Table of Contents

**Introduction** ................................................. 1

Historical Note ............................................. 1

The Contents of the Book .................................... 8

**Standard Notation** ........................................ 12

**I. Preliminaries** .......................................... 13

Topology and Algebra ...................................... 13

1. Notation and Basic Facts ................................. 13
2. Some Properties of Bilinear Forms ...................... 15
3. Vector Bundles, Characteristic Classes and the Index Theorem .... 21

Complex Manifolds ........................................... 23

4. Basic Concepts and Facts ................................. 23
5. Holomorphic Vector Bundles, Serre Duality and Riemann-Roch .... 24
6. Line Bundles and Divisors ................................ 26
7. Algebraic Dimension and Kodaira Dimension ............... 28

General Analytic Geometry ................................. 30

8. Complex Spaces ........................................... 30
9. The $\sigma$-Process ....................................... 34
10. Deformations of Complex Manifolds ...................... 35

Differential Geometry of Complex Manifolds ............... 39

11. De Rham Cohomology .................................... 39
12. Dolbeault Cohomology ................................... 41
13. Kähler Manifolds ........................................ 42
14. Weight-1 Hodge Structures ............................. 48
15. Yau's Results on Kähler-Einstein Metrics ............... 51

Coverings ..................................................... 53

16. Ramification ............................................. 53
17. Cyclic Coverings ........................................ 54
18. Covering Tricks ......................................... 55

Projective-Algebraic Varieties ............................. 57

19. GAGA Theorems and Projectivity Criteria ............... 57
20. Theorems of Bertini and Lefschetz ...................... 58

**II. Curves on Surfaces** .................................. 61

Embedded Curves ............................................ 61

1. Some Standard Exact Sequences ......................... 61
2. The Picard-Group of an Embedded Curve ................. 63
3. Riemann-Roch for an Embedded Curve ................... 65
4. The Residue Theorem ................................... 66
5. The Trace Map .................................................. 68
6. Serre Duality on an Embedded Curve ......................................... 70
7. The $\sigma$-process .......................................................... 73
8. Simple Singularities of Curves ................................................. 78

Intersection Theory ......................................................... 81

9. Intersection Multiplicities ................................................... 81
10. Intersection Numbers ....................................................... 83
11. The Arithmetical Genus of an Embedded Curve ............................... 84
12. 1-Connected Divisors .......................................................... 85

III. Mappings of Surfaces ...................................................... 89

Bimeromorphic Geometry .................................................... 89
1. Bimeromorphic Maps .......................................................... 89
2. Exceptional Curves ........................................................... 90
3. Rational Singularities ........................................................ 93
4. Exceptional Curves of the First Kind ........................................ 97
5. Hirzebruch-Jung Singularities ................................................ 99
6. Resolution of Surface Singularities .......................................... 105
7. Singularities of Double Coverings, Simple Singularities of Surfaces ....... 107

Fibrations of Surfaces ......................................................... 110
8. Generalities on Fibrations .................................................... 110
9. The $n$-th Root Fibration .................................................... 113
10. Stable Fibrations .............................................................. 114
11. Direct Image Sheaves ......................................................... 116
12. Relative Duality ............................................................... 118

The Period Map of Stable Fibrations ......................................... 121
13. Period Matrices of Stable Curves ........................................... 121
14. Topological Monodromy of Stable Fibrations ................................ 122
15. Monodromy of the Period Matrix ........................................... 125
16. Extending the Period Map ................................................... 127
17. The Degree of $f_*\omega_{X/S}$ ................................................ 129
18. Iitaka's Conjecture $C_{2,1}$ ................................................ 131

IV. Some General Properties of Surfaces ..................................... 135

1. Meromorphic Maps, Associated to Line Bundles ............................. 135
2. Hodge Theory on Surfaces ................................................... 137
3. Existence of Kähler Metrics ................................................ 144
4. Deformations of Surfaces .................................................... 154
5. Some Inequalities for Hodge Numbers ...................................... 157
6. Projectivity of Surfaces ...................................................... 159
7. The Nef Cone ................................................................. 162
8. Surfaces of Algebraic Dimension Zero ....................................... 165
9. Almost-Complex Surfaces without any Complex Structure .............. 166
10. Bogomolov's Theorem ........................................................ 168
11. Reider's Method .............................................................. 174
12. Vanishing Theorems on Surfaces ............................................ 179

V. Examples ................................................................. 185

Some Classical Examples ..................................................... 185
1. The Projective Plane $\mathbb{P}_2$ ............................................. 185
2. Complete Intersections ........................................................ 187
3. Tori of Dimension 2 ............................................................ 188

Fibre Bundles ............................................................... 189
4. Ruled Surfaces ............................................................... 189
<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>Elliptic Fibre Bundles</td>
<td>193</td>
</tr>
<tr>
<td>6</td>
<td>Higher Genus Fibre Bundles</td>
<td>199</td>
</tr>
<tr>
<td></td>
<td>Elliptic Fibrations</td>
<td>200</td>
</tr>
<tr>
<td>7</td>
<td>Kodaira’s Table of Singular Fibres</td>
<td>200</td>
</tr>
<tr>
<td>8</td>
<td>Stable Fibrations</td>
<td>202</td>
</tr>
<tr>
<td>9</td>
<td>The Jacobian Fibration</td>
<td>204</td>
</tr>
<tr>
<td>10</td>
<td>Stable Reduction</td>
<td>207</td>
</tr>
<tr>
<td>11</td>
<td>Classification</td>
<td>211</td>
</tr>
<tr>
<td>12</td>
<td>Invariants</td>
<td>212</td>
</tr>
<tr>
<td>13</td>
<td>Logarithmic Transformations</td>
<td>216</td>
</tr>
<tr>
<td></td>
<td>Kodaira Fibrations</td>
<td>220</td>
</tr>
<tr>
<td>14</td>
<td>Finite Quotients</td>
<td>223</td>
</tr>
<tr>
<td>15</td>
<td>The Godeaux Surface</td>
<td>223</td>
</tr>
<tr>
<td>16</td>
<td>Kummer Surfaces</td>
<td>223</td>
</tr>
<tr>
<td>17</td>
<td>Quotients of Products of Curves</td>
<td>224</td>
</tr>
<tr>
<td></td>
<td>Infinite Quotients</td>
<td>225</td>
</tr>
<tr>
<td>18</td>
<td>Hopf Surfaces</td>
<td>225</td>
</tr>
<tr>
<td>19</td>
<td>Inoue Surfaces</td>
<td>227</td>
</tr>
<tr>
<td>20</td>
<td>Quotients of Bounded Domains in $\mathbb{C}^2$</td>
<td>230</td>
</tr>
<tr>
<td>21</td>
<td>Hilbert Modular Surfaces</td>
<td>231</td>
</tr>
<tr>
<td></td>
<td>Coverings</td>
<td>236</td>
</tr>
<tr>
<td>22</td>
<td>Invariants of Double Coverings</td>
<td>236</td>
</tr>
<tr>
<td>23</td>
<td>An Enriques Surface</td>
<td>238</td>
</tr>
<tr>
<td>24</td>
<td>Kummer Coverings</td>
<td>240</td>
</tr>
</tbody>
</table>

VI. The Enriques Kodaira Classification

1. Statement of the Main Result                                  243
2. Characterising Minimal Surfaces whose Canonical Bundle is Nef 247
3. The Rationality Theorem and Castelnuovo’s Criterion           248
4. The Case $a(X) = 2$.                                          252
5. The Case $a(X) = 1$.                                          255
6. The Case $a(X) = 0$.                                          257
7. The Final Step                                                262
8. Deformations                                                  263

VII. Surfaces of General Type

Preliminaries                                                    269
1. Introduction                                                  269
2. Some General Theorems                                         271

Two Inequalities                                                273
3. Noether’s Inequality                                           273
4. The Inequality $c_1^2 \leq 3c_2$                               275

Pluricanonical Maps                                              279
5. The Main Results                                               279
6. Proof of the Main Results                                      281
7. The Exceptional Cases and the 1-Canonical Map                  286

Surfaces with Given Chern Numbers                                 290
8. The Geography of Chern Numbers                                 291
9. Surfaces on the Noether Lines                                  296
10. Surfaces with $q = p_g = 0$                                   299
VIII. K3-Surfaces and Enriques Surfaces .................................. 307

Introduction ................................................................. 307
1. Notation ................................................................. 307
2. The Results ............................................................. 309

K3-Surfaces ........................................................................ 310
3. Topological and Analytical Invariants ............................... 310
4. Digression on Affine Geometry over $\mathbb{F}_2$ .................... 314
5. The Néron-Severi Lattice of Kummer Surfaces .................. 316
6. The Torelli Theorem for Kummer Surfaces ....................... 323
7. The Local Torelli Theorem for K3-Surfaces ...................... 324
8. A Density Theorem ....................................................... 326
9. Behaviour of the Kähler Cone under Deformations ............ 328
10. Degenerations of Isomorphisms between K3-Surfaces ........ 330
11. The Torelli Theorems for K3-Surfaces ............................ 333
12. Construction of Moduli Spaces ....................................... 335
13. Digression on Quaternionic Structures ............................ 337
14. Surjectivity of the Period Map ....................................... 339

Enriques Surfaces .................................................................. 340
15. Topological and Analytic Invariants ............................... 340
16. Divisors on an Enriques Surface $Y$ ............................... 341
17. Elliptic Pencils .......................................................... 343
18. Double Coverings of Quadrics ....................................... 346
19. The Period Map .......................................................... 351
20. The Period Domain for Enriques Surfaces ...................... 353
21. Global Properties of the Period Map .............................. 355

Special Topics ................................................................. 359
22. Projective K3-surfaces and Mirror Symmetry ................... 359
23. Special Curves on K3-Surfaces ...................................... 365

IX. Topological and Differentiable Structure of Surfaces ........ 375

Topology of Simply Connected Compact Complex Surfaces .... 375
1. Freedman’s Results ...................................................... 375
2. Representability of Unimodular Forms ............................ 377

Donaldson Invariants .......................................................... 379
3. Introduction ............................................................... 379
4. The Donaldson Invariant, a Bird’s Eye View ..................... 380
5. Infinitely many Homeomorphic Surfaces which are not Diffeomorphic 383
6. Further Results obtained by the Donaldson Method ............ 390

Seiberg-Witten Invariants .................................................... 391
7. Introduction ............................................................... 391
8. Properties of the Invariants .......................................... 393
9. Surfaces Diffeomorphic to a Rational Surface .................. 395

Bibliography ...................................................................... 401

Notation ............................................................................. 425

Index .................................................................................. 429
Compact Complex Surfaces
Barth, W.; Hulek, K.; Peters, C.; Ven, A. van de
2004, XII, 436 p., Hardcover
ISBN: 978-3-540-00832-3