

1

Review of Probability

In this chapter we shall recall some basic notions and facts from probability theory. Here is a short list of what needs to be reviewed:

- 1) Probability spaces, σ -fields and measures;
- 2) Random variables and their distributions;
- 3) Expectation and variance;
- 4) The σ -field generated by a random variable;
- 5) Independence, conditional probability.

The reader is advised to consult a book on probability for more information.

1.1 Events and Probability

Definition 1.1

Let Ω be a non-empty set. A σ -field \mathcal{F} on Ω is a family of subsets of Ω such that

- 1) the empty set \emptyset belongs to \mathcal{F} ;
- 2) if A belongs to \mathcal{F} , then so does the complement $\Omega \setminus A$;

- 3) if A_1, A_2, \dots is a sequence of sets in \mathcal{F} , then their union $A_1 \cup A_2 \cup \dots$ also belongs to \mathcal{F} .

Example 1.1

Throughout this course \mathbb{R} will denote the set of real numbers. The family of *Borel sets* $\mathcal{F} = \mathcal{B}(\mathbb{R})$ is a σ -field on \mathbb{R} . We recall that $\mathcal{B}(\mathbb{R})$ is the smallest σ -field containing all intervals in \mathbb{R} .

Definition 1.2

Let \mathcal{F} be a σ -field on Ω . A *probability measure* P is a function

$$P : \mathcal{F} \rightarrow [0, 1]$$

such that

- 1) $P(\Omega) = 1$;
- 2) if A_1, A_2, \dots are pairwise disjoint sets (that is, $A_i \cap A_j = \emptyset$ for $i \neq j$) belonging to \mathcal{F} , then

$$P(A_1 \cup A_2 \cup \dots) = P(A_1) + P(A_2) + \dots$$

The triple (Ω, \mathcal{F}, P) is called a *probability space*. The sets belonging to \mathcal{F} are called *events*. An event A is said to occur *almost surely* (a.s.) whenever $P(A) = 1$.

Example 1.2

We take the unit interval $\Omega = [0, 1]$ with the σ -field $\mathcal{F} = \mathcal{B}([0, 1])$ of Borel sets $B \subset [0, 1]$, and *Lebesgue measure* $P = \text{Leb}$ on $[0, 1]$. Then (Ω, \mathcal{F}, P) is a probability space. Recall that Leb is the unique measure defined on Borel sets such that

$$\text{Leb}[a, b] = b - a$$

for any interval $[a, b]$. (In fact Leb can be extended to a larger σ -field, but we shall need Borel sets only.)

Exercise 1.1

Show that if A_1, A_2, \dots is an *expanding* sequence of events, that is,

$$A_1 \subset A_2 \subset \dots,$$

then

$$P(A_1 \cup A_2 \cup \cdots) = \lim_{n \rightarrow \infty} P(A_n).$$

Similarly, if A_1, A_2, \dots is a *contracting* sequence of events, that is,

$$A_1 \supset A_2 \supset \cdots,$$

then

$$P(A_1 \cap A_2 \cap \cdots) = \lim_{n \rightarrow \infty} P(A_n).$$

Hint Write $A_1 \cup A_2 \cup \cdots$ as the union of a sequence of disjoint events: start with A_1 , then add a disjoint set to obtain $A_1 \cup A_2$, then add a disjoint set again to obtain $A_1 \cup A_2 \cup A_3$, and so on. Now that you have a sequence of disjoint sets, you can use the definition of a probability measure. To deal with the product $A_1 \cap A_2 \cap \cdots$ write it as a union of some events with the aid of De Morgan's law.

Lemma 1.1 (Borel–Cantelli)

Let A_1, A_2, \dots be a sequence of events such that $P(A_1) + P(A_2) + \cdots < \infty$ and let $B_n = A_n \cup A_{n+1} \cup \cdots$. Then $P(B_1 \cap B_2 \cap \cdots) = 0$.

Exercise 1.2

Prove the Borel–Cantelli lemma above.

Hint B_1, B_2, \dots is a contracting sequence of events.

1.2 Random Variables

Definition 1.3

If \mathcal{F} is a σ -field on Ω , then a function $\xi : \Omega \rightarrow \mathbb{R}$ is said to be \mathcal{F} -*measurable* if

$$\{\xi \in B\} \in \mathcal{F}$$

for every Borel set $B \in \mathcal{B}(\mathbb{R})$. If (Ω, \mathcal{F}, P) is a probability space, then such a function ξ is called a *random variable*.

Remark 1.1

A short-hand notation for events such as $\{\xi \in B\}$ will be used to avoid clutter. To be precise, we should write

$$\{\omega \in \Omega : \xi(\omega) \in B\}$$

in place of $\{\xi \in B\}$. Incidentally, $\{\xi \in B\}$ is just a convenient way of writing the inverse image $\xi^{-1}(B)$ of a set.

Definition 1.4

The σ -field $\sigma(\xi)$ generated by a random variable $\xi : \Omega \rightarrow \mathbb{R}$ consists of all sets of the form $\{\xi \in B\}$, where B is a Borel set in \mathbb{R} .

Definition 1.5

The σ -field $\sigma\{\xi_i : i \in I\}$ generated by a family $\{\xi_i : i \in I\}$ of random variables is defined to be the smallest σ -field containing all events of the form $\{\xi_i \in B\}$, where B is a Borel set in \mathbb{R} and $i \in I$.

Exercise 1.3

We call $f : \mathbb{R} \rightarrow \mathbb{R}$ a *Borel function* if the inverse image $f^{-1}(B)$ of any Borel set B in \mathbb{R} is a Borel set. Show that if f is a Borel function and ξ is a random variable, then the composition $f(\xi)$ is $\sigma(\xi)$ -measurable.

Hint Consider the event $\{f(\xi) \in B\}$, where B is an arbitrary Borel set. Can this event be written as $\{\xi \in A\}$ for some Borel set A ?

Lemma 1.2 (Doob–Dynkin)

Let ξ be a random variable. Then each $\sigma(\xi)$ -measurable random variable η can be written as

$$\eta = f(\xi)$$

for some Borel function $f : \mathbb{R} \rightarrow \mathbb{R}$.

The proof of this highly non-trivial result will be omitted.

Definition 1.6

Every random variable $\xi : \Omega \rightarrow \mathbb{R}$ gives rise to a probability measure

$$P_\xi(B) = P\{\xi \in B\}$$

on \mathbb{R} defined on the σ -field of Borel sets $B \in \mathcal{B}(\mathbb{R})$. We call P_ξ the *distribution* of ξ . The function $F_\xi : \mathbb{R} \rightarrow [0, 1]$ defined by

$$F_\xi(x) = P\{\xi \leq x\}$$

is called the *distribution function* of ξ .

Exercise 1.4

Show that the distribution function F_ξ is non-decreasing, right-continuous, and

$$\lim_{x \rightarrow -\infty} F_\xi(x) = 0, \quad \lim_{x \rightarrow +\infty} F_\xi(x) = 1.$$

Hint For example, to verify right-continuity show that $F_\xi(x_n) \rightarrow F_\xi(x)$ for any decreasing sequence x_n such that $x_n \rightarrow x$. You may find the results of Exercise 1.1 useful.

Definition 1.7

If there is a Borel function $f_\xi : \mathbb{R} \rightarrow \mathbb{R}$ such that for any Borel set $B \subset \mathbb{R}$

$$P\{\xi \in B\} = \int_B f_\xi(x) dx,$$

then ξ is said to be a random variable with *absolutely continuous distribution* and f_ξ is called the *density* of ξ . If there is a (finite or infinite) sequence of pairwise distinct real numbers x_1, x_2, \dots such that for any Borel set $B \subset \mathbb{R}$

$$P\{\xi \in B\} = \sum_{x_i \in B} P\{\xi = x_i\},$$

then ξ is said to have *discrete distribution* with values x_1, x_2, \dots and *mass* $P\{\xi = x_i\}$ at x_i .

Exercise 1.5

Suppose that ξ has continuous distribution with density f_ξ . Show that

$$\frac{d}{dx} F_\xi(x) = f_\xi(x)$$

if f_ξ is continuous at x .

Hint Express $F_\xi(x)$ as an integral of f_ξ .

Exercise 1.6

Show that if ξ has discrete distribution with values x_1, x_2, \dots , then F_ξ is constant on each interval $(s, t]$ not containing any of the x_i 's and has jumps of size $P\{\xi = x_i\}$ at each x_i .

Hint The increment $F_\xi(t) - F_\xi(s)$ is equal to the total mass of the x_i 's that belong to the interval $[s, t)$.

Definition 1.8

The *joint distribution* of several random variables ξ_1, \dots, ξ_n is a probability measure P_{ξ_1, \dots, ξ_n} on \mathbb{R}^n such that

$$P_{\xi_1, \dots, \xi_n}(B) = P\{(\xi_1, \dots, \xi_n) \in B\}$$

for any Borel set B in \mathbb{R}^n . If there is a Borel function $f_{\xi_1, \dots, \xi_n} : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$P\{(\xi_1, \dots, \xi_n) \in B\} = \int_B f_{\xi_1, \dots, \xi_n}(x_1, \dots, x_n) dx_1 \cdots dx_n$$

for any Borel set B in \mathbb{R}^n , then f_{ξ_1, \dots, ξ_n} is called the *joint density* of ξ_1, \dots, ξ_n .

Definition 1.9

A random variable $\xi : \Omega \rightarrow \mathbb{R}$ is said to be *integrable* if

$$\int_{\Omega} |\xi| dP < \infty.$$

Then

$$E(\xi) = \int_{\Omega} \xi dP$$

exists and is called the *expectation* of ξ . The family of integrable random variables $\xi : \Omega \rightarrow \mathbb{R}$ will be denoted by L^1 or, in case of possible ambiguity, by $L^1(\Omega, \mathcal{F}, P)$.

Example 1.3

The *indicator function* 1_A of a set A is equal to 1 on A and 0 on the complement $\Omega \setminus A$ of A . For any event A

$$E(1_A) = \int_{\Omega} 1_A dP = P(A).$$

We say that $\eta : \Omega \rightarrow \mathbb{R}$ is a *step function* if

$$\eta = \sum_{i=1}^n \eta_i 1_{A_i},$$

where η_1, \dots, η_n are real numbers and A_1, \dots, A_n are pairwise disjoint events.

Then

$$E(\eta) = \int_{\Omega} \eta dP = \sum_{i=1}^n \eta_i \int_{\Omega} 1_{A_i} dP = \sum_{i=1}^n \eta_i P(A_i).$$

Exercise 1.7

Show that for any Borel function $h : \mathbb{R} \rightarrow \mathbb{R}$ such that $h(\xi)$ is integrable

$$E(h(\xi)) = \int_{\mathbb{R}} h(x) dP_{\xi}(x).$$

Hint First verify the equality for step functions $h : \mathbb{R} \rightarrow \mathbb{R}$, then for non-negative ones by approximating them by step functions, and finally for arbitrary Borel functions by splitting them into positive and negative parts.

In particular, Exercise 1.7 implies that if ξ has an absolutely continuous distribution with density f_{ξ} , then

$$E(h(\xi)) = \int_{-\infty}^{+\infty} h(x) f_{\xi}(x) dx.$$

If ξ has a discrete distribution with (finitely or infinitely many) pairwise distinct values x_1, x_2, \dots , then

$$E(h(\xi)) = \sum_i h(x_i) P\{\xi = x_i\}.$$

Definition 1.10

A random variable $\xi : \Omega \rightarrow \mathbb{R}$ is called *square integrable* if

$$\int_{\Omega} |\xi|^2 dP < \infty.$$

Then the *variance* of ξ can be defined by

$$\text{var}(\xi) = \int_{\Omega} (\xi - E(\xi))^2 dP.$$

The family of square integrable random variables $\xi : \Omega \rightarrow \mathbb{R}$ will be denoted by $L^2(\Omega, \mathcal{F}, P)$ or, if no ambiguity is possible, simply by L^2 .

Remark 1.2

The result in Exercise 1.8 below shows that we may write $E(\xi)$ in the definition of variance.

Exercise 1.8

Show that if ξ is a square integrable random variable, then it is integrable.

Hint Use the Schwarz inequality

$$[E(\xi\eta)]^2 \leq E(\xi^2) E(\eta^2) \quad (1.1)$$

with an appropriately chosen η .

Exercise 1.9

Show that if $\eta : \Omega \rightarrow [0, \infty)$ is a non-negative square integrable random variable, then

$$E(\eta^2) = 2 \int_0^\infty tP(\eta > t) dt.$$

Hint Express $E(\eta^2)$ in terms of the distribution function $F_\eta(t)$ of η and then integrate by parts.

1.3 Conditional Probability and Independence

Definition 1.11

For any events $A, B \in \mathcal{F}$ such that $P(B) \neq 0$ the *conditional probability* of A given B is defined by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Exercise 1.10

Prove the *total probability formula*

$$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \dots$$

for any event $A \in \mathcal{F}$ and any sequence of pairwise disjoint events $B_1, B_2, \dots \in \mathcal{F}$ such that $B_1 \cup B_2 \cup \dots = \Omega$ and $P(B_n) \neq 0$ for any n .

Hint $A = (A \cap B_1) \cup (A \cap B_2) \cup \dots$

Definition 1.12

Two events $A, B \in \mathcal{F}$ are called *independent* if

$$P(A \cap B) = P(A)P(B).$$

In general, we say that n events $A_1, \dots, A_n \in \mathcal{F}$ are *independent* if

$$P(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}) = P(A_{i_1})P(A_{i_2}) \dots P(A_{i_k})$$

for any indices $1 \leq i_1 < i_2 < \dots < i_k \leq n$.

Exercise 1.11

Let $P(B) \neq 0$. Show that A and B are independent events if and only if $P(A|B) = P(A)$.

Hint If $P(B) \neq 0$, then you can divide by it.

Definition 1.13

Two random variables ξ and η are called *independent* if for any Borel sets $A, B \in \mathcal{B}(\mathbb{R})$ the two events

$$\{\xi \in A\} \quad \text{and} \quad \{\eta \in B\}$$

are independent. We say that n random variables ξ_1, \dots, ξ_n are *independent* if for any Borel sets $B_1, \dots, B_n \in \mathcal{B}(\mathbb{R})$ the events

$$\{\xi_1 \in B_1\}, \dots, \{\xi_n \in B_n\}$$

are independent. In general, a (finite or infinite) family of random variables is said to be *independent* if any finite number of random variables from this family are independent.

Proposition 1.1

If two integrable random variables $\xi, \eta : \Omega \rightarrow \mathbb{R}$ are independent, then they are *uncorrelated*, i.e.

$$E(\xi\eta) = E(\xi)E(\eta),$$

provided that the product $\xi\eta$ is also integrable. If $\xi_1, \dots, \xi_n : \Omega \rightarrow \mathbb{R}$ are independent integrable random variables, then

$$E(\xi_1\xi_2 \cdots \xi_n) = E(\xi_1)E(\xi_2) \cdots E(\xi_n),$$

provided that the product $\xi_1\xi_2 \cdots \xi_n$ is also integrable.

Definition 1.14

Two σ -fields \mathcal{G} and \mathcal{H} contained in \mathcal{F} are called *independent* if any two events

$$A \in \mathcal{G} \quad \text{and} \quad B \in \mathcal{H}$$

are independent. Similarly, any finite number of σ -fields $\mathcal{G}_1, \dots, \mathcal{G}_n$ contained in \mathcal{F} are *independent* if any n events

$$A_1 \in \mathcal{G}_1, \dots, A_n \in \mathcal{G}_n$$

are independent. In general, a (finite or infinite) family of σ -fields is said to be *independent* if any finite number of them are independent.

Exercise 1.12

Show that two random variables ξ and η are independent if and only if the σ -fields $\sigma(\xi)$ and $\sigma(\eta)$ generated by them are independent.

Hint The events in $\sigma(\xi)$ and $\sigma(\eta)$ are of the form $\{\xi \in A\}$, and $\{\eta \in B\}$, where A and B are Borel sets.

Sometimes it is convenient to talk of independence for a combination of random variables and σ -fields.

Definition 1.15

We say that a random variable ξ is *independent* of a σ -field \mathcal{G} if the σ -fields

$$\sigma(\xi) \quad \text{and} \quad \mathcal{G}$$

are independent. This can be extended to any (finite or infinite) family consisting of random variables or σ -fields or a combination of them both. Namely, such a family is called *independent* if for any finite number of random variables ξ_1, \dots, ξ_m and σ -fields $\mathcal{G}_1, \dots, \mathcal{G}_n$ from this family the σ -fields

$$\sigma(\xi_1), \dots, \sigma(\xi_m), \mathcal{G}_1, \dots, \mathcal{G}_n$$

are independent.

1.4 Solutions

Solution 1.1

If $A_1 \subset A_2 \subset \dots$, then

$$A_1 \cup A_2 \cup \dots = A_1 \cup (A_2 \setminus A_1) \cup (A_3 \setminus A_2) \cup \dots,$$

where the sets $A_1, A_2 \setminus A_1, A_3 \setminus A_2, \dots$ are pairwise disjoint. Therefore, by the definition of probability measure

$$\begin{aligned} P(A_1 \cup A_2 \cup \dots) &= P(A_1 \cup (A_2 \setminus A_1) \cup (A_3 \setminus A_2) \cup \dots) \\ &= P(A_1) + P(A_2 \setminus A_1) + P(A_3 \setminus A_2) + \dots \\ &= \lim_{n \rightarrow \infty} P(A_n). \end{aligned}$$

The last equality holds because the partial sums in the series above are

$$\begin{aligned} P(A_1) + P(A_2 \setminus A_1) + \dots + P(A_n \setminus A_{n-1}) &= P(A_1 \cup \dots \cup A_n) \\ &= P(A_n). \end{aligned}$$

If $A_1 \supset A_2 \supset \dots$, then the equality

$$P(A_1 \cap A_2 \cap \dots) = \lim_{n \rightarrow \infty} P(A_n)$$

follows by taking the complements of A_n and applying De Morgan's law

$$\Omega \setminus (A_1 \cap A_2 \cap \dots) = (\Omega \setminus A_1) \cup (\Omega \setminus A_2) \cup \dots$$

Solution 1.2

Since B_n is a contracting sequence of events, the results of Exercise 1.1 imply that

$$\begin{aligned} P(B_1 \cap B_2 \cap \dots) &= \lim_{n \rightarrow \infty} P(B_n) \\ &= \lim_{n \rightarrow \infty} P(A_n \cup A_{n+1} \cup \dots) \\ &\leq \lim_{n \rightarrow \infty} (P(A_n) + P(A_{n+1}) + \dots) \\ &= 0. \end{aligned}$$

The last equality holds because the series $\sum_{n=1}^{\infty} P(A_n)$ is convergent. The inequality above holds by the subadditivity property

$$P(A_n \cup A_{n+1} \cup \dots) \leq P(A_n) + P(A_{n+1}) + \dots$$

It follows that $P(B_1 \cap B_2 \cap \dots) = 0$.

Solution 1.3

If B is a Borel set in \mathbb{R} and $f : \mathbb{R} \rightarrow \mathbb{R}$ is a Borel function, then $f^{-1}(B)$ is also a Borel set. Therefore

$$\{f(\xi) \in B\} = \{\xi \in f^{-1}(B)\}$$

belongs to the σ -field $\sigma(\xi)$ generated by ξ . It follows that the composition $f(\xi)$ is $\sigma(\xi)$ -measurable.

Solution 1.4

If $x \leq y$, then $\{\xi \leq x\} \subset \{\xi \leq y\}$, so

$$F_\xi(x) = P\{\xi \leq x\} \leq P\{\xi \leq y\} = F_\xi(y).$$

This means that F_ξ is non-decreasing.

Next, we take any sequence $x_1 \geq x_2 \geq \dots$ and put

$$\lim_{n \rightarrow \infty} x_n = x.$$

Then the events

$$\{\xi \leq x_1\} \supset \{\xi \leq x_2\} \supset \dots$$

form a contracting sequence with intersection

$$\{\xi \leq x\} = \{\xi \leq x_1\} \cap \{\xi \leq x_2\} \cap \dots.$$

It follows by Exercise 1.1 that

$$F_\xi(x) = P\{\xi \leq x\} = \lim_{n \rightarrow \infty} P\{\xi \leq x_n\} = \lim_{n \rightarrow \infty} F_\xi(x_n).$$

This proves that F_ξ is right-continuous.

Since the events

$$\{\xi \leq -1\} \supset \{\xi \leq -2\} \supset \dots$$

form a contracting sequence with intersection \emptyset and

$$\{\xi \leq 1\} \subset \{\xi \leq 2\} \subset \dots$$

form an expanding sequence with union Ω , it follows by Exercise 1.1 that

$$\lim_{x \rightarrow -\infty} F_\xi(x) = \lim_{n \rightarrow \infty} F_\xi(-n) = \lim_{n \rightarrow \infty} P\{\xi \leq -n\} = P(\emptyset) = 0,$$

$$\lim_{x \rightarrow \infty} F_\xi(x) = \lim_{n \rightarrow \infty} F_\xi(n) = \lim_{n \rightarrow \infty} P\{\xi \leq n\} = P(\Omega) = 1,$$

since F_ξ is non-decreasing.

Solution 1.5

If ξ has a density f_ξ , then the distribution function F_ξ can be written as

$$F_\xi(x) = P\{\xi \leq x\} = \int_{-\infty}^x f_\xi(y) dy.$$

Therefore, if f_ξ is continuous at x , then F_ξ is differentiable at x and

$$\frac{d}{dx} F_\xi(x) = f_\xi(x).$$

Solution 1.6

If $s < t$ are real numbers such that $x_i \notin (s, t]$ for any i , then

$$F_\xi(t) - F_\xi(s) = P\{\xi \leq t\} - P\{\xi \leq s\} = P\{\xi \in (s, t]\} = 0,$$

i.e. $F_\xi(s) = F_\xi(t)$. Because F_ξ is non-decreasing, this means that F_ξ is constant on $(s, t]$. To show that F_ξ has a jump of size $P\{\xi = x_i\}$ at each x_i , we compute

$$\begin{aligned} \lim_{t \searrow x_i} F_\xi(t) - \lim_{s \nearrow x_i} F_\xi(s) &= \lim_{t \searrow x_i} P\{\xi \leq t\} - \lim_{s \nearrow x_i} P\{\xi \leq s\} \\ &= P\{\xi \leq x_i\} - P\{\xi < x_i\} = P\{\xi = x_i\}. \end{aligned}$$

Solution 1.7

If h is a step function,

$$h = \sum_{i=1}^n h_i 1_{A_i},$$

where h_1, \dots, h_n are real numbers and A_1, \dots, A_n are pairwise disjoint Borel sets covering \mathbb{R} , then

$$\begin{aligned} E(h(\xi)) &= \sum_{i=1}^n h_i E(1_{A_i}(\xi)) = \sum_{i=1}^n h_i P\{\xi \in A_i\} \\ &= \sum_{i=1}^n h_i P_\xi(A_i) = \sum_{i=1}^n \int_{A_i} h(x) dP_\xi(x) = \int_{\mathbb{R}} h(x) dP_\xi(x). \end{aligned}$$

Next, any non-negative Borel function h can be approximated by a non-decreasing sequence of step functions. For such an h the result follows by the monotone convergence of integrals. Finally, this implies the desired equality for all Borel functions h , since each can be split into its positive and negative parts, $h = h^+ - h^-$, where $h^+, h^- \geq 0$.

Solution 1.8

By the Schwarz inequality (1.1) with $\eta = 1$, if ξ is square integrable, then

$$[E(|\xi|)]^2 = [E(1|\xi|)]^2 \leq E(1^2) E(\xi^2) = E(\xi^2) < \infty,$$

i.e. ξ is integrable.

Solution 1.9

Let $F(t) = P\{\eta \leq t\}$ be the distribution function of η . Then

$$E(\eta^2) = \int_0^\infty t^2 dF(t).$$

Since $P(\eta > t) = 1 - F(t)$, we need to show that

$$\int_0^\infty t^2 dF(t) = 2 \int_0^\infty t(1 - F(t)) dt \quad (1.2)$$

First, let us establish a version of (1.2) with ∞ replaced by a finite number a . Integrating by parts, we obtain

$$\begin{aligned} \int_0^a t^2 dF(t) &= \int_0^a t^2 d(F(t) - 1) \\ &= t^2(F(t) - 1)|_0^a - 2 \int_0^a t(F(t) - 1) dt \\ &= -a^2(1 - F(a)) + 2 \int_0^a t(1 - F(t)) dt. \end{aligned} \quad (1.3)$$

We see that (1.2) follows from (1.3), provided that

$$a^2(1 - F(a)) \rightarrow 0, \quad \text{as } a \rightarrow \infty. \quad (1.4)$$

But

$$0 \leq a^2(1 - F(a)) = a^2 P(\eta > a) \leq (n+1)^2 P(\eta > n) \leq 4n^2 P(\eta \geq n),$$

where n is the integer part of a , and

$$E(\eta^2) = \sum_{k=0}^{\infty} \int_{\{k \leq \eta \leq k+1\}} \eta^2 dP < \infty.$$

Hence,

$$n^2 P(\eta \geq n) \leq \int_{\{\eta \geq n\}} \eta^2 dP = \sum_{k=n}^{\infty} \int_{\{k \leq \eta < k+1\}} \eta^2 dP \rightarrow 0 \quad (1.5)$$

as $n \rightarrow \infty$, which proves (1.4).

Solution 1.10

Since $B_1 \cup B_2 \cup \dots = \Omega$,

$$A = A \cap (B_1 \cup B_2 \cup \dots) = (A \cap B_1) \cup (A \cap B_2) \cup \dots,$$

where

$$(A \cap B_i) \cap (A \cap B_j) = A \cap (B_i \cap B_j) = A \cap \emptyset = \emptyset.$$

By countable additivity

$$\begin{aligned} P(A) &= P((A \cap B_1) \cup (A \cap B_2) \cup \dots) \\ &= P(A \cap B_1) + P(A \cap B_2) + \dots \\ &= P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \dots \end{aligned}$$

Solution 1.11

If $P(B) \neq 0$, then A and B are independent if and only if

$$P(A) = \frac{P(A \cap B)}{P(B)}.$$

In turn, this equality holds if and only if $P(A) = P(A|B)$.

Solution 1.12

The σ -fields $\sigma(\xi)$ and $\sigma(\eta)$ consist, respectively, of events of the form

$$\{\xi \in A\} \quad \text{and} \quad \{\eta \in B\},$$

where A and B are Borel sets in \mathbb{R} . Therefore, $\sigma(\xi)$ and $\sigma(\eta)$ are independent if and only if the events $\{\xi \in A\}$, and $\{\eta \in B\}$ are independent for any Borel sets A and B , which in turn is equivalent to ξ and η being independent.



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