1. Let $a, b, c \in \mathbb{C}$. Find $\limsup_{n \to \infty} |a^n + b^n + c^n|^{1/n}$.

2. A function $f \in C([1, +\infty))$ is such that for every $x \geq 1$ there exists a limit

$$
\lim_{A \to \infty} \int_{A}^{Ax} f(u)\,du =: \varphi(x),
$$

$\varphi(2) = 1$, and moreover the function $\varphi$ is continuous at point $x = 1$. Find $\varphi(x)$.

3. A function $f \in C([0, +\infty))$ is such that

$$
f(x) \int_{0}^{x} f^2(u)\,du \to 1, \text{ as } x \to +\infty.
$$

Prove that

$$
f(x) \sim \left(\frac{1}{3x}\right)^{1/3}, \text{ as } x \to +\infty.
$$

4. Find

$$
\sup_{\lambda} \left( \frac{\sum_{k=0}^{n-1} (x_{k+1} - x_k) \sin 2\pi x_k}{\sum_{k=0}^{n-1} (x_{k+1} - x_k)^2} \right),
$$

where the supremum is taken over all possible partitions of $[0, 1]$ of the form $\lambda = \{0 = x_0 < x_1 < \ldots < x_{n-1} < x_n = 1\}$, $n \geq 1$. 

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5. Find general form of a function $f(z)$, which is analytic on the upper half-plane except the point $z = i$, and satisfies the following conditions:
- the point $z = i$ is a simple pole of $f(z)$;
- the function $f(z)$ is continuous and real-valued on the real axis;
- $\lim_{\text{Im} z \to 0} f(z) = A$ ($A \in \mathbb{R}$).

6. Let $\mathcal{D}$ be a bounded connected domain with boundary $\partial \mathcal{D}$, and $f(z), F(z)$ be functions analytic on $\mathcal{D}$. It is known that $F(z) \neq 0$ and $\text{Im} \frac{f(z)}{F(z)} \neq 0$ for every $z \in \partial \mathcal{D}$. Prove that the functions $F(z)$ and $F(z) + f(z)$ have equal number of zeros in $\mathcal{D}$.

7. A linear operator $A$ on a finite-dimensional space satisfies

$$A^{1996} + A^{998} + 1996I = 0.$$ 

Prove that $A$ has an eigenbasis. Here $I$ is the unit operator.

8. Let $A_1, A_2, \ldots, A_{n+1}$ be $n \times n$ matrices. Prove that there exist numbers $a_1, a_2, \ldots, a_{n+1}$ (not all of them equal 0) such that a matrix

$$a_1A_1 + \ldots + a_{n+1}A_{n+1}$$

is singular.

9. The trace of a matrix $A$ equals 0. Prove that $A$ can be decomposed into a finite sum of matrices, such that the square of each of them equals to zero matrix.
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