Chapter 2
Partial Orders and Pontryagin Duality

Abstract Partial orders, supernatural numbers, and Pontryagin duality, are discussed.

2.1 Partial Orders

Definition 2.1 (i) A partially ordered set is a set $S$ with a relation $\preceq$, and the properties:

1. (reflexivity) $a \preceq a$, for all $a \in S$;
2. (antisymmetry) if $a \preceq b$ and $b \preceq a$, then $a = b$;
3. (transitivity) if $a \preceq b$ and $b \preceq c$, then $a \preceq c$.

(ii) A directed partially ordered set $S$, is a partially ordered set with the additional property that for $a, b \in S$, there exists $c \in S$ such that $a \preceq c$ and $b \preceq c$.

$a, b$ are comparable if $a \preceq b$ or $b \preceq a$. A partially ordered set where any pair of elements is comparable, is a chain (total order).

Definition 2.2 Two partially ordered sets $(S, \preceq)$ and $(S', \preceq')$ are order isomorphic, if there is a bijective map $f$ from $S$ to $S'$, and $f(a_1) \preceq' f(a_2)$, if and only if $a_1 \preceq a_2$.

Example 2.1

- the partial order ‘subgroup’ in a set of groups
- the partial order ‘less or equal’ in the set of natural numbers $\mathbb{N}$ (i.e., $a \preceq b$ if $a \leq b$)
- the partial order ‘divisibility’ in the set of natural numbers $\mathbb{N}$ (i.e., $a \preceq b$ if $a|b$)

For simplicity we use the same symbol $\preceq$ for different partial orders, and its precise meaning is clear from the context.

Definition 2.3 An upper bound of a subset $T$ of the partially ordered set $S$, is an element $a \in S$ such that $b \preceq a$ for all $b \in T$. If the set of all upper bounds of $T$ has a smallest element, it is called the supremum of $T$. 
An element \( m \in S \) is called maximal, if there is no element \( k \in S \) such that \( m \prec k \). A partially ordered set might have many maximal elements, or it might have no maximal element.

**Definition 2.4** A partially ordered set \( S \), is called directed-complete partial order (dcpo) if one of the following two statements, which can be proved to be equivalent to each other [1–3], holds:

1. Every directed subset of \( S \) has a supremum.
2. Every chain in \( S \) has a supremum.

A chain which has a supremum, is called a complete chain.

Directed partially ordered sets which are not complete, can sometimes be enlarged into directed-complete partial orders, by adding extra elements.

**Example 2.2** The set \( \mathbb{N} \) of natural numbers, with divisibility as an order is a directed partially ordered set, but it is not a directed-complete partial order. For example the chain \( p, p^2, p^3, \ldots \) where \( p \in \Pi \), has no supremum. \( \mathbb{N} \) has no maximal elements. Below we enlarge this set into the supernatural (Steinitz) numbers, which is a directed-complete partial order.

### 2.2 The Directed-Complete Partial Order of Supernatural (Steinitz) Numbers

The set \( \mathbb{N}_S \) of supernatural (Steinitz) numbers [4, 5] is:

\[
\mathbb{N}_S = \left\{ n = \prod p^{e_p} \mid p \in \Pi; \quad e_p \in \mathbb{Z}_0^+ \cup \{\infty\} \right\}
\]  

(2.1)

The index \( S \) indicates supernatural or Steinitz. Here:

- The exponents can take the value \( \infty \).
- The product might contain an infinite number of prime numbers.

In this set only multiplication is well defined, and by definition

\[
p^\infty p^e = p^\infty; \quad e \in \mathbb{Z}_0^+ \cup \{\infty\}.
\]  

(2.2)

\( \mathbb{N} \) is a subset of \( \mathbb{N}_S \). Indeed, if all \( e_p \neq \infty \) and only a finite number of them are different from zero, the \( \prod p^{e_p} \in \mathbb{N} \).

**Definition 2.5**

- Let \((e_p)\) (where \( p \in \Pi \) and \( e_p \in \mathbb{Z}_0^+ \cup \{\infty\}\)) be an infinite sequence of exponents. The \((e_p) \prec (e'_p)\) indicates that \( e_p \leq e'_p \) for all \( p \). By definition all numbers in \( \mathbb{Z}_0^+ \) are smaller than \( \infty \).
- \( n = \prod p^{e_p} \) is a divisor of \( n' = \prod p^{e_p} \), if \((e_p) \prec (e'_p)\). We denote this as \( n|n' \) or as \( n < n' \).
• $\mathcal{E}$ is the element of $\mathbb{N}_S$, corresponding to the sequence where all $e_p = 1$:

$$\mathcal{E} = \prod_{p \in \Pi} p$$ (2.3)

• $\mathcal{Y}$ is the element of $\mathbb{N}_S$, corresponding to the sequence where all $e_p = \infty$:

$$\mathcal{Y} = \prod_{p \in \Pi} p^\infty$$ (2.4)

Every element of $\mathbb{N}_S$ is a divisor of $\mathcal{Y}$.

The set $\mathbb{N}_S$ ordered by divisibility (as defined above) is a directed-complete partial order, with $\mathcal{Y}$ as supremum. Examples of complete chains in $\mathbb{N}_S$, are

$$p, p^2, ..., p^\infty; \quad p \in \Pi$$

$$p_1 < p_1^2 < ... < p_1^\infty < p_1^\infty p_2 < p_1^\infty p_2^2 < ... < p_1^\infty p_2^\infty$$

$$2 < 2 \cdot 3 < 2 \cdot 3 \cdot 5 < ... < \mathcal{E}$$

$$2^\infty < 2^\infty 3^\infty < 2^\infty 3^\infty 5^\infty < ... < \mathcal{Y}$$ (2.5)

The suprema in these chains are $p^\infty, p_1^\infty p_2^\infty, \mathcal{E}$ and $\mathcal{Y}$, correspondingly. They are examples of the elements that have been added into $\mathbb{N}$, in order to make it the directed-complete partial order $\mathbb{N}_S$.

We use the notation $\mathbb{N}_S(p)$ for the complete chain

$$\mathbb{N}_S(p) = \{p, p^2, ..., p^\infty\}. \quad (2.6)$$

### 2.3 Pontryagin Duality

Let $G$ be an Abelian group and $\tilde{G}$ its Pontryagin dual group, i.e. the group of its characters (we use the notation $\chi$ for characters). For locally compact Abelian groups, the Pontryagin duality theorem states that

$$\tilde{\tilde{G}} \cong G. \quad (2.7)$$

Let $\mathfrak{G}$ be a set of groups, and $\tilde{\mathfrak{G}}$ the set of their Pontryagin dual groups. The partial order subgroup in $\mathfrak{G}$, endows a partial order in $\tilde{\mathfrak{G}}$, where $\tilde{A} < \tilde{G}$ if $A < G$.

**Definition 2.6** Let $A$ be a subgroup of $G$ (we denote this as $A < G$). The annihilator $\text{Ann}_G(A)$ of $A$, is the subgroup of $\tilde{G}$:

$$\text{Ann}_G(A) = \{b \in \tilde{G} \mid \chi_b(a) = 1, \forall a \in A\} \quad (2.8)$$
The groups $G$ relevant to this monograph, together with their Pontryagin dual groups $\tilde{G}$, and the corresponding quantum system are shown in Table 2.1.

<table>
<thead>
<tr>
<th>$G$</th>
<th>$\tilde{G}$</th>
<th>$\Sigma(G, \tilde{G})$</th>
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<tbody>
<tr>
<td>$\mathbb{Z}(d)$</td>
<td>$\mathbb{Z}(d)$</td>
<td>$\Sigma[\mathbb{Z}(d)]$</td>
</tr>
<tr>
<td>$GF(p^\ell)$</td>
<td>$GF(p^\ell)$</td>
<td>$\Sigma[GF(p^\ell)]$</td>
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<tr>
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<td>$\mathbb{Q}_p/\mathbb{Z}_p$</td>
<td>$\Sigma[\mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]$</td>
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<tr>
<td>$\mathbb{Q}$</td>
<td>$\mathbb{Q}/\mathbb{Z}$</td>
<td>$\Sigma[\mathbb{Q}, (\mathbb{Q}/\mathbb{Z})]$</td>
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The following proposition gives the Pontryagin dual group $\tilde{A}$ of a subgroup $A$ of a group $G$, and we present it without proof (e.g., [6]).

**Proposition 2.1** If $A < G$, then the Pontryagin dual group of $A$ is isomorphic to $\tilde{G}/\text{Ann}_{\tilde{G}}(A)$:

$$\tilde{A} \cong \tilde{G}/\text{Ann}_{\tilde{G}}(A).$$

(2.9)

In quantum mechanics $G$ can be used as the group of ‘positions’, and its Pontryagin dual $\tilde{G}$ as the group of ‘momenta’. We denote such a quantum system as $\Sigma(G, \tilde{G})$. For some groups $G \cong \tilde{G}$, and then we use the simpler notation $\Sigma(G)$ for the corresponding quantum system.

**Definition 2.7** $\Sigma(A, \tilde{A})$ is a subsystem of $\Sigma(G, \tilde{G})$ if $A < G$ (in which case the $\tilde{A}$ is related to $\tilde{G}$ as in Eq. (2.9)). We denote this as $\Sigma(A, \tilde{A}) < \Sigma(G, \tilde{G})$.

The groups $G$ relevant to this monograph, together with their Pontryagin dual groups $\tilde{G}$, and the corresponding quantum system, are shown in Table 2.1.

**References**

Finite and Profinite Quantum Systems
Vourdas, A.
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