Chapter 2
The Action Principles in Mechanics

We begin this chapter with the definition of the action functional as time integral over the Lagrangian \( L(q_i(t), \dot{q}_i(t); t) \) of a dynamical system:

\[
S \{ [q_i(t)]; t_1, t_2 \} = \int_{t_1}^{t_2} dt \, L(q_i(t), \dot{q}_i(t); t) .
\]  

(2.1)

Here, \( q_i, i = 1, 2, \ldots, N \), are points in \( N \)-dimensional configuration space. Thus \( q_i(t) \) describes the motion of the system, and \( \dot{q}_i(t) = dq_i/dt \) determines its velocity along the path in configuration space. The endpoints of the trajectory are given by \( q_i(t_1) = q_{i1} \), and \( q_i(t_2) = q_{i2} \).

Next we want to find out what the actual dynamical path of the system is. The answer is contained in the principle of stationary action: in response to infinitesimal variation of the integration path, the action \( S \) is stationary, \( \delta S = 0 \), for variations about the correct path, provided the initial and final configurations are held fixed. On the other hand, if we permit infinitesimal changes of \( q_i(t) \) at the initial and final times, including alterations of those times, the only contribution to \( \delta S \) comes from the endpoint variations, or

\[
\delta S = G(t_2) - G(t_1) .
\]  

(2.2)

Equation (2.2) is the most general formulation of the action principle in mechanics. The fixed values \( G_1 \) and \( G_2 \) depend only on the endpoint path variables at the respective terminal times.

Again, given a system with the action functional \( S \), the actual time evolution in configuration space follows that path about which general variations produce only endpoint contributions. The explicit form of \( G \) is dependent upon the special
representation of the action principle. In the following we begin with the one that is best known, i.e.,

1. Lagrange: The Lagrangian for a point particle with mass \( m \), moving in a potential \( V(x_i, t) \), is

\[
L(x_i, \dot{x}_i; t) = \frac{m}{2} \dot{x}_i^2 - V(x_i, t) .
\] (2.3)

Here and in the following we restrict ourselves to the case \( N = 3 \); i.e., we describe the motion of a single mass point by \( x_i(t) \) in real space. The dynamical variable \( x_i(t) \) denotes the actual classical trajectory of the particle which is parametrized by \( t \) with \( t_1 \leq t \leq t_2 \).

Now we consider the response of the action functional (2.1) with respect to changes in the coordinates and in the time, \( \delta x_i(t) \) and \( \delta t(t) \), respectively. It is important to recognize that, while the original trajectory is being shifted in real space according to

\[
x_i(t) \rightarrow x'_i(t') = x_i(t) + \delta x_i(t)
\] (2.4)

the time-readings along the path become altered locally, i.e., different at each individual point on the varied curve—including the endpoints. This means that our time change is not a global \((\delta t(t) = \text{const.})\) rigid time displacement, equally valid for all points on the trajectory, but that the time becomes changed locally, or, shall we say, gauged, for the transported trajectory. All this indicates that we have to supplement (2.4) by

\[
t \rightarrow t'(t) = t + \delta t(t) ,
\] (2.5)

where the terminal time changes are given by \( \delta t(t_2) = \delta t_2 \), and \( \delta t(t_1) = \delta t_1 \).

To the time change (2.5) is associated the change in the integration measure in (2.1) given by the Jacobi formula

\[
d(t + \delta t) = \frac{d(t + \delta t)}{dt} dt = \left(1 + \frac{d}{dt} \delta t(t)\right) dt
\] (2.6)

or

\[
\delta(dt) := d(t + \delta t) - dt = dt \frac{d}{dt} \delta t(t) .
\] (2.7)

If the time is not varied, we write \( \delta_0 \) instead of \( \delta \); i.e., \( \delta_0 t = 0 \) or \([\delta_0, d/dt] = 0\).

The variation of \( x_i(t) \) is then given by

\[
\delta x_i(t) = \delta_0 x_i(t) + \delta t \frac{d}{dt} (x_i(t))
\] (2.8)
since up to higher order terms we have

$$\delta x_i(t) = x_i'(t') - x_i(t) = x_i'(t + \delta t) - x_i(t) = x_i'(t) + \delta t \frac{dx_i'(t)}{dt} - x_i(t)$$

$$= (x_i'(t) - x_i(t)) + \delta t \frac{dx_i}{dt} =: \delta_0 x_i(t) + \delta t \frac{dx_i}{dt}.$$  

Similarly,

$$\delta \dot{x}_i(t) = \delta_0 \dot{x}_i(t) + \delta t \frac{d}{dt} \dot{x}_i \tag{2.9}$$

$$= \delta_0 \dot{x}_i + \frac{d}{dt}(\delta t \dot{x}_i) - \dot{x}_i \frac{d}{dt} (\delta t)$$

$$= \frac{d}{dt} \left( \delta_0 + \delta t \frac{d}{dt} \right) x_i - \dot{x}_i \frac{d}{dt} \delta t = \frac{d}{dt} (\delta x_i) - \dot{x}_i \frac{d}{dt} \delta t \tag{2.10}. $$

The difference between $\delta$ and $\delta_0$ acting on $t$, $x_i(t)$ and $\dot{x}_i(t)$ is expressed by the identity

$$\delta = \delta_0 + \delta t \frac{d}{dt}. \tag{2.11}$$

So far we have obtained

$$\delta S = \int_{t_1}^{t_2} [\delta (dt)L + dt \delta L] = \int_{t_1}^{t_2} dt \left[ \delta \frac{d}{dt} (L\delta t) \right] + \delta L$$

$$= \int_{t_1}^{t_2} dt \left[ \frac{d}{dt} (L\delta t) + \left( \delta L - \delta t \frac{dL}{dt} \right) \right] = \int_{t_1}^{t_2} dt \left[ \frac{d}{dt} (L\delta t) + \delta_0 L \right], \tag{2.12}$$

since, according to (2.11) we have

$$\delta L = \delta_0 L + \delta t \frac{d}{dt} L \tag{2.13}.$$

The total variation of the Lagrangian is then given by

$$\delta L = \delta_0 L + \delta t \frac{d}{dt} L = \frac{\partial L}{\partial x_i} \delta_0 x_i + \frac{\partial L}{\partial \dot{x}_i} \delta_0 \dot{x}_i + \delta t \frac{dL}{dt}$$

$$= \frac{\partial L}{\partial x_i} \delta_0 x_i + \frac{\partial L}{\partial \dot{x}_i} \delta_0 \dot{x}_i + \delta t \left( \frac{\partial L}{\partial x_i} \dot{x}_i + \frac{\partial L}{\partial \dot{x}_i} \dot{x}_i + \frac{\partial L}{\partial t} \right)$$
\[
\frac{\partial L}{\partial x_i} \left( \delta_0 + \delta t \frac{d}{dt} \right) \dot{x}_i + \frac{\partial L}{\partial \dot{x}_i} \left( \delta_0 + \delta t \frac{d}{dt} \right) \dot{x}_i + \delta t \frac{\partial L}{\partial \dot{t}} = 0
\]

Now we go back to (2.3) and substitute
\[
\frac{\partial L}{\partial x_i} = -\frac{\partial V(x_i, t)}{\partial x_i}, \quad \frac{\partial L}{\partial \dot{x}_i} = m \dot{x}_i, \quad \frac{\partial L}{\partial \dot{t}} = -\frac{\partial V}{\partial \dot{t}},
\]
so that we obtain, with the aid of (2.10):
\[
\delta L = -\frac{\partial V}{\partial \dot{t}} \delta t + \frac{\partial V}{\partial x_i} \delta x_i + m \dot{x}_i \frac{dx_i}{dt} \delta x_i - m \dot{x}_i^2 \frac{d}{dt} \delta t.
\]

Our expression for \( \delta S \) then becomes
\[
\delta S = \int_{t_1}^{t_2} dt \left[ m \dot{x}_i \frac{dx_i}{dt} \delta x_i - \frac{m}{2} \left( \frac{dx_i}{dt} \right)^2 + V \right] \delta t.
\]

We can also write the last expression for \( \delta S \) a bit differently, thereby presenting explicitly the coefficients of \( \delta x_i \) and \( \delta t \):
\[
\delta S = \int_{t_1}^{t_2} dt \left\{ \frac{d}{dt} \left[ m \frac{dx_i}{dt} \delta x_i - \left( \frac{m}{2} \left( \frac{dx_i}{dt} \right)^2 + V \right) \right] \right\}
\]
\[-m \frac{d^2 x_i}{dt^2} \delta x_i - \frac{\partial V}{\partial x_i} \delta x_i - \frac{\partial V}{\partial \dot{x}_i} \delta t + \delta t \frac{d}{dt} \left[ \frac{m}{2} \left( \frac{dx_i}{dt} \right)^2 + V \right] \},
\]
or with the definition
\[
E = \frac{\partial L}{\partial \dot{x}_i} \dot{x}_i - L = \frac{m}{2} \left( \frac{dx_i}{dt} \right)^2 + V(x_i, t),
\]
\[
\delta S = \int_{t_1}^{t_2} dt \frac{d}{dt} \left[ m \frac{dx_i}{dt} \delta x_i - E \delta t \right]
\]
\[-m \frac{d^2 x_i}{dt^2} \delta x_i - \frac{\partial V}{\partial x_i} \delta x_i - \frac{\partial V}{\partial \dot{x}_i} \delta t + \delta t \left( \frac{dE}{dt} - \frac{\partial V}{\partial \dot{t}} \right) \].

Since \( \delta x_i \) and \( \delta t \) are independent variations, the action principle \( \delta S = G_2 - G_1 \) implies the following laws:
\[
\delta x_i : \ m \frac{d^2 x_i}{dt^2} = -\frac{\partial V(x_i, t)}{\partial x_i}, \quad (\text{Newton}),
\]
i.e., one second-order differential equation.

\[ \delta t : \frac{dE}{dt} = \frac{\partial V}{\partial t} , \tag{2.21} \]

so that for a static potential, \( \frac{\partial V}{\partial t} = 0 \), the law of the conservation of energy follows: \( dE/dt = 0 \).

**Surface term:** \( G = m \frac{dx_i}{dt} \delta x_i - E \delta t \). \tag{2.22}

### 2. Hamiltonian:

As a function of the Hamiltonian,

\[ H(x_i, p_i; t) = \frac{p_i^2}{2m} + V(x_i, t) , \tag{2.23} \]

the Lagrangian (2.3) can also be written as (\( p_i := \partial L/\partial \dot{x}_i \)):

\[ L = p_i \frac{dx_i}{dt} - H(x_i, p_i; t) . \tag{2.24} \]

Here, the independent dynamical variables are \( x_i \) and \( p_i \); \( t \) is the independent time-parameter variable. Hence the change of the action is

\[ \delta S = \delta \int_{t_1}^{t_2} dt \left[ p_i \frac{dx_i}{dt} - H(x_i, p_i; t) \right] = \int_{t_1}^{t_2} dt \left[ p_i \frac{d}{dt} \delta x_i + \frac{dx_i}{dt} \delta p_i - \delta H - \frac{d}{dt} \delta t \right] . \tag{2.25} \]

Upon using

\[ \delta H = \left( \frac{\partial H}{\partial x_i} \delta x_i + \frac{\partial H}{\partial p_i} \delta p_i \right) + \frac{\partial H}{\partial t} \delta t , \tag{2.26} \]

where, according to (2.23): \( \partial H/\partial x_i = \partial V/\partial x_i \) and \( \partial H/\partial p_i = p_i/m \), we obtain

\[ \delta S = \int_{t_1}^{t_2} dt \frac{d}{dt} \left[ p_i \delta x_i - H \delta t \right] + \int_{t_1}^{t_2} dt \left[ -\delta x_i \left( \frac{dp_i}{dt} + \frac{\partial V}{\partial x_i} \right) + \delta p_i \left( \frac{dx_i}{dt} - \frac{p_i}{m} \right) + \delta t \left( \frac{dH}{dt} - \frac{\partial H}{\partial t} \right) \right] . \tag{2.27} \]

The action principle \( \delta S = G_2 - G_1 \) then tells us here that

\[ \delta p_i : \frac{dx_i}{dt} = \frac{\partial H}{\partial p_i} = \frac{p_i}{m} , \tag{2.28} \]
Here we recognize the two first-order Hamiltonian differential equations.

\[
\delta x_i : \frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i} = -\frac{\partial V}{\partial x_i} .
\]  

(2.29)

Let us note for later use:

\[
\delta S = G_2 - G_1 = [p_i \delta x_i - H \delta t]_2 - [p_i \delta x_i - H \delta t]_1 .
\]  

(2.32)

Compared with \( x_1 := \{x_i(t_1)\} \), \( x_2 := \{x_i(t_2)\}; i = 1, 2, 3 \)

\[
\delta S = \frac{\partial S}{\partial x_1} \delta x_1 + \frac{\partial S}{\partial x_2} \delta x_2 + \frac{\partial S}{\partial t_1} \delta t_1 + \frac{\partial S}{\partial t_2} \delta t_2
\]  

(2.33)

(2.32) yields

\[
p_1 = -\frac{\partial S}{\partial x_1} , \quad H(x_1, p_1; t_1) = \frac{\partial S}{\partial t_1}
\]  

(2.34)

or

\[
H \left( x_1, -\frac{\partial S}{\partial x_1}, t_1 \right) - \frac{\partial S}{\partial t_1} = 0 .
\]  

(2.35)

In the same manner, it follows that:

\[
p_2 = \frac{\partial S}{\partial x_2} , \quad H \left( x_2, \frac{\partial S}{\partial x_2}, t_2 \right) + \frac{\partial S}{\partial t_2} = 0 .
\]  

(2.36)

Obviously, (2.35) and (2.36) are the Hamilton–Jacobi equations for finding the action \( S \). In this way we have demonstrated that the action (2.1) satisfies the Hamilton–Jacobi equation. (Later on we shall encounter \( S \) again as the generating function of a canonical transformation \( (q_i, p_i) \rightarrow (Q_i, P_i) \) of the \( F_1(q_i, Q_i, t) \)-type.

3. Euler–Maupertuis (Principle of Least Action): This principle follows from the Lagrangian representation of the action principle:

\[
\delta S = \delta \int_{t_1}^{t_2} dt \, L = \left[ m \frac{dx_i}{dt} \delta x_i - E \delta t \right]_1^2 ,
\]  

(2.37)
if we introduce the following restrictions:

a) $L$ should not be explicitly time dependent; then the energy $E$ is a conserved quantity both on the actual and the varied paths; b) for the varied paths, $\delta x_i(t)$ should vanish at the terminal points: $\delta x_i(t_{1,2}) = 0$. What remains is

$$\delta \int_{t_1}^{t_2} dt \; L = -E(\delta t_2 - \delta t_1) .$$  \hfill (2.38)

But under the same restrictions we have, using (2.18),

$$\int_{t_1}^{t_2} dt \; L = \int_{t_1}^{t_2} dt \frac{\partial L}{\partial \dot{x}_i} \dot{x}_i - E(t_2 - t_1) ,$$  \hfill (2.39)

the variation of which is given by

$$\delta \int_{t_1}^{t_2} dt \; L = \delta \int_{t_1}^{t_2} dt \frac{\partial L}{\partial \dot{x}_i} \dot{x}_i - E(\delta t_2 - \delta t_1) .$$  \hfill (2.40)

Comparing (2.40) with (2.38), we get, taking into consideration $p_i := \partial L/\partial \dot{x}_i$:

$$\delta \int_{t_1}^{t_2} dt \; p_i \frac{d\dot{x}_i}{dt} = 0 .$$  \hfill (2.41)

If, in addition, we assume the potential to be independent of the velocity, i.e., that

$$\frac{\partial T}{\partial \dot{x}_i} \dot{x}_i = 2T ,$$  \hfill (2.42)

then (2.41) takes on the form

$$\delta \int_{t_1}^{t_2} dt \; T = 0 .$$  \hfill (2.43)

or

$$\int_{t_1}^{t_2} dt \; T = \text{Extremum} .$$  \hfill (2.44)

Thus the Euler–Maupertuis Principle of Least Action states: The time integral of the kinetic energy of the particle is an extreme value for the path actually selected compared to the neighboring paths with the same total energy which the particle will travel between the initial and final position at any time $-t$ is varied!
This variation in time can also be expressed by writing (2.43) in the form [see also (2.7)]:

$$
\delta \int_{t_1}^{t_2} dt T = \int_{t_1}^{t_2} dt \left( T \frac{d\delta t}{dt} \delta t + \delta T \right). \tag{2.45}
$$

In \(N\)-dimensional configuration space, (2.41) is written as

$$
\delta \int_{t_1}^{t_2} \sum_{i=1}^{N} \frac{\partial L}{\partial q_i} \dot{q}_i \, dt = 0, \tag{2.46}
$$

or

$$
\delta \int_{t_1}^{t_2} \sum_{i=1}^{N} p_i \, dq_i = 0. \tag{2.47}
$$

If we parametrize the path in configuration space between 1 and 2 using the parameter \(\vartheta\), then (2.47) is written

$$
\delta \int_{\vartheta_1}^{\vartheta_2} \sum_{i=1}^{N} p_i \frac{dq_i}{d\vartheta} \, d\vartheta = 0. \tag{2.48}
$$

On the other hand, it follows from the Hamiltonian version of the action principle in its usual form with vanishing endpoint contributions \(\delta q_i(t_{1,2}) = 0, \delta t(t_{1,2}) = 0\) in \(2N\)-dimensional phase space:

$$
\tilde{\delta} \int_{t_1}^{t_2} dt \left[ \sum_{i=1}^{N} p_i \frac{dq_i}{dt} - H \right] = 0. \tag{2.49}
$$

One should note the different role of \(\delta\) in (2.46)—the time is also varied—and \(\tilde{\delta}\), which stands for the conventional virtual (timeless) displacement. With the parametrization \(\vartheta\) in (2.49), the expression

$$
\tilde{\delta} \int_{\vartheta_1}^{\vartheta_2} d\vartheta \left[ \sum_{i=1}^{N} p_i \frac{dq_i}{d\vartheta} - H \frac{dt}{d\vartheta} \right] = 0 \tag{2.50}
$$

can, by introducing conjugate quantities,

$$
q_{N+1} = t, \quad p_{N+1} = -H, \tag{2.51}
$$
be reduced formally to a form similar to (2.48):

\[ \delta \int_{t_1}^{t_2} \sum_{i=1}^{N+1} p_i \frac{dq_i}{d\delta} \, dt = 0. \]  

(2.52)

Besides the fact that in (2.52) we have another pair of canonical variables, the different roles of the two variation symbols \( \delta \) and \( \tilde{\delta} \) should be stressed. \( \delta \) refers to the paths with constant \( H = E \), whereas in the \( \tilde{\delta} \) variation, \( H \) can, in principle, be any function of time. \( \tilde{\delta} \) in (2.52) applies to \( 2N + 2 \)-dimensional phase space, while \( \delta \) in (2.48) applies to configuration space.

If, in the case of the principle of least action, no external forces are involved, i.e., we set without loss of generality \( V = 0 \), then \( E \) as well as \( T \) are constants. Consequently, the Euler–Maupertuis principle takes the form

\[ \delta \int_{t_1}^{t_2} dt = 0 = \delta t_2 - \delta t_1, \]  

(2.53)

i.e., the time along the actual dynamical path is an extremum.

At this point we are reminded of Fermat’s principle of geometrical optics: A light ray selects that path between two points which takes the shortest time to travel. Jacobi proposed another version of the principle of least action. It is always useful when one wishes to construct path equations in which time does not appear. We derive this principle by beginning with the expression for the kinetic energy of a free particle in space:

\[ T = \frac{1}{2} \sum_{i,k=1}^{3} m_{ik} \frac{dx_i}{dt} \frac{dx_k}{dt}, \]  

(2.54)

where \( m_{ik} \) are the elements of the mass tensor, e.g. \( m_{ik} = m\delta_{ik} \).

In generalized coordinates in \( N \)-dimensional configuration space, we then have

\[ T = \frac{1}{2} \frac{(ds)^2}{(dt)^2}, \]  

(2.55)

with the line element

\[ (ds)^2 = \sum_{i,k=1}^{N} m_{ik}(q_1, q_2, \ldots, q_N) dq_i dq_k \]  

(2.56)

and position-dependent elements \( m_{ik} \); for example, from

\[ T = \frac{m}{2} \frac{(dr)^2 + r^2 (d\theta)^2 + (dz)^2}{(dt)^2} \]  

(2.57)
we can immediately see that

\[
\mathbf{\dot{m}} = \begin{pmatrix} m & 0 & 0 \\ 0 & mr^2 & 0 \\ 0 & 0 & m \end{pmatrix}.
\]

The \( m_{ik} \) take over the role of the metric tensor in configuration space. At this point mechanics becomes geometry.

Writing (2.55) in the form \( dt = ds/\sqrt{2T} \) we can restate (2.43) as

\[
\delta \int_{t_1}^{t_2} dt\, T = 0 = \delta \int_{1}^{2} ds \sqrt{T}.
\] (2.58)

Here, we substitute \( T = H - V(q_i) \) to obtain Jacobi’s principle:

\[
\delta \int_{1}^{2} \sqrt{H - V(q_i)} \, ds = 0,
\] (2.59)

or, with (2.56):

\[
\delta \int_{1}^{2} \sqrt{H - V(q_i)} \sqrt{\sum_{i,k=1}^{N} m_{ik}(q_j) dq_i dq_k} = 0.
\] (2.60)

In the integrand, only the generalized coordinates appear. If we parametrize them with a parameter \( \vartheta \), we get

\[
\int_{\vartheta_1}^{\vartheta_2} \sqrt{H - V(q_i)} \sqrt{\sum_{i,k=1}^{N} m_{ik}(q_j) dq_i dq_k} d\vartheta = \text{Extremum}.
\] (2.61)

Since \( \vartheta \) is not constrained in any way, we can construct the Euler equations for the integrand using the conventional variation procedure. The solutions to these equations yield the trajectories in parameter representation.

A comparison of Fermat’s and Jacobi’s principles is appropriate here. If we apply the principle of least time (2.53) to a light ray in a medium with index of refraction \( n(x_i) \) and, due to

\[
\frac{v}{c} = \frac{1}{n(x_i)}, \quad v dt = ds, \quad dt = \frac{n(x_i)}{c} ds
\] (2.62)

get the expression

\[
\delta \int_{1}^{2} ds \, n(x_i) = 0.
\] (2.63)
then it is obvious from a comparison with Jacobi’s principle (2.59) that the quantity $\sqrt{(E-V)}$ can be looked at as “index of refraction” for a massive particle.

4. Schwinger: Here we use $x_i, p_i, t$ and $v_i$ as the variables to be varied. We shall immediately see, however, that $v_i$ does not satisfy an equation of motion, i.e., $dv_i/dt = \ldots$ does not appear; therefore $v_i$ is not a dynamical variable (just like $\phi$ and $B$ in the canonical version of electrodynamics). Schwinger writes

$$L = p_i \left( \frac{dx_i}{dt} - v_i \right) + \frac{1}{2} mv_i^2 - V(x_i, t) \quad (2.64)$$

$$= p_i \frac{dx_i}{dt} - H(x_i, p_i, t) , \quad (2.65)$$

with $H$ given by

$$H = p_i v_i - \frac{1}{2} mv_i^2 + V(x_i, t) . \quad (2.66)$$

The variation of the action now gives

$$\delta S = \int_{t_1}^{t_2} dt \left[ p_i \frac{d}{dt} \delta x_i - \frac{\partial H}{\partial t} \delta t - \frac{\partial V}{\partial x_i} \delta x_i + \left( \frac{dx_i}{dt} - v_i \right) \delta p_i \right.
\quad + \left. (-p_i + mv_i) \delta v_i - \left( p_i v_i - \frac{1}{2} mv_i^2 + V \right) \frac{d}{dt} \delta t \right] ,$$

or

$$\delta S = \int_{t_1}^{t_2} dt \frac{d}{dt} [p_i \delta x_i - H \delta t] + \int_{t_1}^{t_2} dt \left[ -\delta x_i \left( \frac{dp_i}{dt} + \frac{\partial V}{\partial x_i} \right) \right.
\quad + \left. \delta p_i \left( \frac{dx_i}{dt} - v_i \right) + \delta v_i (-p_i + mv_i) + \delta t \left( \frac{dH}{dt} - \frac{\partial H}{\partial t} \right) \right] . \quad (2.67)$$

With the definition of $H$ in (2.66), the action principle yields

$$\delta x_i : \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i} = -\frac{\partial V}{\partial x_i} , \quad (2.68)$$

$$\delta p_i : \quad \frac{dx_i}{dt} = \frac{\partial H}{\partial p_i} = v_i . \quad (2.69)$$

There is no equation of motion for $v_i$: no $dv_i/dt$.

$$\delta v_i : \quad -p_i + mv_i = -\frac{\partial H}{\partial v_i} = 0 , \quad (2.70)$$
\[ \delta t : \frac{dH}{dt} = \frac{\partial H}{\partial t}. \]  
(2.71)

Surface term: \[ G = p_i \delta x_i - H \delta t. \]  
(2.72)

Schwinger’s action principle contains the Lagrangian and Hamiltonian versions as special cases. So when we write

\[
H(x_i, p_i, v_i, t) = p_i v_i - \frac{m}{2} v_i^2 + V(x_i, t)
\equiv \frac{p_i^2}{2m} + V(x_i, t) - \frac{1}{2m} (p_i - m v_i)^2
\]  
(2.73)

and introduce \( v_i = p_i/m \) as definition of \( v_i \), we return to the Hamiltonian description. On the other side we can also write \( L \) in (2.65) as

\[
L = p_i \frac{dx_i}{dt} - p_i v_i + \frac{m}{2} v_i^2 - V(x_i, t) = \frac{m}{2} \left( \frac{dx_i}{dt} \right)^2 - V(x_i, t) + (p_i - m v_i) \left( \frac{dx_i}{dt} - v_i \right) - \frac{m}{2} \left( \frac{dx_i}{dt} - v_i \right)^2,
\]  
(2.74)

and if we now define: \( v_i = dx_i/dt \), then the Lagrangian description follows.

Once again: Schwinger’s realization of the action principle is distinguished by the introduction of additional variables for which no equations of motion exist.

Finally, we should like to briefly discuss the usefulness of the surface terms \( G_{1,2} \). These offer a connection between the conservation laws and the invariants of a mechanical system (Noether).

Let us assume that our variation of the action vanishes under certain circumstances: \( \delta S = 0 \). We then say that the action, which remains unchanged, is invariant under that particular variation of the path. The principle of stationary action then states:

\[
\delta S = 0 = G_2 - G_1,
\]  
(2.75)

i.e., \( G \) has the same value, independent of the initial and final configurations.

In particular, let us assume that the action (Hamiltonian version) is invariant for a variation around the actual path for which it holds that

\[
\delta x_i(t_{1,2}) = 0, \quad \frac{d}{dt}(\delta t) = 0 : \delta t = \text{const.} = \varepsilon.
\]  
(2.76)
Then it follows from the invariance of $S$ under infinitesimal constant time translation:

$$\delta S = 0 = G_2 - G_1 = -H(t_2)\delta t_2 + H(t_1)\delta t_1 = -(H_2 - H_1)\varepsilon,$$  \hspace{1cm} (2.77)

the conservation of energy:

$$H(t_2) = H(t_1), \quad \text{meaning} \quad \frac{dH}{dt} = 0.$$  \hspace{1cm} (2.78)

Similarly, the conservation law for linear momentum follows if we assume that the action of the system is invariant under constant space translation and the change of the terminal times vanishes:

$$\delta x_i = \delta \varepsilon_i = \text{const.}, \quad \delta t(t_{1,2}) = 0.$$  \hspace{1cm} (2.79)

$$\delta S = 0 = G_2 - G_1 = (p_i \delta x_i)_2 - (p_i \delta x_i)_1 = (p_{i2} - p_{i1})\delta \varepsilon_i$$ or

$$p_i(t_2) = p_i(t_1), \quad \text{meaning} \quad \frac{dp_i}{dt} = 0.$$  \hspace{1cm} (2.80)

Now let

$$H = \frac{p_i^2}{2m} + V(r),$$  \hspace{1cm} (2.82)

i.e., the potential may only depend on the distance $r = \sqrt{x_i^2}$. Then no space direction is distinguished, and with respect to rigid rotations $\delta \omega_i = \text{const.}$ and

$$\delta t(t_{1,2}) = 0, \quad \delta x_i = \varepsilon_{ijk}\delta \omega_j x_k.$$  \hspace{1cm} (2.83)

we obtain

$$\delta S = \delta \int_{t_1}^{t_2} dt \left[ p_i \frac{dx_i}{dt} - \frac{p_i^2}{2m} - V(\sqrt{x_i^2}) \right] = 0.$$  \hspace{1cm} (2.84)

Let us prove explicitly that $\delta S = 0$.

$$\delta \left( p_i \frac{dx_i}{dt} \right) - \delta \left( \frac{p_i^2}{2m} \right) = \delta p_i \frac{dx_i}{dt} + p_i \frac{d}{dt} \delta x_i - \frac{p_i}{m} \delta p_i = p_i \frac{d}{dt} \delta x_i.$$
where we used $dx_i/dt = p_i/m$, since our particle travels on the correct classical path; thus we are left with

$$p_i \frac{d}{dt} \delta x_i = p_i \frac{d}{dt} \varepsilon_{ijk} \delta \omega_j x_k = \frac{1}{m} \varepsilon_{ijk} \delta \omega_j p_i p_k = 0 \ .$$  \hspace{1cm} (2.85)

where again, $\dot{x}_k = p_k/m$ has been applied together with the total antisymmetry of $\varepsilon_{ijk}$.

The remaining variation is

$$\delta V = \frac{\partial V}{\partial x_i} \delta x_i = \frac{\partial V}{\partial x_i} \varepsilon_{ijk} \delta \omega_j x_k = \frac{x_i}{r} \frac{\partial V}{\partial r} \varepsilon_{ijk} \delta \omega_j x_k$$

$$= \frac{1}{r} \frac{\partial V}{\partial r} \varepsilon_{ijk} \delta \omega_j x_i x_k = 0 \ .$$  \hspace{1cm} (2.86)

Because

$$\delta S = 0 = G_2 - G_1 = (p_i \delta x_i)_2 - (p_i \delta x_i)_1 = (p_i \varepsilon_{ijk} \delta \omega_j x_k)_2 - (p_i \varepsilon_{ijk} \delta \omega_j x_k)_1$$

$$= \delta \omega_i \{[(r \times p)_i]_2 - [(r \times p)_i]_1\}$$

this implies the conservation of angular momentum:

$$\mathbf{L}(t_2) = \mathbf{L}(t_1) \ , \ \text{meaning} \quad \frac{d\mathbf{L}}{dt} = 0 \ .$$  \hspace{1cm} (2.88)

Conversely, the conservation of angular momentum corresponds to the invariance, $\delta S = 0$, under rigid rotation in space. The generalization of this statement is: if a conservation law exists, then the action $S$ is stationary with respect to the infinitesimal transformation of a corresponding variable. The converse of this statement is also true: if $S$ is invariant with respect to an infinitesimal transformation, $\delta S = 0$, then a corresponding conservation law exists.
Classical and Quantum Dynamics
From Classical Paths to Path Integrals
Dittrich, W.; Reuter, M.
2017, XVI, 489 p. 18 illus., Hardcover
ISBN: 978-3-319-58297-9