Chapter 2
Partial Orders and Pontryagin Duality

Abstract Partial orders, supernatural numbers, and Pontryagin duality, are discussed.

2.1 Partial Orders

Definition 2.1 (i) A partially ordered set is a set \( S \) with a relation \( \prec \), and the properties:

1. (reflexivity) \( a \prec a \), for all \( a \in S \);
2. (antisymmetry) if \( a \prec b \) and \( b \prec a \), then \( a = b \);
3. (transitivity) if \( a \prec b \) and \( b \prec c \), then \( a \prec c \).

(ii) A directed partially ordered set \( S \), is a partially ordered set with the additional property that for \( a, b \in S \), there exists \( c \in S \) such that \( a \prec c \) and \( b \prec c \).

\( a, b \) are comparable if \( a \prec b \) or \( b \prec a \). A partially ordered set where any pair of elements is comparable, is a chain (total order).

Definition 2.2 Two partially ordered sets \((S, \prec)\) and \((S', \prec')\) are order isomorphic, if there is a bijective map \( f \) from \( S \) to \( S' \), and \( f(a_1) \prec' f(a_2) \), if and only if \( a_1 \prec a_2 \).

Example 2.1

• the partial order ‘subgroup’ in a set of groups
• the partial order ‘less or equal’ in the set of natural numbers \( \mathbb{N} \) (i.e., \( a < b \) if \( a \leq b \))
• the partial order ‘divisibility’ in the set of natural numbers \( \mathbb{N} \) (i.e., \( a \prec b \) if \( a|b \))

For simplicity we use the same symbol \( \prec \) for different partial orders, and its precise meaning is clear from the context.

Definition 2.3 An upper bound of a subset \( T \) of the partially ordered set \( S \), is an element \( a \in S \) such that \( b \prec a \) for all \( b \in T \). If the set of all upper bounds of \( T \) has a smallest element, it is called the supremum of \( T \).
An element $m \in S$ is called maximal, if there is no element $k \in S$ such that $m \prec k$. A partially ordered set might have many maximal elements, or it might have no maximal element.

**Definition 2.4** A partially ordered set $S$, is called directed-complete partial order (dcpo) if one of the following two statements, which can be proved to be equivalent to each other [1–3], holds:

1. Every directed subset of $S$ has a supremum.
2. Every chain in $S$ has a supremum.

A chain which has a supremum, is called a complete chain.

Directed partially ordered sets which are not complete, can sometimes be enlarged into directed-complete partial orders, by adding extra elements.

**Example 2.2** The set $\mathbb{N}$ of natural numbers, with divisibility as an order is a directed partially ordered set, but it is not a directed-complete partial order. For example the chain $p, p^2, p^3, \ldots$ where $p \in \Pi$, has no supremum. $\mathbb{N}$ has no maximal elements. Below we enlarge this set into the supernatural (Steinitz) numbers, which is a directed-complete partial order.

### 2.2 The Directed-Complete Partial Order of Supernatural (Steinitz) Numbers

The set $\mathbb{N}_S$ of supernatural (Steinitz) numbers [4, 5] is:

$$\mathbb{N}_S = \left\{ n = \prod p^{e_p} \mid p \in \Pi; \quad e_p \in \mathbb{Z}_0^+ \cup \{\infty\} \right\}$$

(2.1)

The index $S$ indicates supernatural or Steinitz. Here:

- The exponents can take the value $\infty$.
- The product might contain an infinite number of prime numbers.

In this set only multiplication is well defined, and by definition

$$p^\infty p^e = p^\infty; \quad e \in \mathbb{Z}_0^+ \cup \{\infty\}. \quad (2.2)$$

$\mathbb{N}$ is a subset of $\mathbb{N}_S$. Indeed, if all $e_p \neq \infty$ and only a finite number of them are different from zero, the $\prod p^{e_p} \in \mathbb{N}$.

**Definition 2.5**

- Let $(e_p)$ (where $p \in \Pi$ and $e_p \in \mathbb{Z}_0^+ \cup \{\infty\}$) be an infinite sequence of exponents. The $(e_p) \prec (e'_p)$ indicates that $e_p \leq e'_p$ for all $p$. By definition all numbers in $\mathbb{Z}_0^+$ are smaller than $\infty$.
- $n = \prod p^{e_p}$ is a divisor of $n' = \prod p^{e'_p}$, if $(e_p) \prec (e'_p)$. We denote this as $n \mid n'$ or as $n \prec n'$.
2.2 The Directed-Complete Partial Order of Supernatural (Steinitz) Numbers

- $\mathcal{E}$ is the element of $\mathbb{N}_S$, corresponding to the sequence where all $e_p = 1$:
  \[ \mathcal{E} = \prod_{p \in \Pi} p \] (2.3)

- $\mathcal{Y}$ is the element of $\mathbb{N}_S$, corresponding to the sequence where all $e_p = \infty$:
  \[ \mathcal{Y} = \prod_{p \in \Pi} p^\infty \] (2.4)

Every element of $\mathbb{N}_S$ is a divisor of $\mathcal{Y}$.

The set $\mathbb{N}_S$ ordered by divisibility (as defined above) is a directed-complete partial order, with $\mathcal{Y}$ as supremum. Examples of complete chains in $\mathbb{N}_S$, are

\[ \begin{align*}
  p, p^2, \ldots, p^\infty; & \quad p \in \Pi \\
p_1 < p_1^2 < \ldots < p_1^\infty & < p_1^\infty p_2 < p_1^\infty p_2^2 < \ldots < p_1^\infty p_2^\infty \\
2 < 2 \cdot 3 < 2 \cdot 3 \cdot 5 & < \ldots < \mathcal{E} \\
2^\infty < 2^\infty 3^\infty < 2^\infty 3^\infty 5^\infty & < \ldots < \mathcal{Y}
\end{align*} \] (2.5)

The suprema in these chains are $p^\infty$, $p_1^\infty p_2^\infty$, $\mathcal{E}$ and $\mathcal{Y}$, correspondingly. They are examples of the elements that have been added into $\mathbb{N}$, in order to make it the directed-complete partial order $\mathbb{N}_S$.

We use the notation $\mathbb{N}_S(p)$ for the complete chain

\[ \mathbb{N}_S(p) = \{ p, p^2, \ldots, p^\infty \}. \] (2.6)

2.3 Pontryagin Duality

Let $G$ be an Abelian group and $\tilde{G}$ its Pontryagin dual group, i.e. the group of its characters (we use the notation $\chi$ for characters). For locally compact Abelian groups, the Pontryagin duality theorem states that

\[ \tilde{\tilde{G}} \cong G. \] (2.7)

Let $\mathfrak{G}$ be a set of groups, and $\tilde{\mathfrak{G}}$ the set of their Pontryagin dual groups. The partial order subgroup in $\mathfrak{G}$, endows a partial order in $\tilde{\mathfrak{G}}$, where $\tilde{A} < \tilde{G}$ if $A < G$.

**Definition 2.6** Let $A$ be a subgroup of $G$ (we denote this as $A < G$). The annihilator $\text{Ann}_G(A)$ of $A$, is the subgroup of $\tilde{G}$:

\[ \text{Ann}_G(A) = \{ b \in \tilde{G} \mid \chi_b(a) = 1, \forall a \in A \} \] (2.8)
Table 2.1 The groups $G$ relevant to this monograph, together with their Pontryagin dual groups $\tilde{G}$, and the corresponding quantum system

<table>
<thead>
<tr>
<th>$G$</th>
<th>$\tilde{G}$</th>
<th>$\Sigma(G, \tilde{G})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{Z}(d)$</td>
<td>$\mathbb{Z}(d)$</td>
<td>$\Sigma[\mathbb{Z}(d)]$</td>
</tr>
<tr>
<td>$GF(p^\epsilon)$</td>
<td>$GF(p^\epsilon)$</td>
<td>$\Sigma[GF(p^\epsilon)]$</td>
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<tr>
<td>$\mathbb{Z}_p$</td>
<td>$\mathbb{Q}_p/\mathbb{Z}_p$</td>
<td>$\Sigma[\mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]$</td>
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<tr>
<td>$\hat{\mathbb{Z}}$</td>
<td>$\mathbb{Q}/\mathbb{Z}$</td>
<td>$\Sigma[\hat{\mathbb{Z}}, (\mathbb{Q}/\mathbb{Z})]$</td>
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The following proposition gives the Pontryagin dual group $\tilde{A}$ of a subgroup $A$ of a group $G$, and we present it without proof (e.g., [6]).

**Proposition 2.1** If $A < G$, then the Pontryagin dual group of $A$ is isomorphic to $\tilde{G}/\text{Ann}_{\tilde{G}}(A)$:

$$\tilde{A} \cong \tilde{G}/\text{Ann}_{\tilde{G}}(A).$$ (2.9)

In quantum mechanics $G$ can be used as the group of ‘positions’, and its Pontryagin dual $\tilde{G}$ as the group of ‘momenta’. We denote such a quantum system as $\Sigma(G, \tilde{G})$. For some groups $G \cong \tilde{G}$, and then we use the simpler notation $\Sigma(G)$ for the corresponding quantum system.

**Definition 2.7** $\Sigma(A, \tilde{A})$ is a subsystem of $\Sigma(G, \tilde{G})$ if $A < G$ (in which case the $\tilde{A}$ is related to $\tilde{G}$ as in Eq. (2.9)). We denote this as $\Sigma(A, \tilde{A}) \prec \Sigma(G, \tilde{G})$.

The groups $G$ relevant to this monograph, together with their Pontryagin dual groups $\tilde{G}$, and the corresponding quantum system, are shown in Table 2.1.

**References**

Finite and Profinite Quantum Systems
Vourdas, A.
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