$ and $d_n = -\alpha_{n-1}^{(n)}$.

In order to show that $d_N = 0$, we start with (2.5.10) for $n = N$, multiply by y_{N-1} and apply the linear functional Λ to find

$$\Lambda[y_{N+1}y_{N-1}] = \Lambda[xy_N y_{N-1}] - c_N \Lambda[y_N y_{N-1}] - d_N \Lambda[y_{N-1}^2].$$

Since $\Lambda[x_N y_N y_{N-1}] = \Lambda[y_N^2] = 0$ and $\Lambda[y_{N-1}^2] \neq 0$, we get

$$d_N = \frac{\Lambda[y_N^2]}{\Lambda[y_{N-1}^2]} = 0.$$

This completes the proof. \square

Now we will show that a finite system $\{y_n\}_{n=0}^N$ of polynomial solutions of the eigenvalue problem (2.2.1) which satisfies a three-term recurrence relation of the form (2.5.10) can be extended to an infinite system $\{y_n\}_{n=0}^\infty$ of polynomials which satisfies a three-term recurrence relation of the form (2.5.1). First we will prove:

Theorem 2.7. *Let the monic solutions $\{y_n\}_{n=0}^N$ of the eigenvalue problem (2.2.1) satisfy the three-term recurrence relation (2.5.10) and let the regularity condition (2.3.3) with $N \rightarrow \infty$ hold. Then we may write*

$$y_{N+k} = \tilde{y}_k y_N, \quad k = 0, 1, 2, \dots \quad (2.5.12)$$

with monic polynomials \tilde{y}_k of degree k which are solutions of the eigenvalue problem

$$\tilde{\varphi}(\mathcal{A}_{q,\omega}^2 \tilde{y}_k) + \tilde{\psi}(\mathcal{A}_{q,\omega} \tilde{y}_k) = \tilde{\lambda}_k(\mathcal{S}_{q,\omega} \tilde{y}_k), \quad k = 0, 1, 2, \dots, \quad (2.5.13)$$

where $\tilde{\lambda}_k = \lambda_{N+k} - \lambda_N$, $\tilde{\varphi}$ is a polynomial of degree at most 2 with $\tilde{\varphi} \neq 0$ and $\tilde{\psi}$ is a polynomial of degree 1 exactly.

Proof. Since $d_N = 0$, we have from (2.5.10)

$$y_{N+1}(x) = (x - c_N)y_N(x) = \tilde{y}_1(x)y_N(x) \quad \text{with} \quad \tilde{y}_1(x) = x - c_N$$

and

$$\begin{aligned} y_{N+2}(x) &= (x - c_{N+1})y_{N+1}(x) - d_{N+1}y_N(x) \\ &= ((x - c_{N+1})(x - c_N) - d_{N+1})y_N(x) = \tilde{y}_2(x)y_N(x) \end{aligned}$$

with $\tilde{y}_2(x) = (x - c_{N+1})(x - c_N) - d_{N+1}$. Together with $\tilde{y}_0(x) = 1$ this proves (2.5.12) for $k = 0, 1, 2$.

Substitution of $y_{N+k} = \tilde{y}_k y_N$ in the eigenvalue problem (2.2.1) gives

$$\varphi(\mathcal{A}_{q,\omega}^2(\tilde{y}_k y_N)) + \psi(\mathcal{A}_{q,\omega}(\tilde{y}_k y_N)) = \lambda_{N+k}(\mathcal{S}_{q,\omega}(\tilde{y}_k y_N)).$$

By using the product rule (2.1.2), we find

$$\begin{aligned} \mathcal{A}_{q,\omega}(\tilde{y}_k y_N) &= (\mathcal{A}_{q,\omega} y_N)(\mathcal{S}_{q,\omega} \tilde{y}_k) + y_N(\mathcal{A}_{q,\omega} \tilde{y}_k), \\ \mathcal{A}_{q,\omega}^2(\tilde{y}_k y_N) &= (\mathcal{A}_{q,\omega} y_N)\tilde{y}_k + (\mathcal{S}_{q,\omega} y_N)(\mathcal{A}_{q,\omega} \tilde{y}_k) \end{aligned}$$

and

$$\begin{aligned} \mathcal{A}_{q,\omega}^2(\tilde{y}_k y_N) &= (\mathcal{A}_{q,\omega}^2 y_N)(\mathcal{S}_{q,\omega} \tilde{y}_k) + (\mathcal{A}_{q,\omega} y_N)(\mathcal{A}_{q,\omega} \tilde{y}_k) \\ &\quad + (\mathcal{S}_{q,\omega}^2 y_N)(\mathcal{A}_{q,\omega} \tilde{y}_k) + (\mathcal{A}_{q,\omega} \mathcal{S}_{q,\omega} y_N)(\mathcal{A}_{q,\omega} \tilde{y}_k). \end{aligned}$$

Hence

$$\begin{aligned} \varphi(\mathcal{S}_{q,\omega}^2 y_N)(\mathcal{A}_{q,\omega} \tilde{y}_k) + (\varphi(\mathcal{A}_{q,\omega} \mathcal{S}_{q,\omega} y_N) + \varphi(\mathcal{A}_{q,\omega} y_N) + \psi y_N)(\mathcal{A}_{q,\omega} \tilde{y}_k) \\ + (\varphi(\mathcal{S}_{q,\omega}^2 y_N) + \psi(\mathcal{A}_{q,\omega} y_N))(\mathcal{S}_{q,\omega} \tilde{y}_k) = \lambda_{N+k}(\mathcal{S}_{q,\omega} y_N)(\mathcal{S}_{q,\omega} \tilde{y}_k). \end{aligned}$$

Together with

$$\varphi(\mathcal{A}_{q,\omega}^2 y_N) + \psi(\mathcal{A}_{q,\omega} y_N) = \lambda_N(\mathcal{S}_{q,\omega} y_N),$$

we obtain

$$\begin{aligned} \varphi(\mathcal{S}_{q,\omega}^2 y_N)(\mathcal{A}_{q,\omega} \tilde{y}_k) + (\varphi(\mathcal{A}_{q,\omega} \mathcal{S}_{q,\omega} y_N) + \varphi(\mathcal{A}_{q,\omega} y_N) + \psi y_N)(\mathcal{A}_{q,\omega} \tilde{y}_k) \\ = (\lambda_{N+k} - \lambda_N)(\mathcal{S}_{q,\omega} y_N)(\mathcal{S}_{q,\omega} \tilde{y}_k). \end{aligned}$$

This equation holds for $k = 0, 1, 2$. For $k = 1$ this reads

$$(\varphi(\mathcal{A}_{q,\omega} \mathcal{S}_{q,\omega} y_N) + \varphi(\mathcal{A}_{q,\omega} y_N) + \psi y_N) C_1 = (\lambda_{N+1} - \lambda_N)(\mathcal{S}_{q,\omega} y_N)(\mathcal{S}_{q,\omega} \tilde{y}_1)$$

with $C_1 = \mathcal{A}_{q,\omega} \tilde{y}_1 (\neq 0)$. Since $\lambda_{N+1} - \lambda_N \neq 0$, the polynomial on the left-hand side of this equation contains $\mathcal{S}_{q,\omega} y_N$ as a factor. Hence the polynomial $\tilde{\psi}$ can be defined by

$$\tilde{\psi} := \frac{\varphi(\mathcal{A}_{q,\omega} \mathcal{S}_{q,\omega} y_N) + \varphi(\mathcal{A}_{q,\omega} y_N) + \psi y_N}{\mathcal{S}_{q,\omega} y_N}.$$

For $k = 2$ we find

$$\varphi C_2(\mathcal{S}_{q,\omega}^2 y_N) + (\mathcal{A}_{q,\omega} \tilde{y}_2) \tilde{\psi}(\mathcal{S}_{q,\omega} y_N) = (\lambda_{N+2} - \lambda_N)(\mathcal{S}_{q,\omega} y_N)(\mathcal{S}_{q,\omega} \tilde{y}_2)$$

with $C_2 = \mathcal{A}_{q,\omega}^2 \tilde{y}_2 (\neq 0)$. Since $\lambda_{N+2} - \lambda_N \neq 0$, this implies that $\mathcal{S}_{q,\omega} y_N$ divides $\varphi(\mathcal{S}_{q,\omega}^2 y_N)$. Hence the polynomial $\tilde{\varphi}$ can be defined by

$$\tilde{\varphi} := \frac{\varphi(\mathcal{S}_{q,\omega}^2 y_N)}{\mathcal{S}_{q,\omega} y_N}.$$

So we have found that

$$\tilde{\varphi}(\mathcal{A}_{q,\omega}^2 \tilde{y}_k) + \tilde{\psi}(\mathcal{A}_{q,\omega} \tilde{y}_k) = \tilde{\lambda}_k(\mathcal{S}_{q,\omega} \tilde{y}_k), \quad \tilde{\lambda}_k = \lambda_{N+k} - \lambda_N.$$

If (2.3.2) is used, the regularity condition (2.3.3) implies that

$$\tilde{\lambda}_k = \lambda_{N+k} - \lambda_N = \frac{[k]}{q^{N+k}} (e[2N+k-1] + 2\varepsilon) (\neq 0), \quad k = 1, 2, 3, \dots$$

This proves that for (2.5.13) polynomial solutions of all degrees exist. \square

The preceding theorem allows us to prove the existence of a recurrence relation of the form (2.5.1). Let $d_n \neq 0$ for $n = 1, 2, 3, \dots, N-1$ and $d_N = 0$. Then we have the recurrence relation (2.5.10). The preceding theorem shows that this recurrence relation can be continued as long as it is possible to continue the recurrence relation

$$\tilde{y}_{k+1}(x) = (x - \tilde{c}_k)\tilde{y}_k(x) - \tilde{d}_k\tilde{y}_{k-1}(x)$$

for \tilde{y}_k . This recurrence relation either holds for all $k = 0, 1, 2, \dots$ or there is a number $K \in \{1, 2, 3, \dots\}$ such that $\tilde{d}_k \neq 0$ for $k = 0, 1, 2, \dots, K-1$ and $\tilde{d}_K = 0$. Then the preceding theorem can be applied again. This process can be continued until we arrive at:

Theorem 2.8. *If the regularity condition (2.3.3) with $N \rightarrow \infty$ holds, then there exist numbers $c_n, d_n \in \mathbb{C}$ such that the polynomial solutions $\{y_n\}_{n=0}^{\infty}$ of the eigenvalue problem (2.2.1) satisfy a three-term recurrence relation of the form (2.5.1).*

2.6 Explicit Form of the Three-Term Recurrence Relation

To determine the polynomial solutions in section 2.4, we needed a two-term recurrence relation for the coefficients. In this section we will not need such a two-term recurrence relation. Therefore we only need the representation (2.4.3) and the three-term recurrence relation (2.4.4). We will not need (2.4.5) here.

In the case of the differentiation operator D (i.e. $q = 1$ and $\omega = \nu = 0$), we simply use the form

$$y_n(x) = \sum_{k=0}^n a_{n,k} \frac{x^k}{k!}, \quad a_{n,n} \neq 0, \quad n = 0, 1, 2, \dots$$

Since we do not need a two-term recurrence relation for the coefficients, it suffices to take $c = 0$. In that case (2.4.4) reduces to the three-term recurrence relation

$$(n-k)(e(n+k-1) + 2\varepsilon)a_{n,k} - (2fk + \gamma)a_{n,k+1} - ga_{n,k+2} = 0 \quad (2.6.1)$$

for $k = n-1, n-2, n-3, \dots, 0$ with $a_{n,n+1} := 0$.

In the case of the q -derivative operator \mathcal{D}_q (i.e. $q \neq 1$ and $\omega = 0$), we need an extra observation in order to deal with the lack of translation invariance. Recall that (2.2.10) and (2.2.11) imply that

$$(\mathcal{A}_{q,\omega} p(\cdot))(x+c) = (\mathcal{A}_{q,\bar{\omega}} p(\cdot+c))(x)$$

with $\bar{\omega} = \omega + c(q-1)$ and $c \in \mathbb{R}$. For $q \neq 1$ we can put $c = \omega/(1-q)$, which yields $\bar{\omega} = 0$. Hence, since $\mathcal{D}_q := \mathcal{A}_{q,0}$, the operator equation (2.2.1) can be written in terms of the q -derivative operator as

$$\begin{aligned} & \varphi(x+c) (\mathcal{D}_q^2 y_n(\cdot+c))(x) \\ & + \psi(x+c) (\mathcal{D}_q y_n(\cdot+c))(x) = \lambda_n y_n(qx+c), \quad n = 0, 1, 2, \dots \end{aligned} \quad (2.6.2)$$

with $c = \omega/(1 - q)$. This translation does not affect the possible orthogonality. In fact the regularity condition (2.3.3) is preserved since the leading coefficients of $\varphi(x)$ and $\psi(x)$ are equal to the leading coefficients of $\varphi(x + c)$ and $\psi(x + c)$, respectively. Similarly, the recurrence relation (2.5.1) for the polynomials y_n is transformed into the recurrence relation

$$y_{n+1}(x + c) = (x - c_n + c)y_n(x + c) - d_n y_{n-1}(x + c), \quad n = 0, 1, 2, \dots \quad (2.6.3)$$

for the polynomials $y_n(x + c)$. While d_n remains unchanged, c_n is replaced by $c_n - c$. However, c_n is real if and only if $c_n - c$ is real. Together with the theorem by Favard (see the next chapter) this implies that the polynomials y_n are orthogonal in the positive-definite or quasi-definite sense if and only if this is the case for the polynomials $y_n(x + c)$. In the case of the q -derivative operator \mathcal{D}_q , we have $\omega = 0$ and therefore $c = \omega/(1 - q) = 0$ and $v = \omega + c(1 - q) = 0$. If we set $\omega = c = v = 0$ into (2.4.4), we obtain the three-term recurrence relation

$$[n - k](e[n + k - 1] + 2\varepsilon)a_{n,k} - (2f[k] + \gamma)q^{n-k}a_{n,k+1} - gq^{n-k}a_{n,k+2} = 0 \quad (2.6.4)$$

for $k = n - 1, n - 2, n - 3, \dots, 0$ with $a_{n,n+1} := 0$. Note that (2.6.4) for $q = 1$ equals (2.6.1). This implies that the case of the differentiation operator D needs no special treatment.

As a generalization of the difference operator for $q = 1$ and $\omega \neq 0$, we use another observation. From the definition (2.1.1) it follows that

$$\begin{aligned} (\mathcal{A}_{q,\omega} p(\cdot))(\rho x) &= \frac{p(q\rho x + \omega) - p(\rho x)}{q\rho x + \omega - \rho x} \\ &= \frac{p(\rho(qx + \omega/\rho)) - p(\rho x)}{\rho(qx + \omega/\rho - x)} = \frac{1}{\rho} (\mathcal{A}_{q,\omega/\rho} p(\rho \cdot))(x) \end{aligned}$$

for $\rho \neq 0$ and $p \in \mathcal{P}$. For $q = 1$ and $\rho = \omega (\neq 0)$ this yields $\mathcal{A}_{q,\omega/\rho} = \mathcal{A}_{1,1} = \Delta$ and the operator equation (2.2.1) reads

$$\frac{1}{\omega^2} \varphi(\omega x) (\Delta^2 y_n(\omega \cdot))(x) + \frac{1}{\omega} \psi(\omega x) (\Delta y_n(\omega \cdot))(x) = \lambda_n y_n(\omega x + \omega). \quad (2.6.5)$$

Similar to the case $q \neq 1$ and $\bar{\omega} = 0$ above, it can easily be seen by means of the three-term recurrence relation that the possible orthogonality is not affected by the dilatation $x \mapsto \omega x$.

In the case of the difference operator Δ (i.e. $q = 1$ and $\omega = v = 1$), we may use the form

$$y_n(x) = \sum_{k=0}^n a_{n,k} \binom{x}{k}, \quad a_{n,n} \neq 0, \quad n = 0, 1, 2, \dots \quad (2.6.6)$$

Since we do not need a two-term recurrence relation for the coefficients, we may simply take $c = 0$. In that case (2.4.4) reads the three-term recurrence relation

$$\begin{aligned}
& (n-k)(e(n+k-1)+2\varepsilon)a_{n,k} \\
& + \{e(n(n-1)-k(2k-1))+2\varepsilon(n-k)-2fk-\gamma\}a_{n,k+1} \\
& - (ek^2+2fk+g)a_{n,k+2} = 0
\end{aligned} \tag{2.6.7}$$

for $k = n-1, n-2, n-3, \dots, 0$ with $a_{n,n+1} := 0$.

In order to obtain the explicit form of the three-term recurrence relation (2.5.1), we substitute the monic polynomials

$$y_n(x) = \sum_{k=0}^n \alpha_k^{(n)} x^k, \quad \alpha_n^{(n)} = 1, \quad n = 0, 1, 2, \dots \tag{2.6.8}$$

in the recurrence relation (2.5.1) and $y_1(x) = x - c_0$. Comparison of the coefficients of x^n and x^{n-1} yields

$$c_0 = -\alpha_0^{(1)} \quad \text{and} \quad c_n = \alpha_{n-1}^{(n)} - \alpha_n^{(n+1)}, \quad n = 1, 2, 3, \dots \tag{2.6.9}$$

and

$$d_1 = -\alpha_0^{(2)} - c_1 \alpha_0^{(1)} \quad \text{and} \quad d_n = \alpha_{n-2}^{(n)} - \alpha_{n-1}^{(n+1)} - c_n \alpha_{n-1}^{(n)} \tag{2.6.10}$$

for $n = 2, 3, 4, \dots$. Hence it suffices to compute $\alpha_{n-1}^{(n)}$ for $n = 1, 2, 3, \dots$ and $\alpha_{n-2}^{(n)}$ for $n = 2, 3, 4, \dots$

In the case of the q -derivative operator \mathcal{D}_q , we find from (2.6.4) for $k = n-1$

$$(e[2n-2]+2\varepsilon)a_{n,n-1} = q(2f[n-1]+\gamma)a_{n,n}, \quad n = 1, 2, 3, \dots$$

and for $k = n-2$ and $n = 2, 3, 4, \dots$

$$[2](e[2n-3]+2\varepsilon)a_{n,n-2} - q^2(2f[n-2]+\gamma)a_{n,n-1} - q^2ga_{n,n} = 0.$$

Hence, if the regularity condition (2.3.3) holds for $n = 1, 2, 3, \dots$, we find

$$a_{n,n-1} = \frac{q(2f[n-1]+\gamma)}{e[2n-2]+2\varepsilon} a_{n,n}, \quad n = 1, 2, 3, \dots$$

and

$$a_{n,n-2} = \frac{q^3(2f[n-1]+\gamma)(2f[n-2]+\gamma) + q^2g(e[2n-2]+2\varepsilon)}{[2](e[2n-3]+2\varepsilon)(e[2n-2]+2\varepsilon)} a_{n,n}$$

for $n = 2, 3, 4, \dots$. Comparing (2.4.3) and (2.6.8), we find by using (2.4.2) that

$$a_{n,n} = [n]!, \quad \alpha_{n-1}^{(n)} = \frac{a_{n,n-1}}{[n-1]!} = [n] \frac{a_{n,n-1}}{a_{n,n}}, \quad n = 1, 2, 3, \dots$$

with

$$[0]! := 1, \quad [n]! := \prod_{i=1}^n [i], \quad n = 1, 2, 3, \dots$$

and

$$\alpha_{n-2}^{(n)} = \frac{a_{n,n-2}}{[n-2]!} = [n][n-1] \frac{a_{n,n-2}}{a_{n,n}}, \quad n = 2, 3, 4, \dots$$

Hence we have

$$\alpha_{n-1}^{(n)} = \frac{q[n](2f[n-1] + \gamma)}{e[2n-2] + 2\epsilon}, \quad n = 1, 2, 3, \dots$$

and

$$\alpha_{n-2}^{(n)} = \frac{q^2[n][n-1] \{q(2f[n-1] + \gamma)(2f[n-2] + \gamma) + g(e[2n-2] + 2\epsilon)\}}{[2](e[2n-3] + 2\epsilon)(e[2n-2] + 2\epsilon)}$$

for $n = 2, 3, 4, \dots$. By using (2.6.9), we conclude that $c_0 = -\gamma q/2\epsilon$ and

$$\begin{aligned} c_n &= \frac{q[n](2f[n-1] + \gamma)}{e[2n-2] + 2\epsilon} - \frac{q[n+1](2f[n] + \gamma)}{e[2n] + 2\epsilon} \\ &= \frac{q\{[n](2f[n-1] + \gamma)(e[2n] + 2\epsilon) - [n+1](2f[n] + \gamma)(e[2n-2] + 2\epsilon)\}}{(e[2n-2] + 2\epsilon)(e[2n] + 2\epsilon)} \\ &= -\frac{q^n \{(e[n-1] + 2\epsilon)(2f[n](1+q) + \gamma q) - e\gamma[n+1]q^{n-1}\}}{(e[2n-2] + 2\epsilon)(e[2n] + 2\epsilon)} \end{aligned} \quad (2.6.11)$$

for $n = 1, 2, 3, \dots$. By using (2.6.10), we obtain

$$\begin{aligned} d_1 &= -\frac{q^2[2] \{\gamma q(2f + \gamma) + g(e[2] + 2\epsilon)\}}{[2](e + 2\epsilon)(e[2] + 2\epsilon)} + \frac{q \{2\epsilon(2f(1+q) + \gamma q) - e\gamma[2]\}}{2\epsilon(e[2] + 2\epsilon)} \cdot \frac{\gamma q}{2\epsilon} \\ &= \frac{q^2(e[2] + 2\epsilon)(4\epsilon(f\gamma - g\epsilon) - e\gamma^2)}{(2\epsilon)^2(e + 2\epsilon)(e[2] + 2\epsilon)} = \frac{q^2(4\epsilon(f\gamma - g\epsilon) - e\gamma^2)}{4\epsilon^2(e + 2\epsilon)} \end{aligned}$$

and for $n = 2, 3, 4, \dots$

$$\begin{aligned} d_n &= \frac{q^2[n][n-1] \{q(2f[n-1] + \gamma)(2f[n-2] + \gamma) + g(e[2n-2] + 2\epsilon)\}}{[2](e[2n-3] + 2\epsilon)(e[2n-2] + 2\epsilon)} \\ &\quad - \frac{q^2[n][n+1] \{q(2f[n] + \gamma)(2f[n-1] + \gamma) + g(e[2n] + 2\epsilon)\}}{[2](e[2n-1] + 2\epsilon)(e[2n] + 2\epsilon)} \\ &\quad + \frac{q[n](2f[n-1] + \gamma)}{e[2n-2] + 2\epsilon} \\ &\quad \times \frac{q^n \{(e[n-1] + 2\epsilon)(2f[n](1+q) + \gamma q) - e\gamma[n+1]q^{n-1}\}}{(e[2n-2] + 2\epsilon)(e[2n] + 2\epsilon)} \\ &= \frac{q^{n+1}[n](e[n-2] + 2\epsilon)}{(e[2n-3] + 2\epsilon)(e[2n-2] + 2\epsilon)^2(e[2n-1] + 2\epsilon)} \\ &\quad \times \left\{ q^{n-1}(2f[n-1] + \gamma)(2f\{e[n-1] + 2\epsilon\} - q^{n-1}e\gamma) \right. \\ &\quad \left. - g(e[2n-2] + 2\epsilon)^2 \right\}. \end{aligned} \quad (2.6.12)$$

Note that the latter formula also holds for $n = 1$.

In the case of the differentiation operator D , we have $q = 1$. In that case we get

$$c_n = -\frac{2fn(e(n-1)+2\varepsilon) - \gamma(e-\varepsilon)}{2(e(n-1)+\varepsilon)(en+\varepsilon)}, \quad n = 0, 1, 2, \dots \quad (2.6.13)$$

and

$$\begin{aligned} d_n = & \frac{n(e(n-2)+2\varepsilon)}{4(e(2n-3)+2\varepsilon)(e(n-1)+\varepsilon)^2(e(2n-1)+2\varepsilon)} \\ & \times \left\{ \{2f(n-1)+\gamma\} \{2f(e(n-1)+2\varepsilon) - e\gamma\} \right. \\ & \left. - 4g(e(n-1)+\varepsilon)^2 \right\}, \quad n = 1, 2, 3, \dots \quad (2.6.14) \end{aligned}$$

In the case of the difference operator Δ , we find from (2.6.7) with $k = n - 1$

$$(e(2n-2)+2\varepsilon)a_{n,n-1} = (e(n^2-4n+3) - 2\varepsilon + 2f(n-1) + \gamma)a_{n,n}$$

for $n = 1, 2, 3, \dots$, and with $k = n - 2$

$$\begin{aligned} 2(e(2n-3)+2\varepsilon)a_{n,n-2} - (e(n^2-8n+10) - 4\varepsilon + 2f(n-2) + \gamma)a_{n,n-1} \\ - (e(n-2)^2 + 2f(n-2) + g)a_{n,n} = 0 \end{aligned}$$

for $n = 2, 3, 4, \dots$. Hence, if the regularity condition (2.3.3) holds for $n = 1, 2, 3, \dots$, we find

$$a_{n,n-1} = \frac{e(n-1)(n-3) - 2\varepsilon + 2f(n-1) + \gamma}{2(e(n-1)+\varepsilon)} a_{n,n}, \quad n = 1, 2, 3, \dots$$

and for $n = 2, 3, 4, \dots$

$$\begin{aligned} a_{n,n-2} = & \left\{ \frac{e(n-2)^2 + 2f(n-2) + g}{2(e(2n-3)+2\varepsilon)} \right. \\ & + (e(n^2-8n+10) - 4\varepsilon + 2f(n-2) + \gamma) \\ & \left. \times \frac{(e(n-1)(n-3) - 2\varepsilon + 2f(n-1) + \gamma)}{4(e(n-1)+\varepsilon)(e(2n-3)+2\varepsilon)} \right\} a_{n,n}. \end{aligned}$$

Comparing (2.6.8) and (2.6.6), we find that

$$a_{n,n} = n!, \quad \alpha_{n-1}^{(n)} = n \left\{ \frac{a_{n,n-1}}{a_{n,n}} - \frac{n-1}{2} \right\} = \frac{n(2(n-1)(f-e) - (n+1)\varepsilon + \gamma)}{2(e(n-1)+\varepsilon)}$$

for $n = 1, 2, 3, \dots$ and

$$\begin{aligned}
\alpha_{n-2}^{(n)} &= \binom{n}{3} \frac{3n-1}{4} - \frac{n-2}{2} \frac{a_{n,n-1}}{(n-2)!} + \frac{a_{n,n-2}}{(n-2)!} \\
&= n(n-1) \left\{ \frac{(n-2)(3n-1)}{24} - \frac{n-2}{2} \frac{a_{n,n-1}}{a_{n,n}} + \frac{a_{n,n-2}}{a_{n,n}} \right\} \\
&= \frac{n(n-1)}{4(e(n-1) + \varepsilon)(e(2n-3) + 2\varepsilon)} \\
&\quad \times \left\{ \frac{1}{6}(n+1)(3n+2)(e(n-1) + \varepsilon)(e(2n-3) + 2\varepsilon) \right. \\
&\quad \quad - en(n-1)^2(e(2n-3) + 2\varepsilon) + e^2(n-1)^2(n-2)^2 \\
&\quad \quad - 2(2f(n-1) + \gamma)(e(2n-3) + n\varepsilon) - \gamma(e(n-1) + 2\varepsilon) \\
&\quad \quad \left. + (2f(n-1) + \gamma)(2f(n-2) + \gamma) + 2g(e(n-1) + \varepsilon) \right\}
\end{aligned}$$

for $n = 2, 3, 4, \dots$ By using (2.6.9), we conclude that $c_0 = 1 - \gamma/2\varepsilon$ and

$$\begin{aligned}
c_n &= \frac{n(2(n-1)(f-e) - (n+1)\varepsilon + \gamma)}{2(e(n-1) + \varepsilon)} \\
&\quad - \frac{(n+1)(2n(f-e) - (n+2)\varepsilon + \gamma)}{2(en + \varepsilon)} \\
&= \frac{n(e(n-1) + 2\varepsilon)(2(e-f) + \varepsilon) + (e-\varepsilon)(\gamma - 2\varepsilon)}{2(e(n-1) + \varepsilon)(en + \varepsilon)} \tag{2.6.15}
\end{aligned}$$

for $n = 1, 2, 3, \dots$ By using (2.6.10), we find

$$\begin{aligned}
d_1 &= -\frac{1}{2(e+\varepsilon)(e+2\varepsilon)} \left\{ 4(e+\varepsilon)(e+2\varepsilon) - 2e(e+2\varepsilon) \right. \\
&\quad \left. - 2(2f+\gamma)(e+2\varepsilon) - \gamma(e+2\varepsilon) + (2f+\gamma)\gamma + 2g(e+\varepsilon) \right\} \\
&\quad + \frac{2\varepsilon(2(e-f) + \varepsilon) + (e-\varepsilon)(\gamma - 2\varepsilon)}{2\varepsilon(e+\varepsilon)} \cdot \frac{2\varepsilon - \gamma}{2\varepsilon} \\
&= \frac{4\varepsilon(f\gamma - g\varepsilon) - e\gamma^2}{4\varepsilon^2(e+2\varepsilon)}.
\end{aligned}$$

and for $n = 2, 3, 4, \dots$

$$\begin{aligned}
d_n &= \frac{n(n-1)}{4(e(n-1)+\varepsilon)(e(2n-3)+2\varepsilon)} \\
&\quad \times \left\{ \frac{1}{6}(n+1)(3n+2)(e(n-1)+\varepsilon)(e(2n-3)+2\varepsilon) \right. \\
&\quad \quad - en(n-1)^2(e(2n-3)+2\varepsilon) + e^2(n-1)^2(n-2)^2 \\
&\quad \quad - 2(2f(n-1)+\gamma)(e(2n-3)+n\varepsilon) - \gamma(e(n-1)+2\varepsilon) \\
&\quad \quad \left. + (2f(n-1)+\gamma)(2f(n-2)+\gamma) + 2g(e(n-1)+\varepsilon) \right\} \\
&\quad - \frac{n(n+1)}{4(en+\varepsilon)(e(2n-1)+2\varepsilon)} \\
&\quad \times \left\{ \frac{1}{6}(n+2)(3n+5)(en+\varepsilon)(e(2n-1)+2\varepsilon) \right. \\
&\quad \quad - en^2(n+1)(e(2n-1)+2\varepsilon) + e^2n^2(n-1)^2 \\
&\quad \quad - 2(2fn+\gamma)(e(2n-1)+(n+1)\varepsilon) - \gamma(en+2\varepsilon) \\
&\quad \quad \left. + (2fn+\gamma)(2f(n-1)+\gamma) + 2g(en+\varepsilon) \right\} \\
&\quad - \frac{n(2(e-f)+\varepsilon)(e(n-1)+2\varepsilon) + (e-\varepsilon)(\gamma-2\varepsilon)}{2(e(n-1)+\varepsilon)(en+\varepsilon)} \\
&\quad \times \frac{n(2(n-1)(f-e) - (n+1)\varepsilon + \gamma)}{2(e(n-1)+\varepsilon)} \\
&= - \frac{n(e(n-2)+2\varepsilon)}{4(e(2n-3)+2\varepsilon)(e(n-1)+\varepsilon)^2(e(2n-1)+2\varepsilon)} \\
&\quad \times \left\{ e(n-1)^2(e(n-1)+2\varepsilon)^2 \right. \\
&\quad \quad + 2(n-1)(e(n-1)+2\varepsilon)(2eg+2f(\varepsilon-f)-e\gamma) \\
&\quad \quad \left. + 4\varepsilon(g\varepsilon-f\gamma)+e\gamma^2 \right\}. \tag{2.6.16}
\end{aligned}$$

Note that the latter formula also holds for $n = 1$.



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