Chapter 2
Motion Equations of Coriolis Vibratory Gyroscopes

The first and one of the most important steps in analysis of any mechanical system is to derive its motion equations. Such equations, being solved either numerically or in a closed form, allow all forms of dynamics analysis and design optimisation to be performed on the system under consideration. The central mechanical part of any Coriolis vibratory gyroscope is its sensitive element. No matter what specific design has been utilised, the sensitive element must be capable of providing at least two forms of mechanical oscillations: primary and one or more secondary. While the former is intentionally induced into the mechanical structure, the latter will appear due to the Coriolis force when the sensitive element is rotated.

In this chapter, we shall derive differential motion equations for sensitive elements of all commonly used designs and then generalise them to obtain a single set of equations suitable for different CVG designs.

2.1 Translational Sensitive Element Motion Equations

Translational motion sensitive element utilises translational motion for both primary and secondary oscillations. It is also commonly accepted to refer to the translational sensitive element as an LL-gyro (linear primary and linear secondary motion).

Figure 2.1 shows the schematic representation of the structure implementing translational type of kinematics. In the most generalised form, sensitive element consists of the proof mass (m_2), the decoupling frame (m_1) and two sets of elastic elements ("springs") connecting masses to each other and to the base. In addition to that, let us introduce the right-handed orthogonal and normalised reference frame *OXYZ*, in which primary oscillations are excited along the axis Y, then the secondary oscillations occur along the axis X and therefore the third axis Z is considered as a sensitive axis. The latter means that in an ideal case, external rotation around this axis will be sensed by the sensitive element.

Let us define position $\vec{X}_1$ of the decoupling frame and position $\vec{X}_2$ of the proof mass in the reference frame *OXYZ* as

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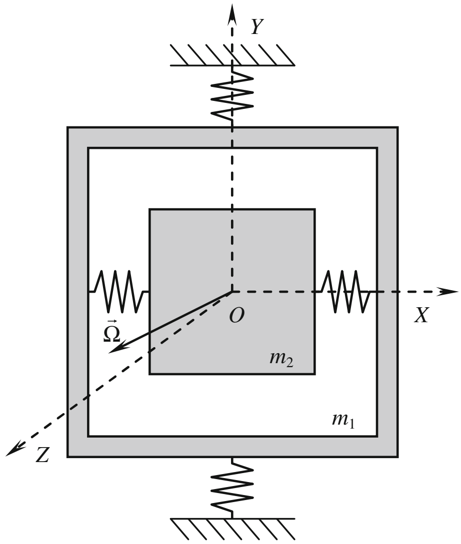

Fig. 2.1 Translational sensitive element of CVG

$$\begin{aligned}\vec{X}_1 &= \{0, x_1, 0\},\\ \vec{X}_2 &= \{x_2, x_1, 0\}.\end{aligned} \tag{2.1}$$

Here x_1 is the displacement of the decoupling frame relatively to the fixed base, and x_2 is the displacement of the proof mass relatively to the decoupling frame. Hereinafter, subscripts 1 and 2 stand for primary and secondary motions of the sensitive element and should not be confused with the axis number.

For the sake of generality, let us assume that the sensitive element rotation is given by an arbitrary angular rate vector $\vec{\Omega}$, which is defined by its projections on the introduced above reference frame as $\vec{\Omega} = \{\Omega_x, \Omega_y, \Omega_z\}$.

In order to derive motion equations of the sensitive element, let us use the Lagrange equation in the following form:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = Q_i. \tag{2.2}$$

Here $L = E_k - E_p$ is the Lagrange's function, E_k and E_p are the total kinetic and potential energies of the sensitive element, Q_i is the generalised force acting on the sensitive element, and subscript i ranges from 1 for the primary motion to the number secondary motions under consideration.

So to use the Lagrange Eq. (2.2), proper expressions for the kinetic and potential energies for the sensitive element must be obtained.

In the most generalised case, kinetic energy of a moving point mass m is

$$E_k = \frac{m}{2}(\vec{V} \cdot \vec{V}), \tag{2.3}$$

where $\vec{v}$ is the absolute velocity vector that can be expressed in terms of the point mass position $\vec{X}$ and angular rate $\vec{\Omega}$ of the reference frame rotation as

$$\vec{V} = \dot{\vec{X}} = \frac{\tilde{\mathrm{d}}}{\mathrm{dt}}\vec{X} + \vec{\Omega} \times \vec{X}. \tag{2.4}$$

Here $\frac{\tilde{\mathrm{d}}}{\mathrm{dt}}\vec{X} = \{\dot{X}_x, \dot{X}_y, \dot{X}_z\}$ is the local derivative of the vector $\vec{X}$ in the rotating frame. Velocities vectors for the decoupling frame and the proof mass positions (2.1) calculated using (2.4) are

$$\begin{aligned} \vec{V}_1 &= \{-x_1\Omega_z, \dot{x}_1, x_1\Omega_x\}, \\ \vec{V}_2 &= \{-x_1\Omega_z + \dot{x}_2, x_2\Omega_z + \dot{x}_1, x_1\Omega_x - x_2\Omega_y\}. \end{aligned} \tag{2.5}$$

Corresponding to (2.5), kinetic energies for primary and secondary motions are as follows:

$$\begin{aligned} E_{k1} &= \frac{m_1}{2}[\dot{x}_1^2 + x_1^2(\Omega_x^2 + \Omega_z^2)], \\ E_{k2} &= \frac{m_2}{2}[(\dot{x}_1 + x_2\Omega_z)^2 + (\dot{x}_2 - x_1\Omega_z)^2 + (x_1\Omega_x - x_2\Omega_y)^2], \\ E_k &= E_{k1} + E_{k2}. \end{aligned} \tag{2.6}$$

Here E_k is the total kinetic energy of the CVG sensitive element as a sum of the respective kinetic energies of the decoupling frame and the proof mass.

The second term in the Lagrange function is the total potential energy of springs in the elastic suspension of the sensitive element

$$E_p = \frac{k_1}{2}x_1^2 + \frac{k_2}{2}x_2^2, \tag{2.7}$$

where k_1 is the total stiffness of the elastic suspension along the axis Y (primary motion) and k_2 is the total stiffness along the axis X (secondary motion).

Combining expressions (2.6) and (2.7) into the Lagrange function and using the result with the Lagrange Eq. (2.2) gives us the following system of two differential equations, describing the motion of the CVG sensitive element:

$$\begin{cases} (m_1 + m_2)\ddot{x}_1 + [k_1 - (m_1 + m_2)(\Omega_x^2 + \Omega_z^2)]x_1 + m_2((\Omega_x\Omega_y + \dot{\Omega}_z)x_2 + 2\Omega_z\dot{x}_2) = Q_1, \\ m_2\ddot{x}_2 + [k_2 - m_2(\Omega_y^2 + \Omega_z^2)]x_2 + m_2(-2\Omega_z\dot{x}_1 + (\Omega_x\Omega_y - \dot{\Omega}_z)x_1) = Q_2. \end{cases}$$

These equations can now be rewritten by dividing both parts of the equations by its corresponding higher order derivative terms coefficients ($m_1 + m_2$ for the first equation and m_2 for the second). The result is

$$\begin{cases} \ddot{x}_1 + (\omega_1^2 - \Omega_x^2 - \Omega_z^2)x_1 + 2d\Omega_z\dot{x}_2 + d(\Omega_x\Omega_y + \dot{\Omega}_z)x_2 = q_1, \\ \ddot{x}_2 + (\omega_2^2 - \Omega_y^2 - \Omega_z^2)x_2 - 2\Omega_z\dot{x}_1 + (\Omega_x\Omega_y - \dot{\Omega}_z)x_1 = q_2, \end{cases} \tag{2.8}$$

where $\omega_1^2 = k_1/(m_1 + m_2)$ and $\omega_2^2 = k_2/m_2$ are the squared natural frequencies of the primary and secondary motions, respectively, $d = m_2/(m_1 + m_2)$ is the dimensionless inertia asymmetry factor, $q_1 = Q_1/(m_1 + m_2)$, $q_2 = Q_2/m_2$ are the generalised accelerations from different external forces that act along respective axes.

Equation (2.8) describes the motion of the generalised CVG with translational sensitive element. Note that if we simply assume that the mass of the decoupling frame is zero ($m_1 = 0$), then $d = 1$, and we can obtain motion equations for the single-mass CVG without decoupling frame, which, for example, corresponds to a vibrating beam design.

Finally, Eq. (2.8) must be completed by adding damping forces terms, producing translational sensitive element motion equations

$$\begin{cases} \ddot{x}_1 + 2\zeta_1\omega_1\dot{x}_1 + (\omega_1^2 - \Omega_x^2 - \Omega_z^2)x_1 + 2d\Omega_z\dot{x}_2 + d(\Omega_x\Omega_y + \dot{\Omega}_z)x_2 = q_1, \\ \ddot{x}_2 + 2\zeta_2\omega_2\dot{x}_2 + (\omega_2^2 - \Omega_y^2 - \Omega_z^2)x_2 - 2\Omega_z\dot{x}_1 + (\Omega_x\Omega_y - \dot{\Omega}_z)x_1 = q_2. \end{cases} \tag{2.9}$$

Here ζ_1 and ζ_2 are the dimensionless damping factors that correspond to the primary and secondary motions of the sensitive element.

Equation (2.9) contain all components of the angular rate vector $\vec{\Omega}$, but only component Ω_z is present as a first-order angular rate term and therefore will be properly measured by translational CVG. Let us now assume that the angular rate coincides with the axis Z, which means $\vec{\Omega} = \{0, 0, \Omega\}$. This assumption leads to the simpler form of motion equations as

$$\begin{cases} \ddot{x}_1 + 2\zeta_1\omega_1\dot{x}_1 + (\omega_1^2 - \Omega^2)x_1 = q_1 - 2d\Omega\dot{x}_2 - d\dot{\Omega}x_2, \\ \ddot{x}_2 + 2\zeta_2\omega_2\dot{x}_2 + (\omega_2^2 - \Omega^2)x_2 = q_2 + 2\Omega\dot{x}_1 + \dot{\Omega}x_1. \end{cases} \tag{2.10}$$

Taking a closer look at (2.10), one can see that in case of an ideal elastic suspension, where there is no cross-coupling between the decoupling frame and the proof mass, primary and secondary motions in the system are coupled only by means of the angular rate Ω. It means that given the absence of any external forces acting on the proof mass along generalised coordinate x_2 (e.g. $q_2 = 0$), any forced displacements in this direction will be caused by the angular rate alone. At the same time, angular rate in the Eq. (2.10) is unknown and, generally speaking, varies in time. This means that although the obtained system of sensitive element motion equations is linear, it contains unknown time-dependent coefficients. Needless to say that it is quite a complicated task to find a closed-form analytical solution for such system. However, the good news is that in order to analyse CVG dynamics and being able to optimise its performances, we do not need to solve these equations with respect to the sensitive element displacements x_1 and x_2.

2.2 Rotational Sensitive Element Motion Equations

Contrary to the translational sensitive element, rotational design utilises rotation for both primary and secondary oscillations. For this reason, it is often referred to as a RR-gyro (rotational primary and rotational secondary motion).

Similar to the translational sensitive element, Fig. 2.2 demonstrates rotational CVG sensitive element kinematics.

Linear (translational) springs are replaced with the rotational elastic torsions, and generalised coordinates now represent angles rather than translational displacements. Here α_1 corresponds to the angle between base and the decoupling frame, and α_2 corresponds to the angle between decoupling frame and the proof mass element. These angles are commonly referred to as Euler angles. Transformations of the reference frame axes after each of these rotations (primary and secondary) are demonstrated in Fig. 2.3.

Here reference frame $OX_1Y_1Z_1$ is the result of rotating the reference frame $OXYZ$ around Y axis by the angle α_1, and is fixed to the decoupling frame. Similarly, reference frame $OX_2Y_2Z_2$ is obtained by rotating frame $OX_1Y_1Z_1$ by the angle α_2, and is fixed to the proof mass element.

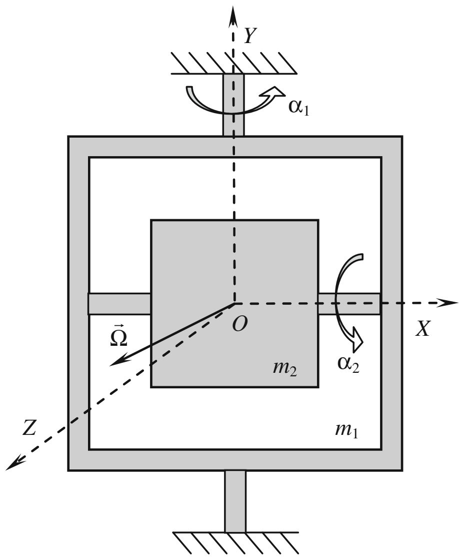

Fig. 2.2 Rotational sensitive element of CVG

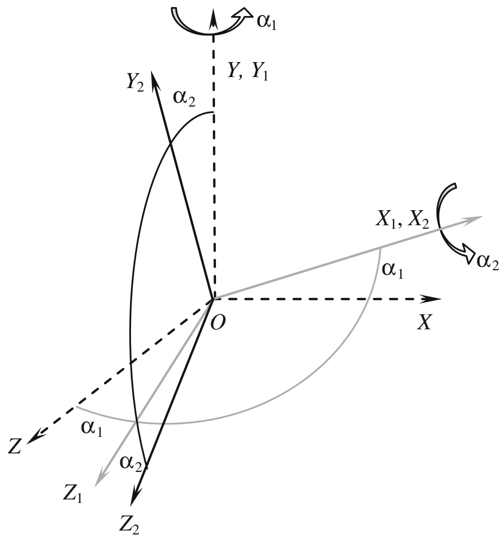

Fig. 2.3 Axes transformations due to the primary and secondary motion

In order to derive rotational CVG sensitive element motion equation, we need to start again with the proper expressions for kinetic and potential energies.

The difference with the previous case is that the motion is rotational now, and the corresponding kinetic energy is expressed in terms of angular rate instead of velocities and moments of inertia instead of mass:

$$E_k = \frac{I}{2}\Omega^2. \tag{2.11}$$

Here I is the moment of inertia of the rotating element around axis, which coincides with the direction of the angular rate vector $\vec{\Omega}$. Apparently, this case is slightly more complicated comparing to the translational motion of the CVG sensitive element, since proper moments of inertia are applied to components of the angular rate in different reference frames.

Initially, external angular rate vector $\vec{\Omega}$ is defined by its components $\vec{\Omega} = \{\Omega_x, \Omega_y, \Omega_z\}$ in the reference frame $OXYZ$. Transformation of its components into the reference frame $OX_1Y_1Z_1$, fixed to the decoupling frame, are given by the following expressions:

$$\begin{aligned} \Omega_{x1} &= \Omega_x \cos\alpha_1 - \Omega_z \sin\alpha_1, \\ \Omega_{y1} &= \Omega_y + \dot{\alpha}_1, \\ \Omega_{z1} &= \Omega_x \sin\alpha_1 + \Omega_z \cos\alpha_1. \end{aligned} \tag{2.12}$$

Further transformation of the angular rate into the reference frame $OX_2Y_2Z_2$ assigned to the proof mass element is performed in a similar way:

$$\begin{aligned}\Omega_{x2} &= \Omega_{x1} + \dot{\alpha}_2,\\ \Omega_{y2} &= \Omega_{y1}\cos\alpha_2 + \Omega_{z1}\sin\alpha_2\\ \Omega_{z2} &= -\Omega_{y1}\sin\alpha_2 + \Omega_{z1}\cos\alpha_2.\end{aligned} \tag{2.13}$$

Kinetic energy (2.11) can now be expressed in terms of different angular rates components (2.12) and (2.13) as

$$E_k = \frac{1}{2}\left(I_{1x}\Omega_{x1}^2 + I_{1y}\Omega_{y1}^2 + I_{1z}\Omega_{z1}^2 + I_{2x}\Omega_{x2}^2 + I_{2y}\Omega_{y2}^2 + I_{2z}\Omega_{z2}^2\right). \tag{2.14}$$

Here I_{ix}, I_{iy}, I_{iz} are the moments of inertia of the ith element ($i = 1$ corresponds to the decoupling frame, $i = 2$ corresponds to the proof mass element) around respective axis of the corresponding reference frame.

Potential energy of the sensitive element is similar to the case of translational motions except for the angular stiffness of the torsions instead of the springs:

$$E_p = \frac{k_1}{2}\alpha_1^2 + \frac{k_2}{2}\alpha_2^2. \tag{2.15}$$

Here k_i are the angular spring constants of the elastic suspension.

Substituting obtained expressions for the kinetic (2.14) and potential (2.15) energies in the Lagrange Eq. (2.2) results in rather big equations that are difficult to analyse. At the same time, these equations are nonlinear due to the presence of sine and cosine functions of the generalised variables α_1 and α_2.

However, taking into consideration that elastic suspensions usually do not allow big angular deflections, it is reasonable to assume that angles α_1 and α_2 are small. Hence, obtained motion equations can be linearized using Taylor series expansion that results in the following linear in terms of angles equations:

$$\begin{cases}(I_{1y} + I_{2y})(\ddot{\alpha}_1 + \dot{\Omega}_y) + [k_1 + (I_{1x} - I_{1z} + I_{2x} - I_{2z})(\Omega_x^2 - \Omega_z^2)]\alpha_1\\ \quad + (I_{2x} + I_{2y} - I_{2z})\Omega_z\dot{\alpha}_2 - (I_{2y} - I_{2z})(\Omega_x\Omega_y - \dot{\Omega}_z)\alpha_2\\ \quad + (I_{1x} - I_{1z} + I_{2x} - I_{2z})\Omega_x\Omega_z = Q_1,\\ I_{2x}(\ddot{\alpha}_2 + \dot{\Omega}_x) + [k_2 + (I_{2y} - I_{2z})(\Omega_y^2 - \Omega_z^2)]\alpha_2\\ \quad - (I_{2x} + I_{2y} - I_{2z})\Omega_z\dot{\alpha}_1 + (I_{2y} - I_{2z})\Omega_x\Omega_y\alpha_1 - I_{2x}\dot{\Omega}_z\alpha_1\\ \quad - (I_{2y} - I_{2z})\Omega_y\Omega_z = Q_2.\end{cases} \tag{2.16}$$

Here Q_1 and Q_2 are the generalised torques acting around primary and secondary rotations. Let us now divide both sides of the equations by the corresponding highest derivative coefficients:

$$\begin{cases} \ddot{\alpha}_1 + \dot{\Omega}_y + \left[\dfrac{k_1}{I_{1y}+I_{2y}} + \dfrac{I_{1x}-I_{1z}+I_{2x}-I_{2z}}{I_{1y}+I_{2y}}(\Omega_x^2-\Omega_z^2)\right]\alpha_1 \\ + \dfrac{I_{2x}+I_{2y}-I_{2z}}{I_{1y}+I_{2y}}\Omega_z\dot{\alpha}_2 - \dfrac{I_{2y}-I_{2z}}{I_{1y}+I_{2y}}(\Omega_x\Omega_y - \dot{\Omega}_z)\alpha_2 \\ + \dfrac{I_{1x}-I_{1z}+I_{2x}-I_{2z}}{I_{1y}+I_{2y}}\Omega_x\Omega_z = \dfrac{Q_1}{I_{1y}+I_{2y}}, \\ \ddot{\alpha}_2 + \dot{\Omega}_x + \left[\dfrac{k_2}{I_{2x}} + \dfrac{I_{2y}-I_{2z}}{I_{2x}}(\Omega_y^2-\Omega_z^2)\right]\alpha_2 \\ - \dfrac{I_{2x}+I_{2y}-I_{2z}}{I_{2x}}\Omega_z\dot{\alpha}_1 + \dfrac{I_{2y}-I_{2z}}{I_{2x}}\Omega_x\Omega_y\alpha_1 - \dot{\Omega}_z\alpha_1 \\ - \dfrac{I_{2y}-I_{2z}}{I_{2x}}\Omega_y\Omega_z = \dfrac{Q_2}{I_{2x}}. \end{cases} \tag{2.17}$$

The following new variables can now be introduced in order to simplify Eq. (2.17):

$q_1 = Q_1/(I_{1y}+I_{2y})$, $q_2 = Q_2/I_{2x}$—are the generalised angular accelerations from the corresponding external torques, $\omega_1^2 = k_1/(I_{1y}+I_{2y})$ and $\omega_2^2 = k_2/I_{2x}$—are the squared natural frequencies of the primary and secondary motions correspondingly, $g_1 = (I_{2x}+I_{2y}-I_{2z})/(I_{1y}+I_{2y})$ and $g_2 = (I_{2x}+I_{2y}-I_{2z})/I_{2x}$—are the gyroscopic Coriolis coefficients, $d_1 = (I_{1x}-I_{1z}+I_{2x}-I_{2z})/(I_{1y}+I_{2y})$, $d_2 = (I_{2y}-I_{2z})/I_{2x}$, $d_3 = (I_{2y}-I_{2z})/(I_{1y}+I_{2y})$—are the coefficients that can be seen as sensitive element design parameters along with the gyroscopic coefficients. All these coefficients allow to rewrite Eq. (2.17) as follows:

$$\begin{cases} \ddot{\alpha}_1 + 2\zeta_1\omega_1\dot{\alpha}_1 + [\omega_1^2 + d_1(\Omega_x^2-\Omega_z^2)]\alpha_1 + g_1\Omega_z\dot{\alpha}_2 \\ - d_3(\Omega_x\Omega_y - \dot{\Omega}_z)\alpha_2 + d_1\Omega_x\Omega_z + \dot{\Omega}_y = q_1, \\ \ddot{\alpha}_2 + 2\zeta_2\omega_2\dot{\alpha}_2 + [\omega_2^2 + d_2(\Omega_y^2-\Omega_z^2)]\alpha_2 - g_2\Omega_z\dot{\alpha}_1 \\ + d_2\Omega_x\Omega_y\alpha_1 - \dot{\Omega}_z\alpha_1 - d_2\Omega_y\Omega_z + \dot{\Omega}_x = q_2. \end{cases} \tag{2.18}$$

Here damping terms were also added that are characterised by the dimensionless damping factors ζ_1 and ζ_2. Finally, similarly to the case of translational CVG sensitive element, we can assume that angular rate coincides with the axis Z (e.g. $\vec{\Omega} = \{0, 0, \Omega\}$). This gives us the much simpler form of the Eq. (2.18):

$$\begin{cases} \ddot{\alpha}_1 + 2\zeta_1\omega_1\dot{\alpha}_1 + (\omega_1^2 - d_1\Omega^2)\alpha_1 = q_1 - g_1\Omega\dot{\alpha}_2 - d_3\dot{\Omega}\alpha_2, \\ \ddot{\alpha}_2 + 2\zeta_2\omega_2\dot{\alpha}_2 + (\omega_2^2 - d_2\Omega^2)\alpha_2 = q_2 + g_2\Omega\dot{\alpha}_1 + \dot{\Omega}\alpha_1. \end{cases} \tag{2.19}$$

Rotational CVG sensitive element motion equations in the form (2.19) have exactly the same structure as the Eq. (2.10) for the translational CVG. The only difference is in the coefficients and meaning of the generalised coordinates, which describe either translational or rotational motion. This fact will allow us later to produce a single set of CVG sensitive element motion equations, leading to unified analysis of both types of designs.

2.3 Tuning Fork Sensitive Element Motion Equations

We have already derived motion equations for CVG sensitive elements utilising either translational or rotational motion for both primary and secondary modes. Let us now move on to the design that uses combination of translational and rotational motions, namely "tuning fork" CVG. Kinematics scheme of the tuning fork sensitive element is shown in Fig. 2.4.

Tuning fork sensitive element consists of two identical proof masses m_1 and m_2 attached using linear springs to the common frame m_3, which is attached to the base by means of elastic torsions. Proof masses are able to move along axis Y, and when the sensitive element rotates around axis Z, due to the Coriolis force the whole sensitive element starts to rotate around axis Z (angle α). For the sake of simplicity, it is also assumed that the external angular rate is perfectly aligned with the axis Z ($\vec{\Omega} = \{0, 0, \Omega\}$. Position of the proof masses in the reference frame fixed to the common frame are given by the following two vectors:

$$\begin{aligned} \vec{X}_1 &= \{0, -r + y_1, 0\}, \\ \vec{X}_2 &= \{0, r + y_2, 0\}. \end{aligned} \tag{2.20}$$

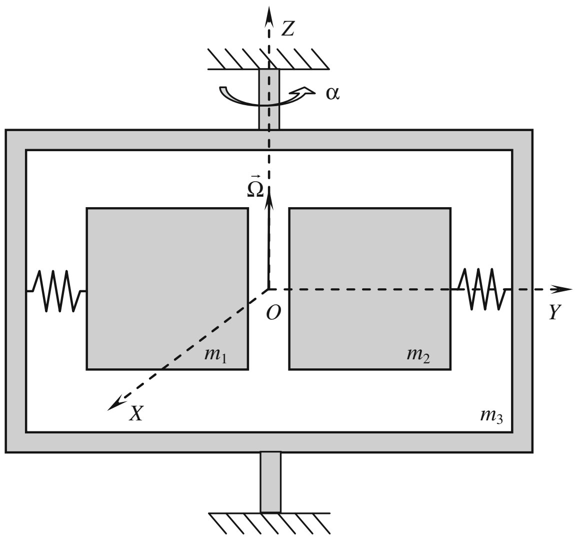

Fig. 2.4 Tuning fork sensitive element

Here r is the constant distance from the axis Z to the centres of the proof masses when they are not moving, and the motion is described by the displacements y_1 and y_2. Total angular rate of the frame rotation is

$$\vec{\Omega}_3 = \{0, 0, \Omega + \dot{\alpha}\}. \tag{2.21}$$

Corresponding to (2.20) velocity vectors are as follows:

$$\begin{aligned} \vec{V}_1 &= \{(r - y_1)(\Omega + \dot{\alpha}), \dot{y}_1, 0\}, \\ \vec{V}_2 &= \{-(r + y_2)(\Omega + \dot{\alpha}), \dot{y}_2, 0\}. \end{aligned} \tag{2.22}$$

Total kinetic energy of the sensitive element obtained using velocities (2.22) for the expression (2.3) (translational motion) and angular rate (2.21) for the expression (2.11) (rotational motion) is

$$\begin{aligned} E_k = &\frac{m_1}{2}\left[(r - y_1)^2(\Omega + \dot{\alpha})^2 + \dot{y}_1^2\right] + \frac{m_2}{2}\left[(r + y_2)^2(\Omega + \dot{\alpha})^2 + \dot{y}_2^2\right] \\ &+ \frac{I_{3z}}{2}(\Omega + \dot{\alpha})^2. \end{aligned} \tag{2.23}$$

Here I_{3z} is the moment of inertia of the common frame around axis Z. Potential energy of the elastic suspension is

$$E_p = \frac{k_1}{2}y_1^2 + \frac{k_2}{2}y_2^2 + \frac{k_3}{2}\alpha^2. \tag{2.24}$$

Finally, substituting energies (2.23) and (2.24) into the Lagrange Eq. (2.2) results in the following tuning fork sensitive element motion equations:

$$\left\{ \begin{aligned} m_1\ddot{y}_1 + (k_1 - m_1\Omega^2)y_1 + 2m_1 r\Omega\dot{\alpha} + m_1 r\Omega^2 &= Q_1, \\ m_2\ddot{y}_2 + (k_2 - m_2\Omega^2)y_2 - 2m_2 r\Omega\dot{\alpha} - m_2 r\Omega^2 &= Q_2, \\ I_z\ddot{\alpha} + k_3\alpha - 2r(m_1\dot{y}_1 - m_2\dot{y}_2)\Omega - 2r(m_1 y_1 - m_2 y_2)\dot{\Omega} + I_z\dot{\Omega} &= Q_3. \end{aligned} \right. \tag{2.25}$$

Here $I_z = I_{3z} + (m_1 + m_2)r^2$ is the total moment of inertia of the sensitive element around Z axis, Q_i are the generalised forces for translational variables and torque for rotational, respectively. Also note that motion Eq. (2.25) are the linear part of the original equations, obtained from (2.2).

Let us now divide both sides of the equations by the corresponding highest derivative coefficients and add the damping terms:

$$\left\{ \begin{aligned} &\ddot{y}_1 + (\omega_1^2 - \Omega^2)y_1 + 2r\Omega\dot{\alpha} + r\Omega^2 = q_1, \\ &\ddot{y}_2 + (\omega_2^2 - \Omega^2)y_2 - 2r\Omega\dot{\alpha} - r\Omega^2 = q_2, \\ &\ddot{\alpha} + \omega_3^2\alpha - 2\frac{r}{I_z}(m_1\dot{y}_1 - m_2\dot{y}_2)\Omega - 2\frac{r}{I_z}(m_1 y_1 - m_2 y_2)\dot{\Omega} + \dot{\Omega} = q_3. \end{aligned} \right. \tag{2.26}$$

Here $\omega_1^2 = k_1/m_1$, $\omega_2^2 = k_2/m_2$ and $\omega_3^2 = k_3/I_z$ are the squared natural frequencies of the corresponding motions; $q_1 = Q_1/m_1$, $q_2 = Q_2/m_2$ and $q_3 = Q_3/I_z$ are the accelerations from the external forces, acting on the corresponding elements of the sensitive element.

System of Eq. (2.26) can be further simplified if the new variable $y = y_1 - y_2$ is introduced and proof masses along with its elastic suspensions are assumed to be identical ($m_1 = m_2 = m$, $k_1 = k_2 = k$). As a result, first two equations are reduced to one, and the tuning fork sensitive element motion equations become

$$\begin{cases} \ddot{y} + 2\zeta_y \omega_y \dot{y} + (\omega_y^2 - \Omega^2)y + 2r\Omega(2\dot{\alpha} + \Omega) = q_y, \\ \ddot{\alpha} + 2\zeta_3\omega_3\dot{\alpha} + \omega_3^2\alpha - 2m\dfrac{r}{I_z}(\Omega\dot{y} + y\dot{\Omega}) + \dot{\Omega} = q_3, \end{cases} \tag{2.27}$$

where $q_y = q_1 - q_2$, $\omega_y^2 = k/m$, ζ_y and ζ_3 are the dimensionless damping factors of the added damping terms.

Analysis of the motions Eq. (2.27) shows that provided masses are set to oscillate in the opposite phases, angular motion of the sensitive element becomes dependent on the external angular rate, which allows its measurement.

2.4 Ring-Shaped Sensitive Element Motion Equations

While all previously considered designs use lumped masses to sense Coriolis acceleration, distributed masses are also used and provide excellent performances in numerous applications. As an example of the distributed masses design, let us consider ring-shaped sensitive element of CVG, shown in Fig. 2.5.

Ring-shaped sensitive element is set to oscillate along axes X and Y (primary ring displacement x_1). These oscillations could be also viewed as a standing wave excited in the ring. When there is no external rotation applied to the sensitive

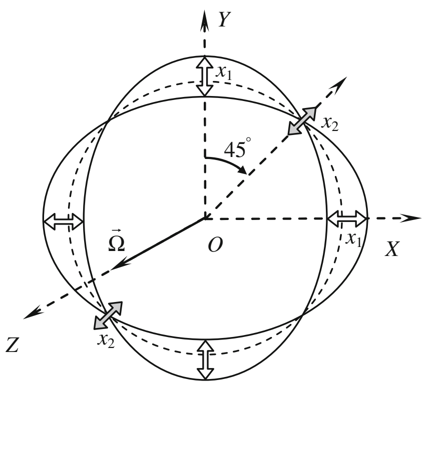

Fig. 2.5 Ring-shaped CVG sensitive element

element, primary wave has four nodes, where the ring displacement is zero. These nodes are located at 45° between axes X and Y.

When the sensitive element rotates around axis Z, the primary wave starts to shift around the ring due to the Coriolis force. As a result, secondary oscillations (ring displacement x_2) start to appear in the nodes, thus allowing measurements of the external angular rate $\vec{\Omega}$.

Despite the fact that there are no lumped masses in the ring sensitive element, its motion equations in terms of the primary and secondary displacements x_1 and x_2 still could be written in a lumped form as

$$\begin{cases} \ddot{x}_1 + 2\zeta\omega\dot{x}_1 + (\omega^2 - \Omega^2)x_1 = q_1 - 2c\Omega\dot{x}_2 - c\dot{\Omega}x_2, \\ \ddot{x}_2 + 2\zeta\omega\dot{x}_2 + (\omega^2 - \Omega^2)x_2 = q_2 + 2c\Omega\dot{x}_1 + c\dot{\Omega}x_1. \end{cases} \tag{2.28}$$

Here ω is the natural frequency of the ring, ζ is the dimensionless damping factor, Ω is the projection of the external angular rate to the axis Z, c is the gyroscopic coupling factor (Brian coefficient), q_1 and q_2 are the effective accelerations from external forces acting along primary and secondary displacements, respectively. Apparently, motion Eq. (2.28) are quite similar to the translational sensitive element equations in case when there is no decoupling frame.

2.5 Generalised Motion Equations

It is no coincidence that motion equations of all considered above sensitive element designs look so similar, since they are all based on the same principle—Coriolis acceleration measurement by vibrating structures. Let us now write down CVG sensitive element motion equations in a form that is applicable to all designs:

$$\begin{cases} \ddot{x}_1 + 2\zeta_1\omega_1\dot{x}_1 + (\omega_1^2 - d_1\Omega^2)x_1 = q_1 - g_1\Omega\dot{x}_2 - d_3\dot{\Omega}x_2, \\ \ddot{x}_2 + 2\zeta_2\omega_2\dot{x}_2 + (\omega_2^2 - d_2\Omega^2)x_2 = q_2 + g_2\Omega\dot{x}_1 + d_4\dot{\Omega}x_1. \end{cases} \tag{2.29}$$

Here x_1 and x_2 are the generalised displacements describing primary and secondary motion of the sensitive element, either translational or rotational; ω_1 and ω_2 are the natural frequencies, ζ_1 and ζ_2 are the damping factors of the primary and secondary motions correspondingly; q_1 and q_2 are the accelerations (either translational or rotational) from external forces/torques, Ω is the external angular rate, orthogonal to the primary and secondary motions. Remaining coefficients are the functions of the sensitive element design parameters and are given in the Table 2.1.

Note that in order to make tuning fork motion Eq. (2.27) compatible with the generalised form (2.29), certain terms were dropped. However, these modifications do not make the generalised form less applicable to analysis of the tuning fork

Table 2.1 Design-dependent generalised equations coefficients

Design:	Beam	LL-gyro	RR-gyro	Tuning fork	Ring
g_1	2	$\frac{2m_2}{m_1+m_2}$	$\frac{I_{2x}+I_{2y}-I_{2z}}{I_{1y}+I_{2y}}$	$4r$	$2c$
g_2	2	2	$\frac{I_{2x}+I_{2y}-I_{2z}}{I_{2x}}$	$\frac{2mr}{I_z}$	$2c$
d_1	1	1	$\frac{I_{1x}-I_{1z}+I_{2x}-I_{2z}}{I_{1y}+I_{2y}}$	1	1
d_2	1	1	$\frac{I_{2y}-I_{2z}}{I_{2x}}$	0	1
d_3	1	$\frac{m_2}{m_1+m_2}$	$\frac{I_{2y}-I_{2z}}{I_{1y}+I_{2y}}$	0	c
d_4	1	1	1	$\frac{2mr}{I_z}$	c

sensitive element, since its effect is either negligibly small or removed by the demodulation of the secondary oscillations.

Let us now finally look at the most simplified form of the generalised equations when the external angular rate Ω is small comparing to natural frequency and slowly varying, e.g. $\Omega^2 \approx 0$, $\Omega \approx \text{const}$, and $\dot{\Omega} \approx 0$. As a result, we obtain the most commonly known form of CVG motion equations

$$\begin{cases} \ddot{x}_1 + 2\zeta_1\omega_1\dot{x}_1 + \omega_1^2 x_1 = q_1 - g_1\Omega\dot{x}_2, \\ \ddot{x}_2 + 2\zeta_2\omega_2\dot{x}_2 + \omega_2^2 x_2 = q_2 + g_2\Omega\dot{x}_1, \end{cases} \tag{2.30}$$

where only Coriolis terms are present. Although system (2.30) is good enough to describe CVG principles of operations, we shall use Eq. (2.29) to analyse sensitive element motion, and to design signal processing and control algorithms. Needless to say that if any results obtained for the Eq. (2.29), they are applicable to CVG of any discussed above design.

Resume

Derived in this chapter, the generalised motion equations of Coriolis vibratory gyroscopes allow us to analyse motion and optimise the design of sensitive elements and the results can be equally applicable to different designs. For apparent reasons, not all possible designs were included in the Table 2.1. However, as demonstrated above, motion equations for any sensitive element design could be derived using Lagrange method, and then its essential terms can be related to the corresponding terms in the generalised form (2.29). As soon as proper entries to the Table 2.1 are identified, all the results and methodologies can be applied to the specific CVG design.

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