Chapter 2
Basic Properties of Quasivarieties

This chapter supplies basic facts concerning quasivarieties and the equational systems associated with quasivarieties. Many of these facts are of syntactical character. An equational logic is an extension of the familiar Birkhoff’s logic. The narrative structure of the book is strictly linked with the properties of lattices of theories of equational logics. Examining these lattice requires formal tools. They are introduced in this part; some of them are new.

2.1 Quasi-Identities

\( \omega \) is the set of natural numbers (with zero). An algebraic signature is a pair \( \tau := (F, a) \), where \( F \) is a set (of operation symbols), and \( a : F \to \omega \) is a function (assigning arity). An algebra of signature \( \tau \) is a pair \( A := (A, F^A) \), where \( A \) is a non-empty set, called the universe of \( A \), and for each \( f \in F \) with \( a(f) = m \), there is \( m \)-ary operation \( f^A : A^m \to A \). The operations \( f^A \) are called the basic (or fundamental) operations of \( A \). If \( a(f) = 0, f \) is also called a constant symbol. \( f^A \) is then an element of \( A \).

An algebra \( A \) of signature \( \tau \) will often be referred to as a \( \tau \)-algebra.

\( \text{Hom}(A, B) \) is the set of all homomorphisms from a \( \tau \)-algebra \( A \) to \( \tau \)-algebra \( B \).

Let \( \tau \) be a fixed algebraic signature and let \( L_\tau \) be the corresponding first-order language with equality \( \approx \). \( \text{Var} = \{v_n : n \in \omega \} \) is the set of individual variables of \( L_\tau \). \( \text{Te}_\tau \) is the algebra of terms of \( L_\tau \) and \( \text{Eq}(\tau) \) is the set of equations of \( L_\tau \).

If \( t = t(x_1, \ldots, x_n) \) is a term in at most \( n \) individual variables \( x = x_1, \ldots, x_n \), and \( a = a_1, \ldots, a_n \) is a sequence of elements of a \( \tau \)-algebra \( A \), then \( t^A(a_1, \ldots, a_n) \) is the value of the term \( t \) for \( a = a_1, \ldots, a_n \) in \( A \). \( t^A(a_1, \ldots, a_n) \) is defined in the standard way by induction on complexity of terms. We shall also use the abbreviation \( t^A(a) \) for \( t^A(a_1, \ldots, a_n) \) often omitting the superscript ‘\( A \)’ when the algebra is clear from context.
A quasi-equation is a formula of the form

\[ \alpha_1 \approx \beta_1 \wedge \ldots \wedge \alpha_n \approx \beta_n \rightarrow \alpha \approx \beta, \]

where \( \alpha_1, \beta_1, \ldots, \alpha_n, \beta_n, \alpha, \beta \) are terms. \( n = 0 \) is possible so every equation qualifies as a quasi-equation.

A universally quantified quasi-equation is called a quasi-identity. As is customary, the universal quantifiers in quasi-identities are usually not explicitly written. It is left to the context to distinguish quasi-identities from quasi-equations.

Any class of algebras defined by a set of quasi-identities is called a quasivariety. If \( Q \) is a quasivariety, then any set \( \Gamma \) of quasi-identities defining \( Q \) is called a base for \( Q \); we then write \( Q = \text{Mod}(\Gamma) \).

The symbols \( I, H, S, P, \) and \( P_u \), respectively, denote the operations of forming isomorphic images, homomorphic images, subalgebras, direct products and ultraproducts. (The class operations \( S, P \) and \( P_u \) are interpreted in the inclusive sense which means that they also comprise isomorphic copies of algebras; for example, \( P_u(\mathcal{K}) \) is the class of all algebras isomorphic to an ultraproduct of a system of algebras from \( \mathcal{K} \).

Let \( Q \) be a class closed under isomorphisms. By a well-known result due to Mal’cev, \( Q \) is a quasivariety if and only if \( Q \) is closed under the operations \( S, P \) and \( P_u \).

If \( \mathcal{K} \) is a class of algebras, then \( Q_v(\mathcal{K}) \) is the smallest quasivariety containing \( \mathcal{K} \); the class \( \mathcal{K} \) is then said to generate the quasivariety \( Q_v(\mathcal{K}) \). \( \text{Va}(\mathcal{K}) \) is the variety generated by \( \mathcal{K} \). The equalities \( Q_v(\mathcal{K}) = \text{SPP}_u(\mathcal{K}) \) and \( \text{Va}(\mathcal{K}) = \text{HSP}(\mathcal{K}) \) holding for any class \( \mathcal{K} \) are classical results of universal algebra.

A quasivariety \( Q \) is finitely generated if \( Q = Q_v(\mathcal{K}) \) for a finite set \( \mathcal{K} \) of finite algebras. In this case \( Q_v(\mathcal{K}) = \text{SP}(\mathcal{K}) \).

### 2.2 Rules of Inference

Let \( \tau \) be a fixed signature. A rule of inference \( r \) in \( \text{Eq}(\tau) \) is a set of pairs \( \langle \Sigma, \sigma \rangle \), where \( \Sigma \) is a (possibly infinite) set of equations and \( \sigma \) is a single equation. (The pair \( \langle \Sigma, \sigma \rangle \) is read: From the set of equations \( \Sigma \) infer the equation \( \sigma \).) Any pair \( \langle \Sigma, \sigma \rangle \) belonging to \( r \) is called an instance of \( r \).

Let \( e \) be a substitution in \( \text{Te}_\tau \), i.e., \( e \) is an endomorphism of the term algebra). \( e \) is fully determined by its values on the set of individual variables of \( \text{Te}_\tau \). Given a set of equations \( \Sigma \) and an equation \( \alpha \approx \beta \), we put: \( e\Sigma = \{ e\rho : \rho \in \Sigma \} \) and \( e(\alpha \approx \beta) := e\alpha \approx e\beta \).

A rule \( r \) is schematic if there exists a single pair \( \langle \Sigma_0, \sigma_0 \rangle \) such that \( r \) is equal to the set of all instances of \( \langle \Sigma_0, \sigma_0 \rangle \), i.e.,

\[
 r = \{ (e\Sigma_0, e\sigma_0) : e \in \text{Hom}(\text{Te}_\tau, \text{Te}_\tau) \}. \tag{1}
\]
2.2 Rules of Inference

The pair \(<\Sigma_0, \sigma_0>\) is then called a scheme of the rule \(r\). A schematic rule \(r\) is finitary (or standard) if for one its schemes \(<\Sigma_0, \sigma_0>\) (equivalently, for all schemes), the set \(\Sigma_0\) is finite. A schematic rule \(r\) is proper (or non-axiomatic) if \(\Sigma_0\) is non-empty; otherwise, \(r\) is called axiomatic.

The equations in \(e\Sigma_0\) are called premises of the rule \(r\) and the equation \(e\sigma_0\) is the conclusion of \(r\), for any \(e\).

Any schematic rule \(r\) is usually identified with its scheme. Therefore any schematic rule \(r\) (with a scheme \(<\Sigma_0, \sigma_0>\)) is often presented in the form

\[
\Sigma_0/\sigma_0
\]

or, more explicitly, as

\[
\{\alpha_i \approx \beta_i : i \in I\}/\alpha \approx \beta
\]

where \(\Sigma_0 = \{\alpha_i \approx \beta_i : i \in I\}\) and \(\sigma_0\) is \(\alpha \approx \beta\).

In particular, following common practice adopted in metalogic, if \(r\) is schematic with a scheme \(<\Sigma_0, \sigma_0>\), where \(\Sigma_0\) is finite, \(\Sigma_0 = \{\alpha_1 \approx \beta_1, \ldots, \alpha_k \approx \beta_k\}\), and \(\sigma_0\) is \(\alpha \approx \beta\), then the standard rule \(r\) is usually presented in the form

\[
\alpha_1 \approx \beta_1, \ldots, \alpha_k \approx \beta_k/\alpha \approx \beta
\]

Thus, formally, (3) is the set consisting of ordered pairs

\[
\{\{e\alpha_1 \approx e\beta_1, \ldots, e\alpha_k \approx e\beta_k\}, e\alpha \approx e\beta\}
\]

called instances of (3), with \(e\) ranging over the set of endomorphisms of \(Te_t\).

If (2) is axiomatic (and hence the set \(I\) is empty), then (2) is written as

\[
/\alpha \approx \beta.
\]

Birkhoff’s rules are the following schematic rules

\[
/x \approx x, \quad \text{(identity axiom)}
\]

\[
x \approx y/y \approx x, \quad \text{(symmetry)}
\]

\[
x \approx y, y \approx z/x \approx z, \quad \text{(transitivity)}
\]

and

\[
x_1 \approx y_1, \ldots, x_m \approx y_m/f(x_1, \ldots, x_m) \approx f(y_1, \ldots, y_m), \quad \text{(functionality)}
\]

for each operation symbol \(f\) of arbitrary arity \(m\). The identity axiom is also referred to as the reflexivity rule. (This rule is axiomatic, with the empty set of premises.)

\[\text{1}A\text{ similar ‘forward slash’ notation is sometimes applied to substitutions in the algebra of terms but this should not lead to confusion.}\]
If the signature $\tau$ is finite, the above set of standard rules is finite as well. The set of Birkhoff’s rules is denoted by $\text{Birkhoff}(\tau)$.

A set of equations $\Sigma$ is closed with respect to a schematic rule $r : \{\alpha_i \approx \beta_i : i \in I\}$ if and only if for every endomorphism $e : T \epsilon \rightarrow T \epsilon$, $\{e\alpha_i \approx e\beta_i : i \in I\} \subseteq \Sigma$ implies that $e\alpha \approx e\beta \in \Sigma$.

If $\Sigma$ and $\Gamma$ are sets of equations, then $\Sigma / \Gamma$ denotes the set of rules $\Sigma / \sigma$, where $\sigma$ ranges over the set $\Gamma$.

Let $R$ be a set of standard rules in $\text{Eq}(\tau)$. We assume that $R$ includes the set of Birkhoff’s rules $\text{Birkhoff}(\tau)$ (This assumption may be overridden, but then one assumes instead that $\text{Birkhoff}(\tau)$ is among secondary rules of the consequence operation $C^eq_R$ determined by $R$—see the definition below.) Let $\delta$ be a set of equations. An $R$-proof from $\delta$ is any finite sequence of equations

$$p_1 \approx q_1, \ldots, p_n \approx q_n$$

satisfying the following condition:

(p1) $p_1 \approx q_1 \in \Sigma$ or $p_1 \approx q_1$ is of the form $x \approx x$ for some variable $x$,

(p2) for every $i$ ($1 < i \leq n$), either $p_i \approx q_i \in \Sigma$ or $p_i \approx q_i$ is of the form $x \approx x$,

or there are indices $i_1, \ldots, i_k$ smaller than $i$ and a rule $r : \alpha_1 \approx \beta_1 \wedge \ldots \wedge \alpha_k \approx \beta_k / \alpha \approx \beta$ in $R$ such that $p_i \approx q_i$ is obtained from $p_{i_1} \approx q_{i_1}, \ldots, p_{i_k} \approx q_{i_k}$ by an application of $r$.

(The phrase “application of a rule” is commonly used in the proof-theoretic parlance. Formally, the meaning of the phrase “$p_i \approx q_i$ is obtained from $p_{i_1} \approx q_{i_1}, \ldots, p_{i_k} \approx q_{i_k}$ by an application of $r$” is that the pair $\{(p_{i_1} \approx q_{i_1}, \ldots, p_{i_k} \approx q_{i_k})\}$ is an instance of $r$.)

Let $p \approx q$ be an equation. An $R$-proof from $\Sigma$ is called an $R$-proof of $p \approx q$ from $\Sigma$ if $p \approx q$ is the last element of this proof. $p \approx q$ is $R$-provable from $\Sigma$ if there exists an $R$-proof of $p \approx q$ from $\Sigma$.

For every set $\Sigma \subseteq \text{Eq}(\tau)$ we define:

$$C^eq_R(\Sigma) := \{p \approx q \in \text{Eq}(\tau) : p \approx q \text{ is } R\text{-provable from } \Sigma\}.$$  

$C^eq_R$ is a structural and finitary consequence operation defined on $\text{Eq}(\tau)$ which validates the set of rules $R$ (see below). In particular $C^eq_R$ validates Birkhoff’s rules for equality $\text{Birkhoff}(\tau)$.

### 2.3 Equational Logics

$\varphi(\text{Eq}(\tau))$ is the power set of $\text{Eq}(\tau)$, i.e., the family of all subsets of $\text{Eq}(\tau)$. A mapping $C^eq : \varphi(\text{Eq}(\tau)) \rightarrow \varphi(\text{Eq}(\tau))$ is a consequence operation on $\text{Eq}(\tau)$ if it satisfies, for all $X, Y \subseteq \text{Eq}(\tau)$:

- (Co1) $X \subseteq C^eq(X)$ (reflexivity)
- (Co2) $C^eq(X) \subseteq C^eq(Y)$ whenever $X \subseteq Y$ (monotonicity)
- (Co3) $C^eq(C^eq(X)) \subseteq C^eq(X)$ (idempotency).
A consequence $C^{eq}$ is finitary if for all $X \subseteq Eq(\tau)$:

(Co4) $C^{eq}(X) = \bigcup \{C^{eq}(X_f) : X_f$ is a finite subset of $X\}$ (finitariness).

(Note that (Co4) already implies (Co2).)

A consequence $C^{eq}$ is structural if for all $X \subseteq Eq(\tau)$:

(Co5) $eC^{eq}(X) \subseteq C^{eq}(eX)$ for every endomorphism $e : T_{\tau} \rightarrow T_{\tau}$ (structurality).

$B_{\tau}$ stands for the consequence operation (on $Eq(\tau)$) determined only by the set of rules Birkhoff($\tau$) in the standard way. $B_{\tau}$ is referred to as Birkhoff’s logic in the signature $\tau$.

By an equational logic we shall understand any structural consequence operation $C^{eq}$ defined on $Eq(\tau)$ which validates Birkhoff’s rules for equality $Birkhoff$, and possibly some other rules. This means that $B_{\tau}(X) \subseteq C^{eq}(X)$, for all $X \subseteq Eq(\tau)$. Birkhoff’s logic $B_{\tau}$ is the least equational logic (in the sense of the above inclusions). Thus the class of equational logics in a given signature $\tau$ comprises exactly all strengthenings of the logic $B_{\tau}$.

The terms “equational logic” and “equational deductive system” will be treated as synonyms.

Any set of equations $\Sigma$ such that $C^{eq}(\Sigma) = \Sigma$ is called a closed theory of $C^{eq}$, or shortly, a theory of $C^{eq}$.

$$Th(C^{eq})$$

is the set of all theories of $C^{eq}$. Since $Th(C^{eq})$ is a closure system, it forms a complete lattice with respect to inclusion. This lattice is denoted by

$$Th(C^{eq})$$

Given a class $K$ of $\tau$-algebras, we let

$$K^{=}$$

denote the consequence operation on the set of $\tau$-equations determined by $K$. Thus, for $\{\alpha_i \approx \beta_i : i \in I\} \cup \{\alpha \approx \beta\} \subseteq Eq(\tau),$

$\alpha \approx \beta \in K^{=}(\{\alpha_i \approx \beta_i : i \in I\})$ if and only if, for every algebra $A \in K$ and every $h \in Hom(T_{\tau}, A)$, $h(\alpha) = h(\beta)$ whenever $h(\alpha_i) = h(\beta_i)$ for all $i \in I$.

$\alpha \approx \beta \in K^{=}(\{\alpha_i \approx \beta_i : i \in I\})$ is read: $\alpha \approx \beta$ follows from $\{\alpha_i \approx \beta_i : i \in I\}$ relative to $K$.

The consequence operation $K^{=}$ is structural, i.e., $\alpha \approx \beta \in K^{=}(\{\alpha_i \approx \beta_i : i \in I\})$ implies that $e\alpha \approx e\beta \in K^{=}(\{e\alpha_i \approx e\beta_i : i \in I\})$ for all endomorphisms $e$ of the term algebra $T_{\tau}$ and all sets $\{\alpha_i \approx \beta_i : i \in I\}$. As $K^{=}$ validates Birkhoff’s rules, $K^{=}$ is an equational logic. Furthermore, if $K$ is closed under the formation of ultraproducts, the consequence $K^{=}{}$ is finitary. Note that $\alpha \approx \beta \in K^{=}{}(\emptyset)$ means that the equation $\alpha \approx \beta$ is valid in the class $K$. 


Following common practice we suppress parentheses as much as possible and in case of finite sets of equations we usually write

$$\alpha \approx \beta \in K^\approx (\alpha_1 \approx \beta_1, \ldots, \alpha_k \approx \beta_k)$$

instead of $$\alpha \approx \beta \in K^\approx (\{\alpha_1 \approx \beta_1, \ldots, \alpha_k \approx \beta_k\}).$$

$$K^\approx (\{\alpha_i \approx \beta_i : i \in I\})$$ is thus the set of all equations $$\alpha \approx \beta$$ which follow from $$\{\alpha_i \approx \beta_i : i \in I\}$$ relative to the class $$K.$$

A schematic rule $$r : \{\alpha_i \approx \beta_i : i \in I\}/\alpha \approx \beta$$ is said to be a rule of the consequence $$K^\approx$$ if $$\alpha \approx \beta \in K^\approx (\{\alpha_i \approx \beta_i : i \in I\}).$$ In this case we also say that $$r$$ is a rule of the class $$K.$$

Every equational logic $$C^eq$$ on $$Eq(\tau)$$ is characterized semantically by some class of algebras, i.e., there exists a class $$K$$ of $$\tau$$-algebras such that $$C^eq = K^\approx,$$ and there always exists the largest such class $$K.$$ Furthermore, if $$C^eq$$ is finitary, $$K$$ can be assumed to be a quasivariety. These, rather simple observations, form the content of the completeness theorem for equational logics. Consequently, when we are dealing with equational logics, we shall mainly consider consequences $$K^\approx$$ already determined by some fixed class $$K$$ of algebras.

Any set of equations $$\Sigma$$ such that $$K^\approx (\Sigma) = \Sigma$$ is called a closed theory of $$K^\approx,$$ or shortly, a theory of $$K^\approx.$$ A set of equations $$\Sigma$$ is a theory of $$K^\approx$$ if and only if $$\Sigma$$ is closed with respect to the set of rules of $$K^\approx.$$ According to the adopted notation, $$Th(K^\approx)$$ is the set of all theories of $$K^\approx.$$ $$Th(K^\approx)$$ forms a complete lattice with respect to inclusion. This lattice is denoted by

$$Th(K^\approx).$$

For any class $$K$$ it is the case that $$K^\approx = \inf\{A^\approx : A \in K\},$$ which means that $$K^\approx (\Sigma) = \bigcap\{A^\approx (\Sigma) : A \in K\},$$ for any set of equations $$\Sigma.$$ (Here $$A^\approx$$ is, in accordance with the definition, $$\{A\}^\approx.$$) Equivalently, $$Th(K^\approx)$$ is the closure system generated by $$\bigcup\{Th(A^\approx) : A \in K\}.$$ There is an obvious translation of $$K^\approx$$ into the language of quasi-identities over $$Te_\tau:$$

$$\alpha \approx \beta \in K^\approx (\alpha_1 \approx \beta_1, \ldots, \alpha_k \approx \beta_k)$$ if and only if the implication

$$\alpha_1 \approx \beta_1 \land \ldots \land \alpha_k \approx \beta_k \Rightarrow \alpha \approx \beta$$

is valid in $$K.$$ This observation enables one to express the properties of the consequence operation $$K^\approx$$ on finite sets in terms of quasi-equations valid in $$K.$$ The definitions given below are formulated in terms of standard rules; but they can be easily reformulated in terms of quasi-identities. The key is in assigning to each standard rule

$$r : \alpha_1 \approx \beta_1 \land \ldots \land \alpha_k \approx \beta_k/\alpha \approx \beta$$

the quasi-identity

$$(r) : \alpha_1 \approx \beta_1 \land \ldots \land \alpha_k \approx \beta_k \Rightarrow \alpha \approx \beta.$$ 

Thus $$r$$ is a rule of $$K^\approx$$ if and only if $$(r)$$ is universally valid in $$K.$$
2.4 Relative Congruences

Although quasi-identities and standard rules of inference are interdefinable concepts, in purely syntactic contexts, when one works with consequence relations defined on equations, the notion of a standard rule of inference often appears to be convenient and useful.

For every quasivariety \( Q \), the logic \( Q^\equiv \) is characterized proof-theoretically by a set \( R \) of standard rules. That is, for every \( Q \) there is a set \( R \) of standard rules including Birkhoff's /FS/ such that

\[
Q^\equiv = C^\text{eq}_R.
\]

(As \( R \) one may take the set consisting of all standard rules of \( Q^\equiv \).) If (1) holds, then \( R \) is called an inferential base for \( Q^\equiv \).

Since the notions of a standard rule of \( Q^\equiv \) and of a quasi-identity of \( Q \) are interdefinable, the fact that a set of rules \( R \) is an inferential base of \( Q^\equiv \) is equivalent to the statement that the set of quasi-equations \( \{ r : r \in R \} \) corresponding to the rules of \( R \) forms an axiomatization of \( Q \).

In what follows we shall interchangeably speak of inferential bases for \( Q^\equiv \) and axiomatic bases for \( Q \). (The latter bases consist of sets of quasi-identities.)

2.4 Relative Congruences

Let \( \tau \) be a signature. Let \( R \) be a binary relation defined on a \( \tau \)-algebra \( A \). \( R \) is closed under a schematic rule \( r : \{ \alpha_i \approx \beta_i : i \in I \} / \alpha \approx \beta \) if, for any \( a \in A^k \), \( R \) contains the pair \( \langle \alpha^A(q), \beta^A(q) \rangle \) whenever it contains the pairs \( \langle \alpha^A_i(q), \beta^A_i(q) \rangle \), for all \( i \in I \).

(Here \( k \) is the length of a sequence \( x = x_1, x_2, \ldots \) which includes every variable occurring in one of the terms of the equations \( \{ \alpha_i \approx \beta_i : i \in I \} \) and \( \alpha \approx \beta \). \( k \) may be infinite. In this case we assume that \( k = \omega \).)

If the rule \( r \) is standard, \( r = \alpha_1 \approx \beta_1, \ldots, \alpha_k \approx \beta_k / \alpha \approx \beta \), then we shall interchangeably use the phrases ‘\( R \) is closed under \( r \)’ and ‘\( R \) is closed under the quasi-equation \( \alpha_1 \approx \beta_1 \land \ldots \land \alpha_k \approx \beta_k \to \alpha \approx \beta \)’.

A binary relation \( R \) on a \( \tau \)-algebra \( A \) is a congruence relation (of \( A \)) if \( R \) is closed under Birkhoff’s rules Birkhoff(\( r \)).

If \( \Phi \) is a congruence of \( A \) and \( a \in A \), then \( a / \Phi \) is the equivalence class of \( a \) with respect to \( \Phi \). \( A / \Phi \) is the quotient algebra whose elements are equivalence classes \( a / \Phi, a \in A \).

If \( \Phi \) is a congruence of \( A \), then \( \Phi \) is closed under the rule \( \alpha_1 \approx \beta_1, \ldots, \alpha_k \approx \beta_k / \alpha \approx \beta \) if and only if the quotient algebra \( A / \Phi \) validates the quasi-identity 

\[
(\forall x)(\alpha_1 \approx \beta_1 \land \ldots \land \alpha_k \approx \beta_k \to \alpha \approx \beta).
\]

Note. Each equation may be identified with a pair of terms. We may therefore identify a set of equations \( \Sigma \) with a set of pairs of terms; \( \Sigma \) is thus a binary relation on the term algebra \( Te_\tau \). Consequently, the fact that \( \Sigma \) is closed with respect to a rule \( r \) is an instance of the above general definition. In particular, \( \Sigma \) is closed
with respect to Birkhoff’s rules $Birkhoff(\tau)$ if and only if the set of ordered pairs \{\{(\alpha, \beta) : \alpha \approx \beta \in \Sigma\}\} is a congruence of the term algebra $\mathcal{T}_\tau$. \hfill \Box

If $A$ is an algebra, then $\text{Con}(A)$ is the set of congruences of $A$. $\text{Con}(A)$ forms an algebraic lattice, denoted by $\text{Con}(A)$. If $\Phi, \Psi \in \text{Con}(A)$, then $\Phi + \Psi$ marks their join in $\text{Con}(A)$. The lattice meet of $\Phi, \Psi$ in $\text{Con}(A)$ coincides with the intersection $\Phi \cap \Psi$. If $X \subseteq A^2$, $\Theta^A(X)$ denotes the least congruence of $A$ that contains $X$.

Let $Q$ be a quasivariety of $\tau$-algebras and $A$ a $\tau$-algebra, not necessarily in $Q$. A congruence $\Phi$ on $A$ is called a $Q$-congruence if $A/\Phi \in Q$. The set of $Q$-congruences is denoted by $\text{Con}_Q(A)$. Thus $\text{Con}_Q(A) = \{\Phi \in \text{Con}(A) : A/\Phi \in Q\}$. If $Q$ is not a variety, the elements of $\text{Con}_Q(A)$ are also called relative congruences. $\text{Con}_Q(A)$ contains the universal congruence $1_A := A^2$ and it contains the smallest $Q$-congruence being the intersection of all $Q$-congruences of $A$. This smallest $Q$-congruence is the identity congruence $0_A (= \text{diagonal relation on } A)$ if and only if $A \in Q$.

It is easy to see that if $\Gamma$ is an axiomatic base for $Q$, then $\Phi$ is a $Q$-congruence if and only if $\Phi$ is closed under every quasi-identity from the base. It follows from this observation that $\text{Con}_Q(A)$ is closed under arbitrary intersections and the union of directed sets; in other words, $\text{Con}_Q(A)$ is a finitary closure system on $A^2$. (This also follows from the fact that $Q$ is closed under subdirect products and ultraproducts.) $\text{Con}_Q(A)$ therefore forms the universe of an algebraic lattice $\text{Con}_Q(A)$ called the lattice of $Q$-congruences.

If $\Phi, \Psi \in \text{Con}_Q(A)$, then $\Phi +_Q \Psi$ denotes their join in $\text{Con}_Q(A)$. $\Phi +_Q \Psi$ is generally larger than $\Phi + \Psi$. The lattice meet of $\Phi, \Psi$ in $\text{Con}_Q(A)$ coincides with their intersection $\Phi \cap \Psi$.

If $V$ is a variety, and $A$ is an algebra of type $\tau$, then $\text{Con}_V(A)$ forms a principal filter in the lattice $\text{Con}(A)$ of all congruences of $A$. But if $A$ is in $V$, then $\text{Con}_V(A)$ coincides with $\text{Con}(A)$.

For any $X \subseteq A^2$, $\Theta^Q_Q(X)$ denotes the least $Q$-congruence of $A$ that contains $X$. Thus

$$\Theta^Q_Q(X) = \bigcap \{\Phi \in \text{Con}_Q(A) : X \subseteq \Phi\}. $$

The congruence $\Theta^Q(X)$ is a subset of $\Theta^Q_Q(X)$.

$\text{Id}(Q)$ denotes the set of all identities valid in $Q$.

The following characterization of $\Theta^Q_Q(X)$ proves convenient in applications:

**Theorem 2.1.** Let $Q$ be a quasivariety of algebras of type $\tau$ and $\Gamma$ a set of quasi-identities which are not identities such that $Q = \text{Mod}(\text{Id}(Q) \cup \Gamma)$). Then for any algebra $A$, any set $X \subseteq A^2$ and any $a, b \in A$,

$$a \equiv b(\Theta^Q_Q(X)) \text{ if and only if there exists a finite sequence } \langle a_1, b_1 \rangle, \ldots, \langle a_n, b_n \rangle \tag{\ast}$$
2.5 Free Algebras

Let $K$ be a class of $\tau$-algebras. $F_K(\omega)$ denotes the free algebra in $K$ freely generated by a countably infinite set of generators. $F_K(\omega)$ is also free in the variety $Va(K)$. $F_K(\omega)$ need not belong to $K$, but $F_K(\omega)$ is in $Qv(K)$. We therefore have that $F_K(\omega) = F_{QV(K)}(\omega) = F_{Va(K)}(\omega)$.

Let $\Theta_0$ be the congruence relation defined on the term algebra $Te_\tau$ as follows: for any terms $\alpha$, $\beta$,

$$\alpha \equiv \beta \pmod{\Theta_0} \iff df \alpha \approx \beta \in K^= (\emptyset) \quad (\iff \alpha \approx \beta \text{ is valid in } K).$$

The equivalence class of any term $t$ with respect to $\Theta_0$ is denoted by $[t]$. (Thus $\alpha \approx \beta$ is valid in $K$ if and only if $[\alpha] = [\beta]$.)

**Proposition 2.3.** The quotient algebra $Te_\tau / \Theta_0$ is free in the class $K$ and hence in the variety $Va(K)$. Moreover $\{[x] : x \in Var\}$ is the set of free generators of $Te_\tau / \Theta_0$. 

Proof. The congruence \( \Omega_0 \) is invariant, i.e., for any terms \( \alpha, \beta \), if \( \alpha \equiv \beta \pmod{\Omega_0} \), then \( e\alpha \equiv e\beta \pmod{\Omega_0} \), for all endomorphisms \( e \) of the term algebra \( T_{\sigma} \). Hence \( T_{\sigma}/\Omega_0 \) is free in \( K \).

A class \( K \) is **trivial** if it contains only one-element algebras; otherwise, \( K \) is **non-trivial**.

We shall identify \( F_k(\omega) \) with \( T_{\sigma}/\Omega_0 \). Since the congruence \( \Omega_0 \) does not paste together different variables (unless \( K \) is trivial), the free generators of \( F_k(\omega) \) are often identified with individual variables.

**Proposition 2.4.** Let \( Q \) be a quasivariety of algebras. For any set \( \Gamma \) of equations and any equation \( \alpha \approx \beta \),

\[
\alpha \approx \beta \in Q^= (\Gamma) \quad \text{if and only if} \quad \langle [\alpha], [\beta] \rangle \in \Theta_Q^F (\{ ([s], [t]) : s \approx t \in \Gamma \}),
\]

where \( F := F_Q(\omega) \).

The proof is easy and is omitted.

Let \( Q \) be a quasivariety and let \( \Omega \) be the mapping which to each (closed) theory \( \Sigma \) of the consequence operation \( Q^= \) assigns the congruence

\[
\Omega (\Sigma) := \{ [\alpha] : \alpha \approx \beta \in \Sigma \}
\]

on the algebra of terms \( T_{\sigma} \). (\( \Omega_0 \) thus coincides with \( \Omega (Q^= (\emptyset)) \)).

In turn, let \( \Xi \) be the mapping which to each (closed) theory \( \Sigma \) of the consequence operation \( Q^= \) assigns the set of pairs

\[
\Xi(\Sigma) := \{ [\alpha], [\beta] : \alpha \approx \beta \in \Sigma \}
\]

of the free algebra \( F_Q(\omega) \). \( \Xi(\Sigma) \) is a congruence of \( F_Q(\omega) \). \( \Xi(\Sigma) \) is equal to the quotient congruence \( \Omega (\Sigma)/\Omega_0 \).

**Proposition 2.5.** Let \( Q \) be a quasivariety. The mapping \( \Xi \) establishes an isomorphism between the lattice of closed theories of \( Q^= \) and the congruence lattice \( \text{Con}_Q(F_Q(\omega)) \).

Proof. Straightforward.

**Proposition 2.6.** Let \( F := F_Q(X) \) be a free algebra in a non-trivial quasivariety \( Q \) with the set \( X \) of free generators and \( Z \subseteq X^2 \). Then \( \Theta_Q^F(Z) = \Theta^F(Z) \), i.e., the least congruence in \( F \) containing \( X \) is a \( Q \)-congruence.

Proof. \( \Theta^F(Z) \) is equal to \( \Theta^F(R(Z)) \), where \( R(Z) \) is the least equivalence relation on \( X \) that includes \( Z \). Let \( Y \) be a set of selectors for the abstraction classes of \( R(Z) \).

Thus, for every \( x \in X \) there is a unique \( y \in Y \) such that \( x R(Z) y \). The quotient algebra \( F/\Theta^F(Z) \) is isomorphic with the free algebra \( F_Q(Y) \). This fact follows from the observation that the mapping \( h_0 : X \to Y \) defined by:

\[
h_0(x) := \text{ the unique } y \in Y \text{ such that } x R(Z) y,
\]
extends to a homomorphism \( h \) from \( F \) to \( F_Q(Y) \) and \( \ker(h) = \Theta^F(R(Z)) \). Since \( F_Q(Y) \) belongs to \( Q \), \( \Theta^F(Z) \) is a \( Q \)-congruence. (See also, e.g., Czelakowski 2001, Chapter Q, Lemma Q.2.3.)

It follows from the above proposition that for any free generators \( x \) and \( y \) of \( F \), \( \Theta(x, y) \) is a \( Q \)-congruence of \( F \). But it is not true that every congruence \( \Phi \subseteq \Theta(x, y) \) is a \( Q \)-congruence. The following example is due to Keith Kearnes\(^2\). Let \( Q \) be the quasivariety of torsion-free Abelian groups. \( Q \) is known to be relatively congruence-modular. The congruence \( \Phi := \Theta(2x, 2y) \) is a proper subset of \( \Theta(x, y) \), and it is not a \( Q \)-congruence. To see this, it is enough to note that \( x - y \) is a nonzero torsion element of \( F/\Phi \), so \( F/\Phi \) is not in \( Q \).

### 2.6 More on Congruences

Let \( A \) and \( B \) be sets and \( h : A \to B \) a function. If \( Y \) is a subset of \( B^2 \), then \( h^{-1}(Y) := \{ (a, b) \in A^2 : \langle ha, hb \rangle \in Y \} \). Similarly, if \( X \) is a subset of \( A^2 \), then \( h(X) := \{ \langle ha, hb \rangle \in B^2 : \langle a, b \rangle \in X \} \).

If \( h : A \to B \) is a homomorphism between algebras \( A \) and \( B \), then

\[
\ker(h) := \{ (a, b) \in A^2 : ha = hb \}.
\]

\( \ker(h) \) is a congruence of \( A \). It is clear that \( \ker(h) = h^{-1}(0_B) \).

**Proposition 2.7.** (The correspondence property). Let \( h : A \to B \) be a homomorphism between arbitrary algebras \( A \) and \( B \). If \( \Phi \in \text{Con}(A) \) and \( \ker(h) \subseteq \Phi \), then \( h^{-1}h(\Phi) = \Phi \).

**Proof.** \( (\supseteq) \). Suppose \( \langle a, b \rangle \in \Phi \). Then \( \langle ha, hb \rangle \in h(\Phi) \). It follows that \( \langle a, b \rangle \in h^{-1}h(\Phi) \).

\( (\subseteq) \). Assume \( \langle a, b \rangle \in h^{-1}h(\Phi) \). Then \( \langle ha, hb \rangle \in h(\Phi) \). It follows that there are \( x, y \in A \) such that \( \langle ha, hb \rangle = \langle hx, hy \rangle \) and \( \langle x, y \rangle \in \Phi \). As \( ha = hx, hb = hy \), we get that \( \langle a, x \rangle, \langle b, y \rangle \in \ker(h) \subseteq \Phi \). Hence \( \langle x, y \rangle, \langle a, x \rangle, \langle b, y \rangle \in \Phi \). This gives that \( \langle a, b \rangle \in \Phi \). \( \square \)

**Note.** Let \( Q \) be a quasivariety. Let \( h : A \to B \) be a homomorphism, where \( A \) and \( B \) are arbitrary algebras. If \( \Phi \in \text{Con}_Q(B) \), then \( h^{-1}(\Phi) \) is a \( Q \)-congruence on \( A \), i.e., \( h^{-1}(\Phi) \in \text{Con}_Q(A) \). This follows from the fact that the relation \( h^{-1}(\Phi) \) is closed under the rules of \( Q \)-equations. Indeed, let \( r : \alpha_1 \approx \beta_1, \ldots, \alpha_n \approx \beta_n/a \approx \beta \) be a rule of \( Q \)-equations and let \( g : Te_t \to A \) be a homomorphism such that \( \langle g\alpha_i, g\beta_i \rangle \in h^{-1}(\Phi) \) for \( i = 1, \ldots, n \). Hence \( \langle hga_i, hgb_i \rangle \in \Theta \) for \( i = 1, \ldots, n \). As \( h : Te_t \to B \) is a homomorphism and \( \Theta \), being a \( Q \)-congruence, is closed with respect to \( r \), we get that \( \langle hga, hgb \rangle \in \Phi \). Hence \( \langle ga, gb \rangle \in h^{-1}(\Phi) \).

\(^2\)Personal correspondence.
In particular, if \( B \in \mathbb{Q} \), then \( \ker(h) (= h^{-1}(0_B)) \) is a \( \mathbb{Q} \)-congruence on \( A \). \( \square \)

**Corollary 2.8.** Let \( \mathbb{Q} \) be a quasivariety of algebras of a signature \( \tau \). Let \( h : A \to B \) be a homomorphism between arbitrary \( \tau \)-algebras and let \( \Phi \in \text{Con}(A) \) be a congruence such that \( \ker(h) \subseteq \Phi \).

(a) If \( h \) is surjective and \( \Phi \in \text{Con}_\mathbb{Q}(A) \), then \( \Phi \in \text{Con}_\mathbb{Q}(B) \).
(b) If \( h(\Phi) \in \text{Con}_\mathbb{Q}(B) \), then \( \Phi \in \text{Con}_\mathbb{Q}(A) \).
(c) If \( h \) is surjective, then \( \Phi \in \text{Con}_\mathbb{Q}(A) \) if and only if \( h(\Phi) \in \text{Con}_\mathbb{Q}(B) \).

Proof. As \( \ker(h) \subseteq \Phi \), we have that \( h^{-1}(\Phi) = \Phi \), by the correspondence property. It follows that the algebra \( A/\Phi \) is embeddable into \( B/h(\Phi) \). (The embedding is established by the mapping \( \phi \) which to each equivalence class \( a/\Phi \in A/\Phi \) assigns the equivalence class \( ha/h(\Phi), a \in A \).

If \( h(\Phi) \in \text{Con}_\mathbb{Q}(B) \), then \( B/h(\Phi) \in \mathbb{Q} \). It follows that \( A/\Phi \in \mathbb{Q} \), because it is isomorphic with a subalgebra of the \( \mathbb{Q} \)-algebra \( B/h(\Phi) \). This proves (b).

If \( h \) is surjective, then the above mapping is an isomorphism between \( A/\Phi \) and \( B/h(\Phi) \). If \( \Phi \in \text{Con}_\mathbb{Q}(A) \), then \( A/\Phi \) belongs to \( \mathbb{Q} \), and hence \( B/h(\Phi) \) belongs to \( \mathbb{Q} \) as well. Hence \( h(\Phi) \in \text{Con}_\mathbb{Q}(B) \). This proves (a).

(c) follows from (a) and (b). \( \square \)

**Proposition 2.9.** Let \( \mathbb{Q} \) be a quasivariety of algebras of a signature \( \tau \). Let \( h : A \to B \) be a homomorphism between arbitrary \( \tau \)-algebras. Then for every set \( X \subseteq A^2 \),

\[
\text{h}(\Theta_A^X(X)) \subseteq \Theta_B^X(h(X)).
\]

Proof. As \( A/h^{-1}(\Theta_B^X(h(X))) \) is isomorphic with a subalgebra of \( B/((\Theta_B^X(h(X)))) \in \mathbb{Q} \), it follows that \( A/h^{-1}(\Theta_B^X(h(X))) \in \mathbb{Q} \). Hence \( h^{-1}(\Theta_B^X(h(X))) \) is a \( \mathbb{Q} \)-congruence on \( A \). Since \( X \subseteq h^{-1}(\Theta_B^X(h(X))) \), we therefore get that \( \Theta_A^X(X) \subseteq h^{-1}(\Theta_B^X(h(X))) \). Consequently, \( h(\Theta_A^X(X)) \subseteq \Theta_B^X(h(X)) \).

(An alternative proof of the above inclusion is based on Theorem 2.1. For let \( \langle a, b \rangle \in \Theta_A^X(X) \) and let \( \langle a_1, b_1 \rangle, \ldots, \langle a_n, b_n \rangle \) be a \( \mathbb{Q} \)-generating sequence of \( \langle a, b \rangle \) from \( X \) in \( A \). Then \( \langle ha_1, hb_1 \rangle, \ldots, \langle ha_n, hb_n \rangle \) is a \( \mathbb{Q} \)-generating sequence of \( \langle ha, hb \rangle \) from \( h(X) \) in \( B \). Hence \( \langle a, b \rangle \in \Theta_B^X(h(X)) \).) \( \square \)

**Note.** Proposition 2.9 implies that for any set of equations \( X \) and any equation \( \alpha \approx \beta \), if \( \alpha \approx \beta \in \mathbb{Q}^{eq(X)} \), then for any \( \tau \)-algebra \( A \) and any homomorphism \( h : \text{Te}_\tau \to A \) it is the case that \( \langle h(\alpha), h(\beta) \rangle \in \Theta_A^X(\{\langle h(\alpha), h(\beta) \rangle : \gamma \approx \delta \in X \}) \). \( \square \)

Let \( \mathbb{Q} \) be a quasivariety of \( \tau \)-algebras, \( A, B \) arbitrary \( \tau \)-algebras, and \( h : A \to B \) a homomorphism. We define:

\[
\ker_\mathbb{Q}(h) := h^{-1}(\Theta_B^X(0_B)).
\]

i.e., \( \ker_\mathbb{Q}(h) \) is the \( h \)-preimage of the least \( \mathbb{Q} \)-congruence of \( B \). As \( A/\ker_\mathbb{Q}(h) \) is isomorphic with a subalgebra of \( B/\Theta_B^X(0_B) \in \mathbb{Q} \), \( \ker_\mathbb{Q}(h) \) is a \( \mathbb{Q} \)-congruence on \( A \). If \( B \in \mathbb{Q} \), then \( \ker_\mathbb{Q}(h) = \ker(h) \), because \( \Theta_B^X(0_B) = 0_B \).
Proposition 2.10. Let $Q$ be a quasivariety of $\tau$-algebras, $A$, $B$ arbitrary $\tau$-algebras, and $h: A \to B$ a surjective homomorphism. Then for any set $X \subseteq A^2$,

$$h(\Theta^A_Q(X) +_Q \ker(h)) = \Theta^B_Q(h(X)). \quad (*)$$

Note. If $B \in Q$, then $\ker(h) = \ker(h)$. Hence $(*)$ implies that

$$h(\Theta^A_Q(X) + \ker(h)) = \Theta^B_Q(h(X)). \quad \square$$

Proof. Since $\ker_Q(h)$ is a $Q$-congruence on $A$, therefore $\Phi := \Theta^A_Q(X) +_Q \ker_Q(h)$ is a well-defined $Q$-congruence on $A$. As $\ker(h) \subseteq \ker_Q(h) \in Con_Q(A)$, Corollary 2.8.(a) implies that $h(\Phi)$ is a $Q$-congruence on $B$. Since $X \subseteq \Phi$, we get that $h(\Theta^A_Q(X) +_Q \ker_Q(h)) = h(\Phi) \supseteq \Theta^B_Q(h(X))$.

On the other hand, as $h(\Theta^A_Q(X) \subseteq \Theta^B_Q(h(X))$ and $h(\ker_Q(h)) = \Theta^B_Q(0_B)$, we get that $h(\Theta^A_Q(X) +_Q \ker_Q(h)) = h(\Theta^A_Q(X \cup \ker_Q(h)) \subseteq \Theta^B_Q(h(X) \cup h(\ker_Q(h))) = \Theta^B_Q(h(X))$, by Proposition 2.9. \quad \square

Corollary 2.11. Let $Q$ be a quasivariety, $A$, $B$ be arbitrary $\tau$-algebras, and $h: A \to B$ a surjective homomorphism. Then for any set $X \subseteq A^2$,

$$h^{-1}(\Theta^B_Q(hX)) = \ker_Q(h) +_Q \Theta^A_Q(X). \quad \square \quad (**)$$

Proof. Since $B \in Q$, $\ker(h) = \ker_Q(h)$. \quad \square

Corollary 2.12. Let $Q$ be a quasivariety, let $A$, $B$ be arbitrary $\tau$-algebras, and $h: A \to B$ a surjective homomorphism. Then for any $\Phi, \Psi \in Con_Q(B)$,

$$h^{-1}(\Phi +_Q \Psi) = h^{-1}(\Phi) +_Q h^{-1}(\Psi).$$

Proof. As $h$ is “onto”, $hh^{-1}(\Phi) = \Phi$ and $hh^{-1}(\Psi) = \Psi$. Moreover $\ker_Q(h) \subseteq h^{-1}(\Phi)$ and $\ker_Q(h) \subseteq h^{-1}(\Psi)$. Applying Corollary 2.11 we get:

$$h^{-1}(\Phi +_Q \Psi) = h^{-1}(\Theta^B_Q(\Phi \cup \Psi)) = h^{-1}(\Theta^B_Q(h^{-1}(\Phi) \cup h^{-1}(\Psi))) = h^{-1}(\Theta^B_Q(h^{-1}(\Phi) \cup h^{-1}(\Psi))) = \ker_Q(h) +_Q \Theta^A_Q(h^{-1}(\Phi) \cup h^{-1}(\Psi)) = \ker_Q(h) +_Q \Theta^A_Q(h^{-1}(\Phi)) +_Q \Theta^A_Q(h^{-1}(\Psi)) = \ker_Q(h) +_Q h^{-1}(\Phi) +_Q h^{-1}(\Psi) = h^{-1}(\Phi) +_Q h^{-1}(\Psi). \quad \square$$

Let $F$ be a non-trivial infinitely generated free algebra and let $h: F \to F$ be an epimorphism. Let $x$ and $y$ be arbitrary free generators of $F$. Assume $F$ is free in a quasivariety $Q$. We know that $\Theta^F(x, y)$ is a $Q$-congruence of $F$. The kernel $\ker(h)$ is also a $Q$-congruence of $F$. Question: Is $\ker(h) + \Theta^F(x, y)$ a $Q$-congruence of $F$? (“+” stands for the least upper bound in the lattice of congruences of $F$.) If $hx$ and $hy$ are free generators, then indeed $\ker(h) + \Theta^F(x, y)$ is a $Q$-congruence. But if $hx$ and $hy$ are not free generators, the answer may be: No! Here is a simple example provided by Keith Kearnes.
Let \( Q \) be the quasivariety in the language with only two constant symbols 0, 1 (and no other operations) that is axiomatized by the quasi-identity \( 0 \approx 1 \rightarrow x \approx y \). Let \( F \) be the free algebra in \( Q \) with free generators \( \{x_0, x_1, \ldots \} \). Then \( F = \{0, 1, x_0, x_1, \ldots \} \).

Let \( h : F \rightarrow F \) be the function which \( x_0 \) and 0 maps to 0, \( x_1 \) and 1 maps to 1, and \( x_{n+1} \) maps to \( x_n \) for all \( n \geq 1 \). \( h \) is an epimorphism and \( \ker(h) = \{0, x_0\}^2 \cup \{1, x_1\}^2 \cup 0_F \).

The join \( \ker(h) + \Theta^F(x_0, x_1) \) equals \( \{0, 1, x_0, x_1\}^2 \cup 0_F \), i.e., it relates the four elements \( 0, 1, x_0, x_1 \) and relates no other pair of distinct elements. But the \( Q \)-join \( \ker(h) + Q \Theta^F(x_0, x_1) \) is \( F \times F \). Hence \( \ker(h) + \Theta^F(x_0, x_1) \) is not a \( Q \)-congruence.


### 2.7 Properties of Equational Theories

The theory of the equationally defined commutator to a large extent uses syntactical tools derived from the properties of the equational consequences associated with quasivarieties. In this subsection we shall present some of these properties.

The facts presented in the above section on relative congruences have their counterparts for the theories of \( Q \wedge \cdot \cdot \cdot \cdot \), where \( Q \) is a quasivariety of a signature \( \tau \).

Let \( e : T_{e\tau} \rightarrow T_{e\tau} \) be a function. If \( X \) is a set of equations, then

\[
e(X) := \{ep \approx eq : p \approx q \in X\}.
\]

\[
e^{-1}(X) := \{p \approx q \in Eq(\tau) : ep \approx eq \in X\}.
\]

We shall mark the theory \( Q \wedge (\Sigma_1 \cup \Sigma_2) \) as

\[
\Sigma_1 + Q \Sigma_2,
\]

thus using the notation applied to \( Q \)-congruences. (\( \Sigma_1 \) and \( \Sigma_2 \) are arbitrary theories of \( Q \wedge \).)

If \( e : T_{e\tau} \rightarrow T_{e\tau} \) is an endomorphism and \( \Sigma \) is a theory of \( Q \wedge \), then \( e^{-1}(\Sigma) \) is a theory of \( Q \wedge \) as well. This follows from the fact that the set of equations \( e^{-1}(\Sigma) \) is closed with respect to the rules of \( Q \wedge \) (cf. Note following Proposition 2.7).

Some care is needed when one wants to define the kernel of an endomorphism \( e \) of the term algebra. The set \( \{p \approx q \in Eq(\tau) : \text{the term } ep \text{ is identical with } eq\} \) is a theory of Birkhoff’s consequence \( B_\tau \). Not much can be said about the properties of this set when one wants to connect it with a non-trivial quasivariety. We therefore adopt the following definition.
If \( e : T_\tau \rightarrow T_\tau \) is an endomorphism, then
\[
\ker_Q(e) := e^{-1}(Q^\models(\emptyset)).
\]
\( \ker_Q(e) \) is called the kernel of the endomorphism \( e \) relative to \( Q \). Thus \( p \models q \in \ker_Q(e) \) if and only if the equation \( ep \models eq \) is valid in \( Q \). One may directly verify that \( \ker_Q(e) \) is closed with respect to the rules of \( Q^\models \) and hence it is a theory of \( Q^\models \). We obviously have that \( f \models Q^\models \) if and only if the equation \( ep \models eq \) is valid in \( Q \).

The consequence operation \( \forall a. Q^\models \) is weaker than \( Q^\models \) but both operations agree on \( \emptyset \), that is, \( Q^\models(\emptyset) = \forall a. (Q^\models(\emptyset)) \). From the purely inferential viewpoint, \( \forall a. Q^\models \) is the consequence operation determined by the set of all \( Q \)-valid equations and the rules of inference of the Birkhoff’s logic \( B_\tau \). In other words, \( \forall a. Q^\models \) is an axiomatic strengthening of Birkhoff’s logic \( B_\tau \) (in the signature of \( Q \)).

As \( Q^\models(\emptyset) = \forall a.(Q^\models(\emptyset)) \), it follows that
\[
\ker_Q(e) = \ker_{\forall a. Q^\models}(e).
\]

**Proposition 2.13.** (The correspondence property for equational theories). Let \( Q \) be a quasivariety of \( \tau \)-algebras and \( e : T_\tau \rightarrow T_\tau \) an endomorphism. If \( \Sigma \) is a theory of \( Q^\models \) and \( \ker_Q(e) \subseteq \Sigma \), then \( e^{-1}(\Sigma) = \Sigma \).

**Corollary 2.14.** Let \( Q \) be a quasivariety of \( \tau \)-algebras and \( e : T_\tau \rightarrow T_\tau \) an endomorphism. Let \( \Sigma \) be a theory of Birkhoff’s logic \( B_\tau \) in \( T_\tau \), i.e., \( \Sigma \in \text{Th}(B_\tau) \), such that \( \ker_Q(e) \subseteq \Sigma \).

(a) If \( e \) is surjective and \( \Sigma \in \text{Th}(Q^\models) \), then \( e(\Sigma) \in \text{Th}(Q^\models) \).
(b) If \( e(\Sigma) \in \text{Th}(Q^\models) \), then \( \Sigma \in \text{Th}(Q^\models) \).
(c) If \( e \) is surjective, then \( \Sigma \in \text{Th}(Q^\models) \) if and only if \( e(\Sigma) \in \text{Th}(Q^\models) \).

**Proposition 2.15.** Let \( Q \) be a quasivariety of \( \tau \)-algebras and \( e : T_\tau \rightarrow T_\tau \) an epimorphism. Then
\[
e(Q^\models(X) + Q \ker_Q(e)) = Q^\models(e(X))
\]
for any set of equations \( X \).

**Corollary 2.16.** Let \( Q \) be a quasivariety of \( \tau \)-algebras and let \( e : T_\tau \rightarrow T_\tau \) be an epimorphism. Then for any set of equations \( X \),
\[
e^{-1}(Q^\models(eX)) = \ker_Q(e) + Q Q^\models(X).
\]

**Corollary 2.17.** Let \( Q \) be a quasivariety of \( \tau \)-algebras and \( e : T_\tau \rightarrow T_\tau \) an epimorphism. Then for any theories \( \Sigma_1 \) and \( \Sigma_2 \) of \( Q^\models \),
\[
e^{-1}(\Sigma_1 + Q \Sigma_2) = e^{-1}(\Sigma_1) + Q e^{-1}(\Sigma_2)
\]
in the term algebra \( T_\tau \).
Proposition 2.18. Let $X$ be a set of equations of variables, i.e., $X = \{x_i \approx y_i : i \in I\}$, where $x_i, y_i (i \in I)$ are individual variables. Then $\text{Va}(Q)\vdash (X)$ is a theory of $Q\vdash$ for any non-trivial quasivariety $Q$. $\Box$

$\text{Va}(Q)\vdash (X)$ is a theory of the consequence $\text{Va}(Q)\vdash$. The proposition states that $\text{Va}(Q)\vdash (X)$ is a closed theory of the stronger consequence operation $Q\vdash$.

The proofs of the above facts are easy modifications of the corresponding results from the preceding subsection.

2.8 Epimorphisms and Isomorphic Embeddings of the Lattice of Theories

Let $Q$ be a quasivariety of $t$-algebras and let $e : T_\tau \rightarrow T_\tau$ be an endomorphism. We define the function $f_e : Th(Q\vdash) \rightarrow Th(Q\vdash)$ by

$$f_e(\Sigma) := e^{-1}(\Sigma) \quad \text{for all } \Sigma \in Th(Q\vdash).$$

As $e^{-1}(\Sigma)$, the $e$-preimage of $\Sigma$, is a closed theory, $f_e$ is well-defined.

The following fact is immediate:

Lemma 2.19. For any epimorphism $e : T_\tau \rightarrow T_\tau$, the function $f_e$ is an isomorphic embedding of the lattice $Th(Q\vdash)$ into $Th(Q\vdash)$.

Proof. Let $\Sigma_1$ and $\Sigma_2$ be theories of $Q\vdash$. Corollary 2.17 yields that

$$f_e(\Sigma_1 + Q \Sigma_2) = e^{-1}(\Sigma_1 + Q \Sigma_2) = e^{-1}(\Sigma_1) + Q e^{-1}(\Sigma_2) = f_e(\Sigma_1) + Q f_e(\Sigma_2),$$

$$f_e(\Sigma_1 \cap Q \Sigma_2) = e^{-1}(\Sigma_1 \cap Q \Sigma_2) = e^{-1}(\Sigma_1) \cap e^{-1}(\Sigma_2) = f_e(\Sigma_1) \cap f_e(\Sigma_2).$$

The verification that $f_e$ is one-to-one is also straightforward. $\Box$

We define: $Th^e(Q\vdash) := \{\Sigma \in Th(Q\vdash) : \ker_Q(e) \subseteq \Sigma\}$. The set $Th^e(Q\vdash)$ forms a sublattice $Th^e(Q\vdash)$ of $Th(Q\vdash)$. The theory $\ker_Q(e)$ is the least element of $Th^e(Q\vdash)$.

Corollary 2.20. For any epimorphism $e : T_\tau \rightarrow T_\tau$, the function $f_e$ is an isomorphism between the lattices $Th(Q\vdash)$ and $Th^e(Q\vdash)$.

Proof. It suffices to check that $f_e$ is a surjection from $Th(Q\vdash)$ onto $Th^e(Q\vdash)$. Suppose $Y \in Th^e(Q\vdash)$. We claim that $Y = f_e(X)$ for some $X \in Th(Q\vdash)$. We put: $X := Q\vdash (eY)$. Then, by Corollary 2.16, $f_e(X) = e^{-1}(X) = e^{-1}(Q\vdash (eY)) = \ker_Q(e) + Q\vdash (Y) = \ker_Q(e) + Q\vdash Y = Y$. $\Box$

It follows from the correspondence property (Proposition 2.13) that the function $g_e : Th^e(Q\vdash) \rightarrow Th(Q\vdash)$ given by

$$g_e(\Sigma) := e(\Sigma) \quad (= \{e(\sigma) : \sigma \in \Sigma\}), \quad \text{for all } \Sigma \in Th^e(Q\vdash),$$

is the inverse of the isomorphism $f_e$. 

The function \( k_e : \text{Th}(Q^\vdash) \to \text{Th}^e(Q^\vdash) \) given by

\[
k_e(\Sigma) := \ker_Q(e) +_Q \Sigma, \quad \Sigma \in \text{Th}(Q^\vdash),
\]
is a retraction, that is \( k_e \) is a surjection and \( k_e \) is the identity map on the sublattice \( \text{Th}^e(Q^\vdash) \). \( k_e \) need not be a lattice homomorphism: \( k_e \) preserves joins but does not preserve meets of theories.

### 2.9 The Kernels of Epimorphisms

Let \( Q \) be a quasivariety of \( \tau \)-algebras. \( \text{Var} \) is the (countably infinite) set of individual variables of \( \text{Te}_\tau \).

We know that for every endomorphism \( e : \text{Te}_\tau \to \text{Te}_\tau \)

\[
\ker_Q(e) := e^{-1}(Q^{\vdash}(\emptyset))
\]
is a closed theory of \( Q^{\vdash} \). As \( Q^{\vdash}(\emptyset) = \text{Va}(Q)^{\vdash}(\emptyset) \), \( \ker_Q(e) \) is also a theory of \( \text{Va}(Q)^{\vdash}(\emptyset) \). (\( \text{Va}(Q)^{\vdash} \) is the consequence operation determined by the set of all \( Q \)-valid equations and the rules of inference of Birkhoff’s logic \( B_\tau \).

Let \( e : \text{Te}_\tau \to \text{Te}_\tau \) be an endomorphism. We define:

\[
V_e := \{ x \in \text{Var} : e(x) \in \text{Var} \}.
\]

Thus \( V_e := e^{-1}(\text{Var}) \).

From now on we assume \( e : \text{Te}_\tau \to \text{Te}_\tau \) is a fixed epimorphism, i.e., a surjective endomorphism. Then the set \( V_e \) is infinite and \( e \) surjectively maps \( V_e \) onto \( \text{Var} \). Moreover, \( e \) assigns a compound term to each variable \( x \in \text{Var} \setminus V_e \).

\( \text{Te}_\tau(V_e) \) is the set of all terms of \( \text{Te}_\tau \) in the variables \( V_e \) and \( \text{Te}_\tau(V_e) \) is the corresponding subalgebra of \( \text{Te}_\tau \).

For each variable \( x \in \text{Var} \) we mark the term \( e(x) \) as \( t_x \). If \( x \in V_e \), then \( t_x \) is a variable, not necessarily in \( V_e \). It is clear that then there is a variable \( s_x \in V_e \) such that \( e(s_x) = t_x (= e(x)) \), viz., \( s_x := x \). If \( x \in \text{Var} \setminus V_e \), the term \( t_x \) is compound, and we write \( t_x = t_x(y) \), where \( y = y_1, \ldots, y_n \) is the list of variables occurring in \( t_x \). There are (different) variables \( \bar{x} = x_1, \ldots, x_n \) in \( V_e \) such that \( y = e(\bar{x}) \), that is, \( y_1 = e(x_1), \ldots, y_n = e(x_n) \). Then \( e(x) = t_x = t_x(y) = t_x(e(\bar{x})) = e(t_x(\bar{x})) \).

Hence the term \( e(x) \) is identical with \( e(t_x(\bar{x})) \). Let us denote the term \( t_x(\bar{x}) \) by \( s_x \). We thus get:

**Fact 1.** For every variable \( x \in \text{Var} \) there is a term \( s_x \in \text{Te}_\tau(V_e) \) such that the term \( e(x) \) is identical with \( e(s_x) \). If \( x \in V_e \), then \( s_x \) is the variable \( x \). If \( x \in \text{Var} \setminus V_e \), the term \( s_x \) is compound.

To simplify notation, we shall omit parentheses in the symbols like ‘\( e(x) \)’.
e surjectively maps \( V_e \) onto \( \text{Var} \) but it may glue some variables of \( V_e \). We then put:

\[
A_0 := \{ x \approx s_x : x \in \text{Var} \setminus V_e \} \cup \{ x \approx y : x, y \in V_e \text{ and } ex = ey \} \cup \\
\{ x \approx y : x, y \in \text{Var} \setminus V_e \text{ and } ex = ey \},
\]

where, for each \( x \in \text{Var} \setminus V_e \), \( s_x \) is an arbitrary but fixed term in \( Te_t(V_e) \) such that \( ex = es_x \).

The following observation, which seems to have not been considered in the literature, provides a canonical characterization of sets of equations generating the kernel of an arbitrary epimorphism of the term algebra. This characterization will be applied in Chapters 5 and 6.

**Theorem 2.21.** \( \ker Q(e) = \text{Var}(Q)^=\langle A_0 \rangle \).

**Notes.**

1. If \( e \) injectively maps \( V_e \) onto \( \text{Var} \), the set \( \{ x \approx y : x, y \in V_e \text{ and } ex = ey \} \), being a component of \( A_0 \), reduces to \( \{ x \approx x : x \in V_e \} \) and therefore it may be disregarded in the above definition. Similarly, if \( e \) does not paste together different variables in \( \text{Var} \setminus V_e \), the set \( \{ x \approx y : x, y \in \text{Var} \setminus V_e \text{ and } ex = ey \} \) may be omitted too.

2. The theorem implies that \( \ker Q(e) = Q^=\langle A_0 \rangle \), because \( \ker Q(e) \) is a theory of \( Q^=\).

**Proof.** To simplify notation we put \( C := Q^=\langle A_0 \rangle \). We also put: \( C_0 := \text{Var}(Q)^=\langle A_0 \rangle \). The definitions of \( A_0 \) and \( \ker Q(e) \) immediately give that \( A_0 \subseteq \ker Q(e) \). Hence \( C_0(A_0) \subseteq \ker Q(e) \). We shall show that \( \ker Q(e) \subseteq C_0(A_0) \).

**Claim 1.** If \( x, y \in \text{Var} \setminus V_e \) and \( ex \) is identical with \( ey \), then \( s_x \approx s_y \in C_0(A_0) \).

**Proof (of the claim).** We have: \( x \approx s_x, y \approx s_y \in A_0 \). Moreover \( x \approx y \in A_0 \), because \( ex = ey \). It follows that \( s_x \approx s_y \in C_0(A_0) \).

**Claim 2.** For each term \( t \in Te_t \) there is a term \( s_t \in Te_t(V_e) \) such that \( t \approx s_t \in C_0(A_0) \). Moreover, \( et \) is identical with \( es_t \).

**Proof.** We use induction on the complexity of terms. Assume \( t \) is a variable \( x \). If \( x \in V_e \), then \( s_x \) is identical with \( x \). Hence trivially \( x \approx s_x \in C_0(A_0) \). If \( x \in \text{Var} \setminus V_e \), we take the compound term \( s_x \in Te_t(V_e) \) corresponding to \( x \) and defined as above. Then \( ex = es_x \) and \( x \approx s_x \in A_0 \), by the definition of \( A_0 \). Hence \( x \approx s_x \in C_0(A_0) \).

Let \( t \) be a compound term \( F(t_1 \ldots t_k) \), where \( F \) is a \( k \)-ary function symbol, and assume the thesis holds for the terms \( t_1, \ldots, t_k \). There are terms \( s_{t_1}, \ldots, s_{t_k} \in Te_t(V_e) \) such that \( t_1 \approx s_{t_1}, \ldots, t_k \approx s_{t_k} \in C_0(A_0) \) and \( et_1 = es_{t_1}, \ldots, et_k = es_{t_k} \). We put \( s_t := F(s_{t_1} \ldots s_{t_k}) \). We have: \( s_t \in Te_t(V_e), t \approx s_t \in C_0(t_1 \approx s_{t_1}, \ldots, t_k \approx s_{t_k}) \in C_0(A_0) \) by functionality rules, and \( et = es_t \).

It should be noted that if \( t \) is in \( Te_t(V_e) \), then it follows from the above definitions that \( t \) is identical with \( s_t \).
Claim 3. Let \( r \) and \( s \) be terms in \( T \varepsilon_T(V_e) \) such that \( er \approx es \in C_0(\emptyset) \). Then \( r \approx s \in C_0(A_0) \).

Proof. Write \( r = r(\bar{x}) \), \( s = s(\bar{y}) \), where \( \bar{x} = x_1, \ldots, x_n \). As \( r \) and \( s \) are in \( T \varepsilon_T(V_e) \), the substitution \( e \) merely replaces the variables of \( \bar{x} \) by a block of other variables \( y = ex_1, \ldots, ex_n \) (but not necessarily in a one-to-one way). As \( er \approx es \) is identical with \( r(y) \approx s(y) \), we see that \( r(y) \approx s(y) \in C_0(\emptyset) \) implies that \( r(\bar{x}) \approx s(\bar{x}) \in C_0(\{x_i \approx x_j : 1 \leq i < j \leq n \text{ and } ex_i = ex_j\}) \). Since \( \{x_i \approx x_j : 1 \leq i < j \leq n \text{ and } ex_i = ex_j\} \in A_0 \), it follows that \( r \approx s \in C_0(A_0) \).

Claim 4. Let \( p \) and \( q \) be arbitrary terms in \( T \varepsilon_T \) such that \( ep \approx eq \in C_0(\emptyset) \). Then \( p \approx q \in C_0(A_0) \).

Proof. According to Claim 2, there are terms \( s_p, s_q \in T \varepsilon_T(V_e) \) such that \( p \approx s_p \), \( q \approx s_q \in C_0(A_0) \) and \( ep = es_p, eq = es_q \). As \( ep \approx eq \in C_0(\emptyset) \), we trivially get that \( es_p \approx es_q \in C_0(\emptyset) \). Hence, by Claim 3, \( s_p \approx s_q \in C_0(A_0) \). This fact together with \( p \approx s_p, q \approx s_q \in C_0(A_0) \) yields that \( p \approx q \in C_0(A_0) \).

From the above claim the theorem follows.

Example. Let \( p(y) \) and \( q(y) \) be two different compound terms in variables \( y \) and \( x \) be different variables. Let \( e : T \varepsilon_T \rightarrow T \varepsilon_T \) be an epimorphism such that \( ex = p, ey = q \) and \( e \) bijectively maps \( Var \setminus \{x, y\} \) onto \( Var \). Then \( V_e = Var \setminus \{x, y\} \). Let \( \bar{x} \) be the set of variables of \( V_e \) such that \( ex = y \).

Let \( Q \) be an arbitrary quasivariety. It follows from the above theorem that the set

\[
    A_0 := \{x \approx p(\bar{x}), y \approx q(\bar{x})\}
\]

generates the kernel \( \ker_Q(e) \), that is,

\[
    \ker_Q(e) = V\!a(Q)^=\!(A_0).
\]

Note. In reference to Theorem 2.21 we also note the following distributivity law involving the kernels of epimorphisms.

The following notions will be used in the chapters devoted to the equationally defined commutator. A set of equations of variables is any set \( \{x_i \approx y_i : i \in I\} \), where \( x_i \) and \( y_i \) are variables for \( i \in I \) and the variables occurring in the equations \( x_i \approx y_i \) are all pairwise different. Two sets \( X \) and \( Y \) of equations of variables are separated if the equations of \( X \) and \( Y \) do not share a common variable.

Theorem 2.22. Let \( Q \) be a quasivariety and \( e : T \varepsilon_T \rightarrow T \varepsilon_T \) an epimorphism. Define the set \( V_e \) as above. Let \( X \) and \( Y \) be separated sets of equations of variables from \( V_e \). If \( e \) is injective on the set of variables occurring in \( X \cup Y \), then

\[
    (\ker_Q(e)^=\!(X) \cap \ker_Q(e)^=\!(Y)) = \ker_Q(e)^=\!(X) \cap \ker_Q(e)^=\!(Y).
\]
The above theorem can be proved rather easily by applying Theorem 2.21 and working with the kernel $\ker_{Q}(e)$ (cf. also Lemma 5.2.10). Another proof of the theorem is presented in Section 3.3 (see Theorem 3.3.7).

The thesis of Theorem 2.22 holds for any relatively congruence-distributive quasivariety $Q$ (without any restrictions imposed on $X$, $Y$ and $e$).
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