Properties of a Polycentric Oval

In this chapter we sum up the well-known properties of an oval and add new ones, in order to have the tools for the various constructions illustrated in Chap. 3 and for the formulas linking the different parameters, derived in Chap. 4. All properties are derived by means of mathematical proofs based on elementary geometry and illustrated with drawings.

We start with the definition of what we mean in this book by oval:

A polycentric oval is a closed convex curve with two orthogonal symmetry axes (or simply axes) made of arcs of circle subsequently smoothly connected, i.e. sharing a common tangent.

The construction of a polycentric oval is straightforward. On one of two orthogonal lines meeting at a point $O$, choose point $A$ and point $C_1$ between $O$ and $A$ (Fig. 2.1). Draw an arc of circle through $A$ with centre $C_1$, anticlockwise, up to a point $H_1$ such that the line $C_1H_1$ forms an acute angle with $OA$. Choose then a point $C_2$ on line $H_1C_1$, between $C_1$ and the vertical axis, and draw a new arc with centre $C_2$ and radius $C_2H_1$, up to a point $H_2$ such that the line $C_2H_2$ forms an acute angle with $OA$. Proceed in the same way eventually choosing a centre on the vertical axis—say $C_4$. The next arc will have as endpoint, say $H_4$, the symmetric to the previous endpoint w. r. t. the vertical line, say $H_3$, and centre $C_4$. From now on symmetric arches w.r.t. the two axes can be easily drawn using symmetric centres. The result is a 12-centre oval. This way of proceeding allows for common tangents at the connecting points $H_n$. The number of centres involved is always a multiple of four, which means that the simplest ovals have four centres, and to these most of this book is dedicated, considering that formulas and constructions of the latter can already be quite complicated. It is also true that eight-centre ovals have been used by architects in some cases in order to reproduce a form as close as possible to that of an ellipse. Chapter 8 is devoted to those shapes and to the possibility of extending properties of four-centre ovals to them.
The above definition implies that this is a polycentric curve, in the sense that it is made of arcs of circle subsequently connected at a point where they share a common tangent (and this may include smooth or non-smooth connections). In [2] a way of forming polycentric curves using a more general version of a property of ovals is presented.

One of the basic tools by which ovals can be constructed and studied is the Connection Locus. This is a set of points where the connection points for two arcs can be found. It was conjectured by Felice Ragazzo (see [4, 5]) for ovals, egg-shapes and generic polycentric curves. A Euclidean proof of its existence was the main topic in [2], along with constructions of polycentric curves making use of it. In this book the version for four-centre ovals is displayed, along with all the implications, the main one being that the oval is a curve defined by at least three independent parameters, even when a four-centre oval is chosen.

To make this clearer let us consider an ellipse with half-axes $a$ and $b$. These two positive numbers are what is needed to describe the ellipse, whose equation in a
Cartesian plane with axes coincident to those of the curve and the origin as symmetry centre, is

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1; \]

this curve is perfectly inscribed in a rectangle with sides $2a$ and $2b$. However, a four-centre oval inscribed in the same rectangle is not uniquely determined, an independent extra parameter is needed, and in the case of eight-centre ovals the extra parameters needed are three, giving a total of five. This means more freedom in the construction and at the same time more uncertainty in the recognition of the form (see Fig. 2.2).

2.1 Four-Centre Ovals

Let us devote our attention to four-centre ovals, from now on called *simple ovals* or just *ovals* (Fig. 2.3).

The double symmetry implies that it is enough to deal with the top right-hand section, a quarter-oval. In this respect, using Fig. 2.4 as reference, we define

- $O$ as the intersection of the two symmetry axes
- $A$ and $B$ as the intersection points between the quarter-oval and the horizontal and vertical axes. Let $OA > OB$
- $K$ and $J$ as the centres respectively of the small and big circles, with radii $r_1$ and $r_2$, whose arcs form the quarter-oval
- $H$ as the connecting point of the two arcs.

The above definitions and well-known Euclidean geometry theorems imply that:

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Footnote:

1Parts of this section have already been published in the Nexus Network Journal (see [2]).
A further well-known property or condition on $K$ is proved by the following argument, inspired by Bosse’s construction of an oval in [1], which we will present in Chap. 3. Draw from $K$ the parallel to the chord $BH$ and let $F$ be the intersection with the line $OB$ (see Fig. 2.4). Since $\overline{JH} = \overline{JB}$, we have that $\overline{JB} = \overline{JH}$, which means that $BFKH$ is an isosceles trapezoid, implying that $\overline{BF} = \overline{KH}$. Since $F$ has to be inside the segment $OB$, we have the following:

$$\overline{OB} > \overline{FB} = \overline{KH} = \overline{AK} = \overline{AO} - \overline{OK},$$

the first and the last terms yielding that $\overline{OK} > \overline{AO} - \overline{OB}$, which is a condition on point $K$.

Finally let us assume that for the same $K$ a different second centre $J'$ existed (and thus a different oval). Then—see the bottom of Fig. 2.4—the connecting point would be $H'$, and the same construction as the one just performed would yield a different chord $BH'$ and thus a different point $F'$ inside $OB$, but this is impossible,
because it would still have to be $FB = KH = AK$. This means that the choice of a feasible $K$ uniquely determines the quarter oval.

Felice Ragazzo’s work on ovals [4], and, later on, the one on polycentric curves [5], made it possible to find new properties and to solve un-tackled construction problems. His 1995 paper presents a property of any connection point and a conjecture on the locus described by all the possible connection points of an oval with given axis measures.

We will present the first property in the form of a theorem.

**Theorem 2.1** In a quarter oval (see Fig. 2.5) the connection point $H$, the end point $B$ of the smaller axis, and the point $P$ (inside the rectangle inscribing the quarter
oval) lying on the smaller circle and on the parallel to the smaller axis through the centre $K$, are co-linear.

**Proof** Let us connect $B$ and $H$. Since $BH$ is not a tangent either to the circle through $H$ with centre $K$ or to the circle through $H$ with centre $J$, it will have an extra common point with both. For the bigger circle this is clearly $B$. According to Archimedes’ Lemma\(^2\) the intersection with the smaller circle has to lie on the parallel to $BJ$ through $K$. Since it has to lie on the same side of $H$ and $B$, it has to be $P$. □

The same argument can be used to prove that $H$, $A$ and $Q$ are also co-linear, $Q$ being the intersection between the bigger circle and the parallel to the longer axis through its centre $J$ (see again Fig. 2.5).

Ragazzo’s conjecture on ovals reduces the problem to *quarter* ovals inside rectangles. It is the following (our translation from the Italian):

*The locus of the connection points for the arcs [of a quarter oval [Ed.]] [...] is the circle, which we will call Connection Locus, defined through the following three points [points 1, 2 and 3 in Ragazzo’s original drawing (Fig. 2.6)]

- intersection between the half major axis and the half minor side [of the rectangle inscribing the oval [Ed.]]
- endpoint of the half minor side transported compass-wise onto the major side
- intersection between the half major side and the half minor axis

\(^2\)Given two tangent circles, any secant line through the tangency point intersects the two circles at points whose connecting lines with the corresponding centres are parallel.
In other words, any quarter oval inscribed in a given rectangle will have the corresponding connection point on the above defined Connection Locus, vice versa any point on it (as long as it lies between points 1 and 2, as we will see later) can be chosen as connection point for a quarter oval inscribed in the given rectangle.

We first wrote the proof of this conjecture, then managed to extend it to egg-shapes and to polycentric curves. So it has become in [3] a corollary of one of the theorems contained in [2]. In the framework of those works the chosen tool was that of Euclidean geometry. Another proof based on analytic geometry was presented by Ghione in the appendix to [4]. We present here both proofs, to show the two different possible approaches to the study of ovals, with just a couple additions to the second one to make it equivalent to the first.

**Theorem 2.2** Let $OATB$ be a rectangle with $BT > AT$. Let $S$ be the point on $BT$ such that $ST = \frac{1}{3}AT$.

1. The necessary and sufficient condition for $H$ to be the connecting point for two arcs of circle with centres inside $BT$ and tangent in $H$, one tangent to the line $AT$ in $A$ and the other one tangent to the line $TB$ in $B$, is for $H$ to belong to the open arc $AS$ of the circle through $A$, $B$ and $S$.
2. The angle originating at the centre $C$ of such a circle corresponding to the arc $AB$ is a right angle.
3. The line $OC$ forms a $\frac{\pi}{4}$ angle with $OA$.

**Proof 1 (Mazzotti 2014)** Let us consider Fig. 2.7. We will start from the second part of the theorem. Let $C$ be the centre of the circle through $A$, $B$ and $S$; since $ST = AT$ we have that $\hat{TS}A = \frac{\pi}{4}$ which implies that $\hat{CSA} + \hat{CSB} = \hat{ASB} = \frac{3}{4}\pi$ and so, since...
\[ CbSA = CbAS \quad \text{and} \quad CbSB = CbBS, \quad CbAS + CbBS = \frac{3}{4}\pi; \]
looking at the quadrilateral \( CASB \), we can deduce that
\[ A\widehat{CB} = 2\pi - \frac{3}{4}\pi - \frac{3}{4}\pi = \frac{\pi}{2}. \quad (2.1) \]

For the necessary condition we will show that two generic arcs of circle enjoying the properties stated have their connection point on the described circle. If \( K \) on \( OA \) is the centre of one of the two, we have (see the remarks preceding the present theorem) \( OA - OB < OK < OA \).

The connection point belongs to the circle with centre \( K \) and radius \( KA \). We will prove that it is the other intersection between this circle and the one through \( A, \) \( B \) and \( S \). Let \( H \) be this intersection and detect point \( J \) lying on both \( KH \) and \( OB \); if one proves that \( JH = JB \) then the second circle uniquely determined by \( A, \) \( B \) and \( K \) (see again the remarks preceding the present theorem), having its centre in \( J \) and radius equal to \( BJ \), will be found; the connection point of the two arcs is then clearly \( H \).

Summing up, if \( H \) is the (second) intersection of the circle through \( A, \) \( B \) and \( S \), with the circle with centre \( K \) and radius \( KA \), we have (see Fig. 2.7):
Hypothesis : \[
\begin{align*}
BC &= CS = CA = CH \\
TA &= TS \\
KA &= KH
\end{align*}
\]  
Thesis : \( \overline{HI} = \overline{JB} \) \hspace{1cm} (2.2)

Equality of the triangles \( CKA \) and \( CKH \) implies that \( KHC = KAC \), which added to \( CHB = CBH \) yields

\[
KHB = KAC + CBH. \tag{2.3}
\]

Equality (2.1) implies that triangles \( BOF \) and \( FAC \) are similar, and so that \( \overline{JC} = \overline{KA} \); by substitution of this equality in (2.3) we obtain that \( KHB = JC + CBH = JC \), hence the thesis.

With regard to the sufficient condition it is enough to choose a point \( H \) on arc \( AS \) and determine the intersection \( K \) of the axis of the segment \( HA \) with \( OA \); the circle with centre \( K \) and radius \( KH \) will then pass through \( A \) and have \( AT \) as tangent; once we show that \( K \) exists and is such that

\[\overline{OA} - \overline{OB} < \overline{OK} < \overline{OA}\] \hspace{1cm} (2.4)

calling \( J \) the intersection of \( KH \) with \( OA \), the proof that \( H \) is the connection point proceeds formally as the second part (2.2) of the proof of the necessary condition.

In order to prove (2.4) we will call \( V \) the intersection of the parallel to \( OB \) through \( S \) with \( OA \); we start by pointing out that the equality of triangles \( AVC \) and \( VCS \) implies that \( VC \) divides \( ZVO \) in two equal parts, and so that

\[OVC = VZ = \frac{\pi}{4}\] \hspace{1cm} (2.5)

Point \( C \) is on the axis of \( BS \), thus inside the strip determined by \( BO \) and \( SV \). If it were on the same side as \( H \) w.r.t. \( OA \), we would have a non-tangled quadrilateral \( AOBC \) having the angle in \( O \) measuring \( \frac{\pi}{2} \) and the one in \( C \) measuring \( \frac{3}{2} \pi \)—the concave version of \( A\hat{C}B = \frac{\pi}{2} \) (notice that (2.1) has been proved independently of \( K \))—which is unacceptable. This implies, since \( C \) belongs to \( KQ \) (see Fig. 2.7), that point \( K \) exists and lies between \( A \) and \( M \) (intersection of the axis of the segment \( BS \) with \( OA \)), and so that \( \overline{OK} < \overline{OA} \). On the other hand \( K \equiv V \) is no possible option, since if \( \overline{AV} = \overline{VS} \), we would have \( \overline{AK} = \overline{KS} \) and the circle with centre \( K \) would have \( S \) in common with the circle through \( A, B \) and \( S \), i.e. \( H \equiv S \), impossible according to our hypotheses. Moreover, it is also impossible that \( \overline{OM} < \overline{OK} < \overline{OV} \), otherwise \( \overline{OK}C \) would be an external angle of the triangle \( KVC \), and for a known theorem it would be \( \overline{OK}C > \overline{OV}C \); this would imply, since \( A\hat{K}Q = \overline{OK}C \), that \( A\hat{K}Q > \overline{OV}C \), and thus—observing the triangle \( QAK \)—that
$Q\hat{A}K = \frac{\pi}{2} - \hat{A}\hat{K}Q < \frac{\pi}{2} - O\hat{V}C$ \hspace{1cm} (2.6)

But $S\hat{A}K$ is a part of $Q\hat{A}K$, and so $Q\hat{A}K > S\hat{A}K = \frac{\pi}{4}$; comparing this with (2.6) we obtain $\frac{\pi}{2} - O\hat{V}C > \frac{\pi}{4}$ implying $O\hat{V}C < \frac{\pi}{4}$, in contrast with (2.5).

Finally, using (2.5), since $OMC = VMC$, we get that $C\hat{O}A$ also measures $\frac{\pi}{4}$, and the third part has also been proved. □

Proof 2 (Ghione 1995) In order to use Cartesian coordinates, a better reference is that of Fig. 2.8. $ATBD$ is the rectangle—with $BD < AD$—where we are going to inscribe a quarter-oval with symmetry centre $D$, where $A$ is made to be the origin of an orthogonal coordinate system, $AD$ and $AT$ are chosen as the $x$-axis and the $y$-axis, and is set equal to one. In addition let $S$ be the point on the segment $TB$ such that $TS = TA$. After having chosen the centre of the smaller circle $K$ on $AD$, we have the following points:

![Diagram](image)

Fig. 2.8 Proving Ragazzo’s conjecture by means of analytic geometry (Ghione)
\( A = (0, 0), \ B = (1, b), \ S = (b, b) \) and \( K = (k, 0) \), where \( 0 < k < b < a \).

\( TA \) is tangent to the circle with centre \( K \) and radius \( AK \), which is one of the two circles needed to draw the quarter-oval. We draw the parallel to \( AT \) through \( K \) and call \( P \) the intersection with the circle, then the line through \( B \) and \( P \) to detect point \( H \) as its intersection with the circle. Theorem 2.1 tells us now that \( H \) has to be the connecting point for the (only) quarter-oval that can be inscribed in \( ATBD \) with \( K \) as one of the centres.

The circle has centre \( K = (k, 0) \) and radius \( k \), thus the equation

\[
x^2 + y^2 = 2kx; \tag{2.7}
\]

point \( P \) has coordinates \( P = (k, k) \), and the line through \( P \) and \( B \) can be described, in parametric form, by the equations

\[
\begin{align*}
x &= k + (1 - k) \cdot t \\
y &= k + (b - k) \cdot t
\end{align*} \tag{2.8}
\]

where \( t \) varies over real numbers, and \( P \) and \( B \) are obtained when \( t = 0 \) and \( t = 1 \) respectively. By substitution of these equations in that of the circle (2.7) we obtain the two points corresponding to the values of the parameter \( t = 0 \) (yielding point \( P \)) and

\[
t = \frac{-2k(b - k)}{(b - k)^2 + (1 - k)^2},
\]

which corresponds to the connection point \( H \). Substituting these values into (2.8) we get the following coordinates of \( H \):

\[
\begin{align*}
x &= \frac{(b - 1)^2k}{(b - k)^2 + (1 - k)^2} \\
y &= \frac{k(b - 1)(2k - b - 1)}{(b - k)^2 + (1 - k)^2}
\end{align*}
\]

which allow for different positions of \( H \) when a different point \( K \) is chosen (with \( 0 < k < b \)). To get an equation of the locus of the set of \( H \) points for different feasible positions of \( K \), we eliminate \( k \) from the last equations: dividing the first one by the second one we get

\[
y = \frac{2k - (b + 1)}{(a - 1)} \cdot x, \quad \text{which implies} \ k = \frac{(b - 1)y + (b + 1)x}{2x}.
\]

Substitution of the latter into the first in (2.8) yields
\[ x^2 + y^2 - (b + 1)x - (b - 1)y = 0, \]  

which is clearly the equation of a circle, and it is solved by the coordinates of all three points

\[ A = (0, 0), B = (1, b) \text{ and } S = (b, b). \]

As \( k \) varies continuously in the open interval \( ]0; b[ \) the coordinates of \( H \) vary continuously according to formulas (2.8); performing the two limits for \( k \rightarrow 0 \) and for \( k \rightarrow b \) we get that

\[ \lim_{k \rightarrow 0} (x, y) = (0, 0) \equiv A \text{ and that } \lim_{k \rightarrow b} (x, y) = (b, b) \equiv S, \]

which means that all points in the open arc \( AS \) of the circle through \( A, B \) and \( S \) are connecting points for some quarter-oval, and that there are no other such points outside it. The preceding argument works thus as proof of both the necessary and the sufficient condition.

From (2.9) we derive the following coordinates of the centre \( C \) of the circle through \( A, B \) and \( S \):

\[ x_C = \frac{(b + 1)}{2} \text{ and } y_C = \frac{(b - 1)}{2}; \]

calculating the slope of the lines \( CB \) and \( CA \) we get

\[ m_{CB} = \frac{b + 1}{2} - b = \frac{b + 1}{b - 1} \text{ and } m_{CA} = \frac{b + 1}{2} - 0 = \frac{b - 1}{b + 1} \]

which implies, since \( m_{CB} \cdot m_{CA} = -1 \), that \( B\hat{C}A = \frac{\pi}{2} \).

Finally we observe that the coordinates of \( C \), for any value of \( b \) belong to the bisector of \( ADJ \) whose equation is \( y = x - 1 \). The proof is now complete.\( \square \)

The fact that \( CO \) bisects \( J\hat{O}A \) is another result already conjectured by Ragazzo and proved by Ghione in [4].

**Definition** From now on we will call the circle through \( A, B \) and \( S \), following Ragazzo’s work (1995), Connection Locus (our translation), or simply CL (as in [3]).

Rosin found the same set of points after Ragazzo did (see [6]) proving its properties, and called it Locus of continuous tangent joints. Simpson used it in 1745, to solve a problem posed by Stirling (in [7]) without realizing its full properties.

If \( H \equiv S \) the closed convex curve is not an oval anymore, since the arcs of the bigger circle degenerate into segments, and becomes what we call a running track. In Sect. 3.4 we will talk about such a shape.
If $H \equiv A$ the curve cannot be considered an oval because the arcs of the small circles disappear and the shape becomes pointed.

We now observe in Fig. 2.7 that $QK$, the axis of the chord $AH$ of the circle with centre $K$ has to be the same as the axis of the chord $AH$ of the circle with centre $C$, therefore $CQ$ divides the angle $HKA$ into two equal parts, and does the same with the opposite angle $OKJ$. Noting also that (2.5) holds for point $V$ (see again Fig. 2.7), and that $AVT = \frac{\pi}{4}$, we have proved the following theorem.

**Theorem 2.3** *The centre $C$ of the Connection Locus for a quarter oval is the incentre of triangle $OKJ$. Moreover the line connecting $C$ with $T$—the point where the perpendiculars to the axes through $A$ and $B$ meet—forms with $OA$ an angle of $\frac{\pi}{4}$.*

We can now sum up all the properties listed and proved in the previous pages, in order to get a complete picture. These are the properties that have been used to derive parameter formulas (in Chap. 4) and to find easier or new constructions (in Chap. 3), especially in the case where the axis lines are not known.

We use for this purpose Fig. 2.9, where the important points in a full oval have been highlighted, although the properties will be listed only for the points needed to draw a quarter-oval.

*In a quarter oval with the segments $OA$ and $OB$ as half-axes (with $OB < OA$), $T$ as fourth vertex of the rectangle having $O$, $A$ and $B$ as vertices, $K$ as the centre of the smaller arc, $J$ as the centre of the bigger arc and $H$ as the connection point, in addition to the properties listed at the beginning of this section, the following hold:*

**Fig. 2.9** Properties of an oval and its points
– $\overline{AO} - \overline{OB} < \overline{OK} < \overline{OA}
– if S is the point on segment BT such that $\overline{TS} = \overline{TA}$, then H belongs to the open arc AS of the circle through A, B and S—the Connection Locus
– the centre C of the Connection Locus is the incentre of triangle OKJ
– $\triangle ABC$ is a right angle, which implies that the radius of the Connection Locus is equal to the side of a square with the same diagonal as that of the inscribing rectangle OATB
– if $P$ is the intersection between the parallel to OB through K and the smaller circle, then $P, H$ and $B$ are co-linear
– if $Q$ is the intersection between the parallel to OA through J and the bigger circle, then $Q, H$ and $A$ are co-linear
– since $\overline{A\hat{H}B}$ and $\overline{A\hat{S}B}$ cut the same chord $\overline{AB}$ in the CL circle, we have that $A\hat{H}B = \frac{3}{4}\pi$

As we will see in Chap. 3, there are some hidden constraints on the numbers involved, for example, there does not exist a corresponding quarter-oval for any points $A, B$ and $J$ satisfying the above conditions. Moreover other parameters involved, including angles for example, do play a part in the possibility of drawing an oval subject to specific constraints.

References
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