Using Closed Sets to Model Cognitive Behavior

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Abstract. We introduce closed sets, which we will call knowledge units, to represent tight collections of experience, facts, or skills, *etc.* Associated with each knowledge unit is the notion of its generators consisting of those attributes which characterize it.

Using these closure concepts, we then provide a rigorous mathematical model of learning in terms of continuous transformations. We illustrate the behavior of transformations by means of closure lattices, and provide necessary and sufficient criteria for simple transformations to be continuous. By using a rigorous definition, one can derive necessary alternative properties of cognition which may be more easily observed in experimental situations.

1 Introduction

We are concerned with modeling intelligence and learning, but not *artificial* intelligence or *machine* learning. Rather we want to model these phenomena as they might occur in a human mind. It is generally accepted that mental cognition occurs in the brain, which is itself comprised of a network of neurons, axons, and synapses. Neuroscientists have a rather clear understanding of the physical layout of the brain, including which portions are responsible for individual mental functions [6]. But, how mental processes actually occur is still elusive. Nevertheless, it is clear that the response to external stimuli occurs in a reactive network. Thus if we want to model cognitive behavior we must, at some level, come to grips with network behavior.

In Sect. 2, we will introduce the idea of an experiential operator, ρ, which expresses a relationship between the elements of a network. The elements can be raw visual stimuli, at a lower level, or concepts and ideas, at a higher level.

In Sect. 3 we introduce the concept of *closure*, which identifies closely related elements. Closure is central to our mathematics. Then, for want of a better term we call closed sets, *knowledge units*. Properties of these knowledge units are developed in Sect. 4.

It is not until Sect. 5 that we actually encounter network transformations that correspond to learning situations, and define the concept of *continuity*. We will examine several continuous network transformations and provide necessary and sufficient conditions for a simple transformation to be continuous. This section is the meat of the paper.

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2 The Experiential Operator

Let U denote a finite universe of awarenesses, sensations, *etc.* that an individual might experience, $U = \{\ldots, w, x, y, z\}$.[1] We denote sets by $\{\ldots\}$ and by upper case letters. Thus $Y = \{x, z\}$ is a set of two possible experiences in U. Y is said to be a subset of (or contained in) U, denoted $Y \subseteq U$.

Experiences are related to one another. If z is related to x, say for example that z can be experienced having once experienced x, we denote the relationship by $x \ \rho \ z$. Relationships may, or may not, be symmetric; we need not have $z \ \rho \ x$. Based on known neural morphology [6], most neural cells have many inputs and relatively few outputs, so we can assume many relationships will be asymmetric. Relationships come in a great many varieties. Experiential events can be simultaneous or sequenced in time; can be adjacent or distant in space; can be synonyms or antonyms in a lexical space; or can be friendly or threatening in an emotional space. But for this paper we assume only one generic relationship. By ρ we mean that some relationship exists. Throughout this paper we are going to let the term "experience" be generic. We might have related visual stimuli comprising a visual object, or related skills comprising a skill set, or related facts comprising an area of knowledge. All will be regarded as experiential.

Relationships are frequently visualized by means of graphs, or networks, such as Fig. 1. Here an edge between q and s denotes $q \ \rho \ s$. If no arrow head is present, it is assumed that the relation is symmetric.

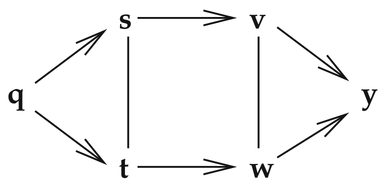

Fig. 1. A very small network depicting the relationships, ρ, between 6 experiential elements.

While network graphs can provide a valuable intuition, we actually prefer to regard relationships as operators that map subsets of U onto other subsets in U. Thus we will denote $q \ \rho \ s$ by the expression $\{q\}.\rho = \{s\}$, that is, ρ operating on q yields s, or because we tacitly assume q is related to itself, and because $q \ \rho \ t$, $\{q\}.\rho = \{q, s, t\}$. In Fig. 1, $\{s\}.\rho = \{s, t, v\}$ and $\{t\}.\rho = \{s, t, w\}$. Using this kind of suffix notation is a bit unusual, but it has value. One reason for preferring an operator notation is that in order to experience y it may be necessary to first experience both v and w, that is, $y \in \{v, w\}.\rho$, but $y \notin \{v\}.\rho$. For example, for a neuron y to respond, it may need signals from both v and w. So properly,

[1] This finiteness constraint can be relaxed somewhat, but there is relatively little yield for the resulting complexity.

ρ is a function on sets, not individual elements of U. A second reason is that in later sections we will compose the functional operators, and suffix notation lets us read the composition in a natural left to right manner.

To formalize this, we let 2^U denote all possible combinations of "experiences" in the universe U. Mathematically, it is called the **power set** of U. The relationship operator, ρ, maps subsets $Y \subseteq U$ into other subsets, $Z = Y.\rho \subseteq U$. By convention we assume that every experience is related to itself, so that, for all Y, $Y \subseteq Y.\rho$. Consequently, ρ is an **expansive** operator. This is precisely what we want; ρ denotes the possibility of expanding one's realm of experiences. For example, having the experiences x and y, it may be possible to also experience z, or $\{x, y\}.\rho = \{x, y, z\}$.

We will also assume that a greater collection of experience will permit a greater awareness of possible new experience. That is, $X \subseteq Y$ implies $X.\rho \subseteq Y.\rho$. Then ρ is said to be a **monotone** operator.

3 Closure Operators and Knowledge Units

Certain collections of experiences, of facts, of abilities, appear to be more robust than others. They go by many names in the literature. A cluster of perceived visual stimuli may be called an *external entity*, or *object*. If the granularity of the base experiential elements, U, is coarser, say that of *skills* or *facts*, we might call a cluster of abilities an *area of expertise*, such as *horseshoeing*; or a cluster of facts might be regarded as a *discipline*, such as *medieval history* or *high school algebra*. With so many possible terms and interpretations, we choose to use a more neutral term. We will call such clusters *knowledge units* without trying to specify precisely what such a unit is. In this section we will postulate that this organizing process can be approximately modeled by a mathematical *closure* operator.

An operator φ is said to be a **closure operator** if for all $X, Y \subseteq U$,

$Y \subseteq Y.\varphi$	φ is **expansive**,
$X \subseteq Y$ implies $X.\varphi \subseteq Y.\varphi$	φ is **monotone**, and
$Y.\varphi.\varphi = Y.\varphi$	φ is **idempotent**.

There is an extensive literature on closure and closure operators of which [2,5,9,12,14] are only representative.

Since ρ is both expansive and monotone, it is almost a closure operator itself. But, ρ need not be idempotent. In Fig. 1, we have $\{q\}.\rho = \{qst\} \subset \{qstuv\} = \{q\}.\rho.\rho$. However, we can define a closure operator φ_ρ with respect to ρ. Let,

$$Y.\varphi_\rho = \bigcup_{z \in Y.\rho} \{\{z\}.\rho \subseteq Y.\rho\}. \tag{1}$$

Readily, if $z \in Y$ then $z.\rho \subseteq Y.\rho$, so $Y \subseteq Y.\varphi$. We call φ_ρ the **experiential closure** because it is determined by the experiential operator ρ. Note that any relationship, ρ, of any type can give rise to a closure operator, φ_ρ.

Proposition 1. *φ_ρ is a closure operator.*

Proof. Readily, $Y \subseteq Y.\varphi_\rho$ by definition. Let $X \subseteq Y$ and let $z \in X.\varphi_\rho$. By (1) $z.\rho \subseteq X.\rho \subseteq Y.\rho$ hence $z \in Y.\varphi_\rho$. Now let $z \in Y.\varphi_\rho.\varphi_\rho$. Then $z.\rho \subseteq Y.\varphi_\rho.\rho = \bigcup_{z \in Y.\varphi_\rho}\{z.\rho \subseteq Y.\rho\}$, hence $z \in Y.\varphi_\rho$. □

In the network of Fig. 1, observe that $\{y\}$ is closed, but $\{v\}$ is not, because $\{y\}.\rho = \{y\} \subseteq \{vwy\} = \{v\}.\rho$, so $\{v\}.\varphi_\rho = \{vwy\}$. Neither is $\{w\}$ closed, because $\{w\}.\varphi_\rho = \{vwy\} = \{v\}.\varphi_\rho$. So, singleton elements need not be closed.

A set Y is said to be **closed** if $Y = Y.\varphi$. Because φ is expansive, U itself must be closed. The empty set, $\emptyset$, is most often closed, but need not be. (Here, $\emptyset$ denotes an "empty set" that contains no elements.)

Normally, we omit the subscript ρ from the closure symbol φ because most results are valid for all closure operators. Only if some property of the relational closure is required will we use the symbol φ_ρ.

By a **knowledge unit**, K_i, we mean a set closed with respect to φ_ρ in U. That is, the elements of K_i are a tightly bound collection of related experiences that will be regarded as a unit of *knowledge awareness*. In Fig. 1, because $\{st\}$ is closed, it is a knowledge set, K_1. The set $\{qst\}$ is also closed, and thus also a knowledge unit, K_2. Here, $K_1 = \{st\} \subset \{qst\} = K_2$. We can think of increasing knowledge awareness with increasing experience or capability.

3.1 An Example of Experiential Closure

The formal definition of experiential closure, φ_ρ, as well as the more general definition with respect to expansive, monotonicity, and idempotency, conveys little intuitive sense of its being. Here we will examine an example which could occur in human cognition.

Consider the retina of the eye, where the close packing of cells (frequently called "pixels", and here shown as hexagonal, even though the retina is never quite so regular) endows each receptive cell with 6 neighbors. Figure 2 illustrates a simulated portion of the retinal structure with a mottled pattern of 43 excited cells (black dots) which we will denote by Y. We seek an experiential closure of Y based on an adjacency relation, ρ. The pixels, or neural cells, containing an $\times$ in Fig. 3(a) denote the extent of $Y.\rho$. If all the neighbors of an $\times$-cell are also $\times$-cells, then it is in $Y.\varphi_\rho$ which is shown as Fig. 3(b). Surely, a process that can extract more "solid" objects in a natural kind of mottled camouflage will convey survival benefit, and might be "built-in".

It was shown in [16] that this spatial closure operator can be implemented in parallel by "expanding" each stimulated element in Y then expanding its complement thus contracting Y.

Since it is assumed that virtually all processing of information passing back from the retina to the visual cortex occurs in parallel; that spatial retinal relationships are preserved in some of this visual pathway; and that this pathway consists of alternating odd/even cell layers [18], it is plausible to regard this example as an actual, but vastly oversimplified, cognitive process.

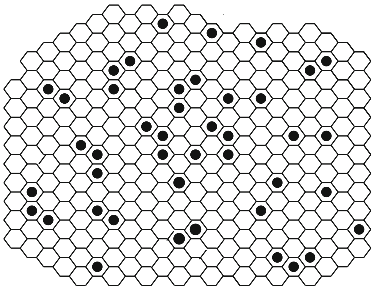

Fig. 2. A mottled pattern on a simulated retina.

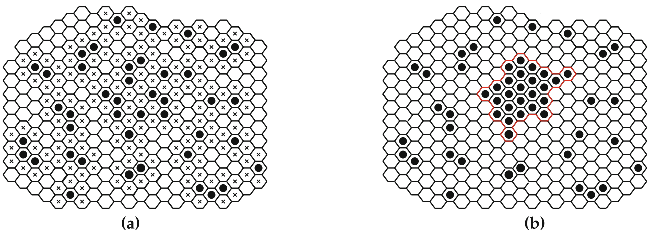

Fig. 3. Closure within the mottled pattern of Fig. 2

This example of a closure operator has been set within the context of visual cognition. It does not necessarily imply that this black and white "cartoon" example mimics an actual visual process. Real visual cognition is far more complex, for example, we see in multiple frequencies (color). But, it does establish that closure concepts are compatible with known aspects of visual physiology, and illustrates how a closure operator can extract "identifiable" objects from a pattern.

A well-known property of closure systems is that if X and Y are closed then their intersection $X \cap Y$ must be closed; or equivalently, $X.\varphi \cap Y.\varphi = (X \cap Y).\varphi$. Readily, we encounter many different kinds of experiential relationships in the real world, say $\rho_1, \rho_2, \ldots, \rho_n$. We can show by counter example that $X.\varphi_{\rho_1 \& \rho_2} \neq X.\varphi_{\rho_1} \cap X.\varphi_{\rho_2}$. But, for all X, $X.\varphi_{\rho_1} \cap X.\varphi_{\rho_2} = X.\varphi_{\rho_1 \cdot \rho_2}$. That is, the intersection of closed sets corresponds to closure based on concatenated, rather than concurrent, relationships, which seems to be what occurs in the visual pathway.

4 Generators and Knowledge Lattices

If K is a closed knowledge unit there exists at least one set $Y \subseteq K$ such that $Y.\varphi = K$. (It may be K itself.) Y is said to be a **generator** of K. A reasonable interpretation of generating sets is that these are a set of features of K that serve to characterize K.

Readily, the set Y is a generator of $Y.\varphi$, as is any set Z, $Y \subset Z \subseteq Y.\varphi$. If for all $X \subset Y$, $X.\varphi \subset Y.\varphi = K$ then Y is said to be a **minimal generator** of K.[2] In general, a closed set K may have several minimal generating sets, denoted $K.\Gamma = \{Y_1, \ldots Y_m\}$ where $Y_i.\varphi = K, 1 \leq i \leq m$. For example, in Fig. 1, $\{qv, qw\}$ are both minimal generators of $\{qstvwy\}$.

4.1 Knowledge Lattices

It is assumed that our knowledge is structured. One way of doing this is to partially order the knowledge units by containment to form a lattice. Because U itself must be closed (φ is expansive) and because $X \cap Y$ must be closed, any collection of discrete closed sets can be partially ordered by containment to form a complete lattice. We call them **knowledge lattices**, denoted $\mathcal{L}_\varphi$. Figure 4 illustrates the knowledge lattice, $\mathcal{L}_\varphi$, associated with the experiential operator, ρ, of Fig. 1. Doignon and Falmange called such lattices "knowledge spaces" [4]. This idea of *knowledge spaces* has generated a considerable amount of psychological literature.[3] Ganter and Wille [5] regard a lattice of closed sets as a "concept lattice". In both theories the lattice structure is central; for us, it will be important, but ancillary.

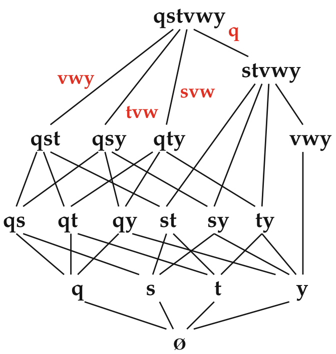

Fig. 4. Closed set lattice, $\mathcal{L}_\varphi$, of Fig. 1. Four set differences have been labeled.

A closed set K_m in $\mathcal{L}_\varphi$ is said to **cover** K_i if $K_i \subset K_m$ and there exists no set K_j such that $K_i \subset K_j \subset K_m$. That is, K_m is the next set above K_i in the lattice.[4] We can think of the **difference**, $K_m - K_i$, as being the skill/experience set differentiating an individual with knowledge unit K_i from one with K_m.

[2] If for all closed sets K, there is a *unique* minimal generating set, the closure operator is said to be *antimatroid*. While antimatroid knowledge systems, such as [4,5], are mathematically most interesting, they seem, in practice, to be most rare.

[3] Over 400 references can be found at the web site <cord.hockemeyer@uni-graz.at>.

[4] Because U is discrete, there always is a "next" set above K_i in $\mathcal{L}$, unless $K_i = U$, the maximal element.

In Fig. 4, $\{gstvwy\}-\{qst\} = \{vwy\}$ and $\{qstvwy\}-\{stvwy\} = \{q\}$. Explicitly showing the set differences as we have done in 4 instances in Fig. 4 can be an aid to understanding Proposition 2 which follows.

Proposition 2. *If a closed set K covers the closed sets $K_1, \ldots, K_m$ in $\mathcal{L}_\rho$, then X is a generator of K if and only if $X \cap (K - K_i) \neq \emptyset$ for all $1 \leq i \leq m$.*

Proof. A rigorous proof can be found in [8], here we present a more intuitive argument.

A knowledge unit is the smallest closed set containing some set, X, of experiences. Suppose X is a generator of K. Now, if X does not embrace at least one element from $K-K_i$ then $X.\varphi = K_i$, not K.

Conversely, $\mathcal{L}_\rho$ contains a number of knowledge units, K_i, and if X includes at least one experience that differentiates each one from K, then X must characterize K; it must be a generator. $\square$

That is, the generators of a knowledge unit are precisely those features which differentiate it from other knowledge units in the lattice. By Proposition 2, if one knows the generators of a closed knowledge unit, one knows the closed sets it covers, and conversely given the lattice of closed sets one can determine all the generators. It is worthwhile convincing oneself of this unusual result by actual trial. In Fig. 4, $\{qstvwy\}$ covers $\{qst\}$, $\{qsy\}$, $\{qty\}$, and $\{stvwy\}$ with respective differences being $\{vwy\}$, $\{tvw\}$, $\{svw\}$, and $\{q\}$. Using Fig. 1, convince yourself that both of the sets $\{qv\}$ and $\{qw\}$, each of which intersect all four set differences are actually generators of $\{qstvwy\}$.

Suppose U consists of visual stimuli. If X generates K, a closed set of related stimuli, constituting a visual object, then X consists of those visual attributes that characterize the object; and differentiate it from other similar objects, K_i. On the other hand, if K represents an ability level in high-school algebra, as in [4], then X represents those skills necessary to advance from lesser sets of algebraic abilities, K_i to K. Finally, if K represents knowledge of the Napoleonic wars, then questions embodying the facts found in a generator, X, would comprise an excellent test of the student's knowledge. The concept of generators resonates with many educational themes depending on the network granularity.

Experiential networks are real. The neural networks of the mind are real; our social networks are real; the related collections of facts we call knowledge are real. Our rendition of these real networks by ρ may be an over simplification; but it is an abstract depiction of real phenomena. In contrast, these *knowledge lattices* are **not** real. They have no existential counterpart that we know of. They are purely a mathematical construct designed to help us understand the organization and structure of real networks; and in the next section, to help us understand how their structure can change under dynamic transformation. This is an important distinction. While in this section, and the next, we may seem to be fixated on these *knowledge lattices*. We are really most concerned about the underlying network of experiential relationships.

Do the concepts of *closure* and *generators* correspond to real phenomena? Even though we have no compelling proof, we believe they do. It seems clear that our minds are capable of identifying and labeling, in some fashion, related collections of experiential input. Several cognitive psychologists have emphasized this fact. Objects that are linguistic nouns appear to invariably behave as closed concepts, with adjectives often fulfilling the role of generating features. Replacing a cluster of primitive experiential elements with a single label can optimize neuron use because it facilitates the internal representation at a coarser granularity. It seems necessary for "abstract" thought.

Similarly, it seems apparent that the mind, on many levels, apprehends objects and abstractions of the real world by abbreviated collections of salient features. This, too, represents an economical use of neurons — which must be important to all organisms. Whether *generators* exactly model this phenomenon is unclear; but surely they represent an abstraction of this capability.

Our imposition of a formal lattice structure as a mathematical device to comprehend the organization of experiential networks may be a major contribution of this paper. In the following sections we will see where this leads us.

5 Transformation as Learning

The notion of *transformation* is a familiar one in educational psychology; for example, the process of *internalization* has been described by the Russian psychologist, Lev Vygotsky, as a "series of transformations" [3]. In this section we will develop the idea of transformation as a mathematical function. Most of us are familiar with polynomial functions, which describe numerical change — the speed of a falling object is a quadratic function of it's time of flight. But now, we let a transformation be a function that describes a change of *structure*. It requires a different mathematical mind set. It is one reason we use suffix notation.

By a **transformation**, $U \xrightarrow{f} U'$, we mean a function f which for every set $Y \subseteq U$, assigns a set $Y.f = Y' \subseteq U'$. (We use Y' to denote the image of Y in U'). Of most interest will be the effect, $K.f$ of transforming closed knowledge units, and how the transformation will affect their relationship with other knowledge units, $K_i.f$. The importance of using a power set as the domain and codomain of a transformation is that elements can be functionally inserted or removed from the system. For example, consider the transformation f depicted by Fig. 5 which adds a completely new element, r, to the network of Fig. 1. That is, $\emptyset.f = \{r\}$, so $\{y\}.f = \{ry\}$, and all closed sets containing q now contain $\{qr\}$.

In the mathematics of the real line, the behavior of functions is typically visualized by the familiar graph plotting the value $y = f(x)$ for all x along the x-axis. When the function is defined on sets of discrete elements a different approach must be taken. We prefer to illustrate its behavior by what happens to the closed set/knowledge lattice. Although f must be defined for all sets, $Y \subseteq U$, we use only these closed sets to visualize the process. In Fig. 5 the lower transformation $\mathcal{L}_\varphi \xrightarrow{f*} \mathcal{L}'_{\varphi'}$ illustrates its behavior with respect to the knowledge lattice. This transformation, f, is a classic example of a smooth, well-behaved lattice morphism.

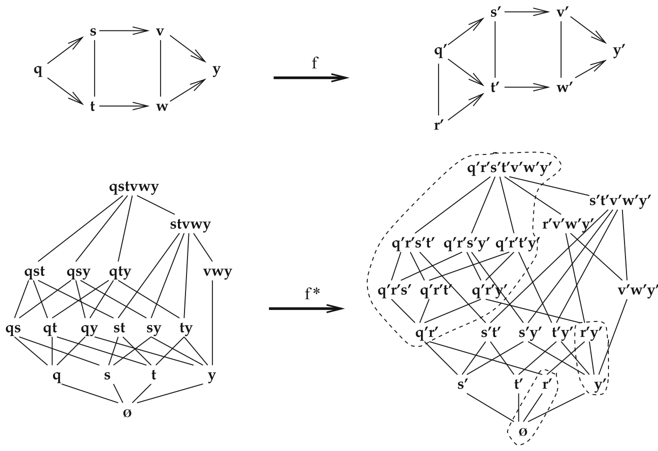

Fig. 5. A transformation $U \xrightarrow{f} U'$ that adds a completely new element r' to the network of Fig. 1.

A transformation $U \xrightarrow{f} U'$ is said to be **monotone** if for all sets X, Y in U, $X \subseteq Y$ implies $X.f \subseteq Y.f$. Monotonicity is essential throughout the following mathematical approach.[5] Observe that the transformation f of Fig. 5 is monotone, in that $K_i \subseteq K_m$ in $\mathcal{L}_\varphi$ implies $K_i.f \subseteq K_m.f$ in $\mathcal{L}'_{\varphi'}$.

5.1 Continuous Transformations

In high school we are told that a "continuous" function, $f(x)$, is one whose graph can be drawn without lifting one's pencil from the paper. The more precise definition encountered in real analysis is quite analogous to the definition that follows.[6] A discrete transformation, $U \xrightarrow{f} U'$, is said to be **continuous** if for all $Y \subseteq U$,

$$Y.\varphi.f \subseteq Y.f.\varphi' \tag{2}$$

This is the traditional definition of continuity for functions on discrete spaces [9–11,20,21]. Yet this short equation conveys little intuitive sense of its import. The transformation f of Fig. 5 is continuous; it is "smooth". Continuity takes on additional importance when viewed as a function on knowledge lattices.

[5] In artificial intelligence (A.I.), learning is said to be "monotonic" if no new piece of information can invalidate any existing "knowledge" as represented by a set of rules. That concept of knowledge involves a notion of logical contradiction, not just the simple inclusion or deletion of experiential input. There is an abundance of literature about A.I. architectures which support both monotonic and non-monotonic reasoning [13,17]. Our use of the term is rather different.

[6] A real function $y = f(x)$ is said to be continuous if for any open set O_y containing y, there exists an open set O_x containing x such that $f(O_x) \subseteq O_y = O_{f(x)}$, or using suffix notation $x.O.f \subseteq y.f.O'$.

It effectively asserts that if a learning transformation is continuous, it only expands the knowledge units of an individuals experiential awareness. That is, if $K = Y.\varphi$ then $K.f \subseteq Y.f.\varphi_{\rho'} = K'$.

Before considering more fully what comprises continuous transformations in a cognitive context it can be valuable to examine the purely formal characteristics of continuity.

Proposition 3. *Let* $(U, \varphi) \xrightarrow{f} (U', \varphi')$, $(U', \varphi') \xrightarrow{g} (U'', \varphi'')$ *be monotone transformations. If both* f *and* g *are continuous, then so is* $U \xrightarrow{f \cdot g} U''$.

Proof. We have $X.\varphi.f \subseteq X.f.\varphi'$ for any $X \in U$ and $Y.\varphi'.g \subseteq Y.g.\varphi''$ for any $Y \in U'$. Consequently, as g is monotone, $X.\varphi.f.g \subseteq X.f.\varphi'.g \subseteq X.f.g.\varphi''$. Thus $f \cdot g$ is continuous. □

Proposition 4. *Let* $(U, \varphi) \xrightarrow{f} (U', \varphi')$ *be monotone, continuous and let* $Y.f = Y'$ *be closed. Then* $Y.\varphi.f = Y'$.

Proof. Let $Y.f$ be closed in U'. Because f is continuous $Y.\varphi.f \subseteq Y.f.\varphi' = Y.f$, since $Y.f$ is closed. By monotonicity, $Y.f \subseteq Y.\varphi.f$, so $Y.\varphi.f = Y.f$. □

Proposition 5. *Let* $(U, \varphi) \xrightarrow{f} (U', \varphi')$ *be monotone. Then* f *is continuous if and only if* $X.\varphi = Y.\varphi$ *implies* $X.f.\varphi' = Y.f.\varphi'$.

Proof. Let f be continuous, and let $X.\varphi = Y.\varphi$. By monotonicity and continuity, $X.f \subseteq X.\varphi.f = Y.\varphi.f \subseteq Y.f.\varphi'$. Similarly, $Y.f \subseteq X.f.\varphi'$. Since $Y.f.\varphi'$ is the smallest closed set containing $X.f$ and $X.f.\varphi'$ is the smallest closed set containing $Y.f$, $X.f.\varphi' = Y.f.\varphi'$.

Conversely, assume f is not continuous. So there exists Y with $Y.\varphi.f \not\subseteq Y.f.\varphi'$ There exists $X \in Y.\varphi^{-1}$. $X.f \subseteq X.\varphi.f = Y.\varphi.f \not\subseteq Y.f.\varphi'$, so $X.f.\varphi' \neq Y.f.\varphi'$, contradicting the condition. □

Corollary 1. *If* $(U, \varphi) \xrightarrow{f} (U', \varphi')$ *is a monotone, continuous transformation and* X *generates* K ($X.\varphi = K$) *then* $X.f$ *generates* $K.f.\varphi'$.

Note that even though f is monotone and continuous, and K is closed with respect to φ, $K.f$ need not be closed with respect to φ'. However, by Corollary 1, $K.f$ must be a generating set of $K.f.\varphi'$.

Continuous transformations are very well-behaved with other demonstrable properties, *c.f.* [11]. It is our conjecture that continuous transformations of a human's experiential network (as exemplified by ρ) corresponds to our "natural" reaction to new experience and stimuli. It is an, almost automatic, response to novel experiences.

5.2 Small Incremental Change

The key to continuous learning is not just exposure to new experience, but how that new experience is integrated with other related experience. It has been suggested that new experience, new stimuli, is integrated into our memory, or knowledge structure, as we sleep. Apparently this occurs through the creation of new axons and synaptic connections [1]. Some researchers believe that the elimination of connections may be as equally important as creating new ones [19].

It was shown in [14], that if a discontinuity exists, it will manifest itself at a single experiential event.

Proposition 6. *If there exists Y such that $Y.\varphi.f \not\subseteq Y.f.\varphi'$ then there exists a singleton set $\{y\} \subseteq Y.\rho$ such that $\{y\}.\varphi.f \not\subseteq \{y\}.f.\varphi'$.*

This makes testing for continuity viable.

The following two propositions characterize continuous transformations that add, or delete, edges/relationships within a network. In both Propositions 7 and 8, we assume that $U' = U$, and that f is the identity function on $\mathcal{L}_\varphi$, and that $y' = \{y\}.f$ denotes the same node, but within the new structure of $\mathcal{L}'_{\varphi'}$. In the statement of these propositions we use the term $x.\eta$. By $Y.\eta$, which we call the **neighborhood** of Y, we mean the set $Y.\eta = Y.\rho - Y$, that is, the immediate neighbors of Y with respect to ρ.[7]

In Proposition 7 we show that new links can be continuously created between two experiential events x and z if there already exists a reasonably close relationship. Granovetter [7], and many other sociologists have observed this phenomenon.

Proposition 7. *Let $U \xrightarrow{f} U'$ be the identity transformation. If f adds an edge (x', z') to create a network ρ', it will be continuous at x if and only if for all $y \in x.\eta$, if $x \in y.\varphi$ then $z \in y.\rho$.*

Proof. Assume that $\exists y \in x.\eta, x \in y.\varphi$ but $z \notin y.\eta$. Since $x \in y.\varphi$, $x.\eta \subseteq y.\rho$. But, because $z \notin y.\eta$, $x'.\eta' \not\subseteq y'.\rho'$ and $y.\varphi.f \not\subseteq y.f.\varphi'$.

Conversely, assume f is discontinuous. First, we observe that $x.\varphi.f \subseteq x.f.\varphi'$, since the addition of an edge (x', z') cannot reduce the closure $x'.\varphi'$. So, f must be discontinuous at $y \in x.\eta$; that is, $\exists w \in y.\varphi$ such that $w' \notin y'.\varphi'$, because $w'.\eta \not\subseteq y'.\rho'$. Readily $w' = x'$ (or z'). After adding the edge (x', z'), $x'.\eta' \not\subseteq y'.\rho'$ only if $z' \notin y'.\eta$, that is $z \notin y.\eta$. $\square$

We say f is "discontinuous at x" even though the actual *discontinuity* may occur at $y \in \{x\}.\eta \subseteq \{x\}.\rho$ as noted in Proposition 6. This slight abuse of terminology allows us to focus on the structure surrounding the node x before (x', z') is created.

Observe that the creation of the link (t', v') in Fig. 6 is continuous because for $\{s, w\} \subseteq t.\rho$, we have $t \notin s.\varphi$ and $t \notin w.\varphi$, so Proposition 7 is satisfied vacuously.

Next we show that a link between two experiential events x and z can be continuously deleted if they are not too closely connected.

Proposition 8. *Let $U \xrightarrow{f} U'$ be the identity transformation. If f deletes an edge (x, z) from ρ', it will be discontinuous at x if and only if either*

(a) $z \in x.\varphi$ and $z.\varphi \neq x.\varphi$ or

(b) there exists $y \in x.\varphi$, with $z \in y.\eta$.

[7] Note that the η operator is normally neither expansive nor monotone.

Proof. Suppose (a), $z \in x.\varphi$. Since (x, z) is being deleted $z' \notin x'.\eta'$. Consequently, $\{x\}.\varphi.f \not\subseteq \{x'\}.f.\varphi'$. The last conjunct $x.\varphi \neq z.\varphi$ of condition (a) covers the special case described in [15].

Suppose (b) that $\exists y \in x.\varphi$ and $z \in y.\eta$. $\{y'\} \subseteq x.\varphi.f$, but $z' \notin x'.\eta'$ implies that $y'.\eta' \not\subseteq x'.\eta'$, hence $y' \notin x'.\varphi' = x.f.\varphi'$. Now, $\{x\}.\varphi.f \not\subseteq \{x\}.f.\varphi'$, and f is discontinuous.

Conversely, suppose f is not continuous at x. Then by Proposition 6, either (1) $\{x\}.\varphi.f \not\subseteq \{x\}.f.\varphi'$ or (2) for some $y \in \{x\}.\eta$, $\{y\}.\varphi.f \not\subseteq \{y\}.f.\varphi'$.

Assume the former, then $\exists$ some $w \in \{x\}.\varphi$ such that $w' = w.f \notin \{x\}.f.\varphi'$. Since (x, z) is the only edge being deleted, w must be z.

Now assume the latter. If $y \in \{x\}.\varphi$ then $y.\eta \subseteq x.\rho$. If $z \notin y.\eta$ then $\{y\}.\varphi.f \subseteq \{y\}.f.\varphi'$; but f is assumed to be discontinuous, so $z \in y.\eta$. $\square$

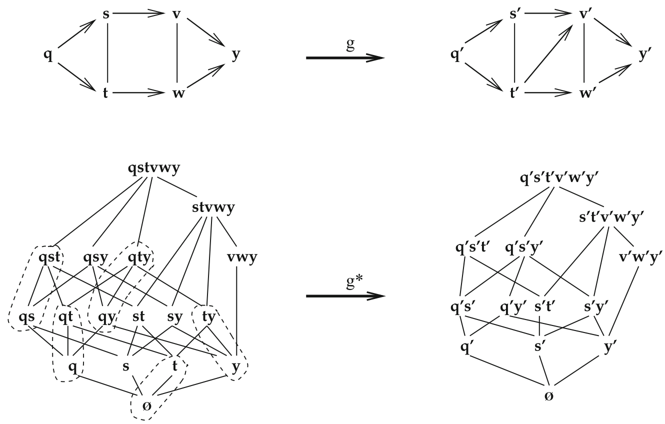

Fig. 6. A transformation g that adds a new connection (t', v') to the network of Fig. 1.

In Fig. 6, consider the inverse function, g^{-1} which removes the edge (t', v'). By Proposition 8, it is not continuous because $s' \in t'.\varphi'$ and $v' \in s'.\eta'$ satisfying condition (b) for discontinuity. We can verify the discontinuity, because $t'\varphi' = \{s't'\}$ in $\mathcal{L}'$, so $t'.\varphi'.g^{-1} = \{st\} \not\subseteq \{t\} = \{t'\}.g^{-1}.\varphi$.

If f and g are both continuous single edge additions or deletions, then by Proposition 3, their composition $f \cdot g$ is as well. It would be mathematically satisfying, if conversely every continuous restructuring of ρ could be decomposed into primitive single edge transformations; but in [14], it is shown that this need not be true.

6 Summary

Our goal has been to explore whether properties of closure operators and closed set systems can be relevant to modeling cognitive processes. We have presented ρ

as an *experiential* operator. We have considered closed sets as units of *knowledge* that can be characterized by their generators and partially ordered to form a *knowledge lattice* We have couched *learning* in terms of transformations.

Proposition 8 provides necessary and sufficient conditions for a specific kind of transformation which removes a link in a relationship to be continuous. It seems to be a widely held contention that *learning* involves the acquisition of more experiences and more data. In early childhood when our neural capabilities are growing this would seem so. But, even at an early age, children appear to be condensing raw stimuli into abstract identifiable concepts. In the process of *learning*, deletion seems to be as valuable as addition. In many forms of autism, it is the inability to delete and control an overload of raw sensory images that is problematic.

We believe we have demonstrated that an approach to network comprehension based on closed sets and continuous transformation can be a potentially valuable tool for modeling cognitive behavior. It will certainly take further refinement, including consideration of multiple experiential relationships, and considerable experimental testing to validate that claim.

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