This book is based on the notes of both authors for a course called “Higher Algebra,” a graduate level course. Its purpose was to offer the basic abstract algebra that any student of mathematics seeking an advanced degree might require. Students may have been previously exposed to some of the basic algebraic objects (groups, rings, vector spaces, etc.) in an introductory abstract algebra course such as that offered in the classic book of Herstein. But that exposure should not be a hard requirement as this book proceeds from first principles. Aside from the far greater theoretical depth, perhaps the main difference between an introductory algebra course, and a course in “higher algebra” (as exemplified by classics such as Jacobson’s Basic algebra [1, 2] and Van der Waerden’s Modern Algebra [3]) is an emphasis on the student understanding how to construct a mathematical proof, and that is where the exercises come in.

The authors rotated teaching this one-year course called “Higher Algebra” at Kansas State University for 15 years—each of us generating his own set of notes for the course. This book is a blend of these notes.

Listed below are some special features of these notes.

1. (Combinatorial Background) Often the underlying combinatorial contexts—partially ordered sets etc.—seem almost invisible in a course on modern algebra. In fact they are often developed far from home in the middle of some specific algebraic context. Partially ordered sets are the natural context in which to discuss the following:

   a) Zorn’s Lemma and the ascending and descending chain conditions,
   b) Galois connections,
   c) The modular law,
   d) The Jordan Hölder Theorem,
   e) Dependence Theories (needed for defining various notions of “dimension”).

The Jordan Hölder Theorem asserts that in a lower semimodular semilattice, any semimodular function from the set of covers (unrefinable chains of length one)
to a commutative monoid extends to an interval measure on all algebraic intervals (those intervals containing a finite unrefinable chain from the bottom to the top). The extension exists because the multiset of values of the function on the covers in any two unrefinable chains connecting \( a \) and \( b \) must be the same. The proof is quite easy, and the applications are everywhere. For example, when \( G \) is a finite group and \( P \) is the poset of subnormal subgroups, one notes that \( P \) is a semimodular lower semilattice and the reading of the simple group \( A/B \) of a cover \( A < B \) is a semimodular function on covers by a fundamental theorem of homomorphisms of groups. By the theorem being described, this function extends to an interval measure with values in the additive monoid of multisets on the isomorphism classes of simple groups. The conclusion of the combinatorial Jordan-Hölder version in this context becomes the classical Jordan-Hölder Theorem for finite groups. One needs no “Butterfly Lemma” or anything else.

2. (Free Groups) Often a free group on generators \( X \) is presented in an awkward way—by defining a “multiplication” on ‘reduced words’ \( r(w) \), where \( w \) is a word in the free monoid \( M(X \cup X^{-1}) \). ‘Reduced’ means all factors of the form \( xx^{-1} \) have been removed. Here are the complications: First the reductions, which can often be performed in many ways, must lead to a common reduced word. Then one must show \( r(w_1 \circ w_2) = r(r(w_1) \circ r(w_2)) \) to get “multiplication” defined on reduced words. Then one needs to verify the associative law and the other group axioms. In this book the free group is defined to be the automorphism group of a certain labelled graph, and the universal mapping properties of the free group are easily derived from the graph. Since full sets of automorphisms of an object always form a group, one will not be wasting time showing that an awkwardly-defined multiplication obeys the axioms of a group.

3. (Universal Mapping Properties) These are always instances of the existence of an initial or terminal object in an appropriate category.

4. (Avoiding Determinants of Matrices) Of course one needs matrices to describe linear transformations of vector spaces, or to record data about bilinear forms (the Grammian). It is important to know when the rows or columns of a matrix are linearly dependant. One can calculate what is normally called the determinant by finding the invariant factors. For an \( n \times n \) matrix, that process involves roughly \( n^3 \) steps, while the usual procedure for evaluating the determinant using Lagrange’s rule, involves exponentially many steps. One of the standard proofs that the trace mapping \( \text{tr} : K \to F \) of a finite separable field extension \( F \subseteq K \) is nonzero proceeds as follows: First, one forms the normal closure \( L \) of the field \( K \). One then invokes the theorem that \( L = F(\theta) \), a simple extension, with the algebraic conjugates of \( \theta \) as an \( F \)-basis of \( L \). And then one reaches the conclusion by observing that a van der Monde determinant is non-zero. Perhaps it is an aesthetic quibble, but one does not like to see a nice “soft” algebraic proof about “soft” algebraic objects reduced to a matrix calculation. In Sect. 11.7 the proof that the trace is non-trivial is accomplished using only the Dedekind Independence Lemma and an elementary fact about bilinear forms.
In general, in this book, the determinant of a transformation $T$ acting on an $n$-dimension vector space $V$ is defined to be the scalar multiplication it induces on the $n$-th exterior product $\wedge^n(V)$. Of course there are historical reasons for making a few exceptions to any decree to ban the usual formulaic definition of determinants altogether. Our historical discussion of the discriminant on page 395 is such an exception.

In addition, we have shaped the text with several pedagogical objectives in mind.

1. (Catch-up opportunities) Not infrequently, the teacher of a graduate course is expected to accommodate incoming transfer students whose mathematical preparation is not quite the same as that of current students of the program, or is even unknown. At the same time, this accommodation should not sacrifice course content for the other students. For this reason we have written each chapter at a gradient—with simplest explanations and examples first, before continuing at the level the curriculum requires. This way, a student may “catch up” by studying the introductory material more intensely, while a more brief review of it is presented in class. Students already familiar with the introductory material have merely to turn the page.

2. (Curiosity-driven Appendices) The view of both authors has always been that a course in Algebra is not an exercise in cramming information, but is instead a way of inspiring mathematical curiosity. Real learning is basically curiosity-driven self-learning. Discussing what is already known is simply there to guide the student to the real questions. For that reason we have inserted a number of appendices which are largely centered around incites connected with proofs in the text. Similarly, in the exercises, we have occasionally wandered into open problems or offered avenues for exploration. Mathematics education is not a catechism.

3. (Planned Redundancy) Beside its role as a course guide, a textbook often lives another life as a source book. There is always the need of a student or colleague in a nearby mathematical field to check on some algebraic fact—say, to make sure of the hypotheses that accompany that fact. He or she does not need to read the whole book. But occasionally one wanders into the following scenario: one looks up topic A in the index, and finds, at the indicated page, that A is defined by further words B and C whose definition can be deciphered by a further visit to the index, which obligingly invites one to further pages at which the frustration may be enjoyed once again. It becomes a tree search. In order to intercept this process, we have tried to do the following: when an earlier-defined key concept re-inserts itself in a later discussion, we simply recall the definition for the reader at that point, while offering a page number where the concept was originally defined. Nevertheless we are introducing a redundancy. But in the

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1If we carried out this process for the most common concepts, pages would be filled with re-definitions of rings, natural numbers, and what the containment relation is. Of course one has to limit these reminders of definitions to new key terms.
view of the authors’ experience in Kansas, redundancy is a valuable tool in teaching. Inspiration is useless if the student cannot first understand the words—and no teacher should apologize for redundancy. Judiciously applied, it does not waste class time; it actually saves it.

Of course there are many topics—direct offshoots of the material of this course—that cannot be included here. One cannot do justice to such topics in a brief survey like this. Thus one will not find in this book material about (i) Representation Theory and Character Theory of Groups, (ii) Commutative Rings and the world of Ext and Tor, (iii) Group Cohomology or other Homological Algebra, (iv) Algebraic Geometry, (v) Really Deep Algebraic Number Theory and (vi) many other topics. The student is better off receiving a full exposition of these courses elsewhere rather than being deceived by the belief that the chapters of this book provide such an expertise. Of course, we try to indicate some of these points of departure as we meet them in the text, at times suggesting exterior references.

A few words are inserted here about how the book can be used.

As mentioned above, the book is a blend of the notes of both authors who alternately taught the course for many years. Of course there is much more in this book than can reasonably be covered in a two-semester course. In practice a course includes enough material from each chapter to reach the principle theorems. That is, portions of chapters can be left out. Of course the authors did not always present the course in exactly the same way, but the differences were mainly in the way focus and depth were distributed over the various topics. We did not “teach” the appendices to the chapters. They were there for the students to explore on their own.

The syllabus presented here would be fairly typical. The numbers in parenthesis represent the number of class-hours the lectures usually consume. A two-semester course entails 72 class-hours. Beyond the lectures we normally allowed ourselves 10–12 h for examinations and review of exercises.

1. Chapter 1: (1 or 2) [This goes quickly since it involves only two easy proofs.]
2. Chapter 2: (6, at most) [This also goes quickly since, except for three easy proofs, it is descriptive. The breakdown would be: (a) 2.2.1–2.2.9 (skip 2.2.10), 2.2.10–2.2.15 (3 h), (b) 2.3 and 2.5 (2 h) and (c) 2.6 (1 h).]
3. Chapter 3: (3)
4. Chapter 4: (3) [Sometimes omitting 4.2.3.]
5. Chapter 5: (3 or 4) [Sometimes omitting 5.5.]
6. Chapter 6: (3) [Omitting the Brauer-Ree Theorem [6.4] but reserving 15 minutes for Sect. 6.6.]
7. Chapter 7: (3) Mostly examples and few proofs. Section 7.3.6 is often omitted.
8. Chapter 8: (7 or 8) [Usually (a) 8.1 (2 or 3 h), and (b) 8.2–8.4 (4 h). We sometimes omitted Sect. 8.3 if behind schedule.]
9. Chapter 9: (4) [One of us taught only 9.1–9.8 (sometimes omitting the local characterization of UFDs in 9.6.3) while the other would teach all of 9.9–9.12 (Dedekind’s Theorem and the ideal class group.)]
10. Chapter 10: (6) [It takes 3 days for 10.1–10.5. The student is asked to read 10.6, and 3 days remain for Sect. 10.7 and autopsies on some of the exercises.]

11. Chapter 11: (11) [The content is rationed as follows: (a) 11.1–11.4 (2 h), sometimes omitting 11.4.2 if one is short a day, (b) 11.5–11.6 (3 h) (c) 11.7 [One need only mention this.] (d) 11.8–11.9 (3 h) (e) [One of us would often omit 11.10 (algebraic field extensions are often simple extensions). Many insist this be part of the Algebra Catechism. Although the result is vaguely interesting, it is not needed for a single proof in this book.] (f) 11.11 (1 h) (g) [Then a day or two would be spent going through sampled exercises.]]

12. Chapter 12: (5 or 6) [Content divided as (a) 12.1–12.3 (2 h) and (b) 12.4–12.5 (2 h) with an extra hour wherever needed.]

13. Chapter 13: (9 or 10) [Approximate time allotment: (a) 13.1–13.2 (1 h) (only elementary proofs here), (b) 13.3.1–13.3.2 (1 h), (c) 13.3.3–13.3.4 (adjunct functors) (1 h), (d) 13.4–13.5 (1 h), (e) 13.6–13.8 (1 or 2 h), (f) 13.9 (1 h), 13.10 (2 h) and 13.8 (3 h).]

The list above is only offered as an example. The book provides ample “wiggle room” for composing alternative passages through this course, perhaps even re-arranging the order of topics. The one invariant is that Chap. 2 feeds all subsequent chapters.

Beyond this, certain groups of chapters may serve as one semester courses on their own. Here are some suggestions:

GROUP THEORY: Chaps. 3–6 (invoking only the Jordan Holder Theorem from Chap. 2).

THEORY OF FIELDS: After an elementary preparation about UFD’s (their maximal ideals, and homomorphisms of polynomial rings in Chap. 6), and Groups (their actions, homomorphisms and facts about subgroup indices from Sects. 3.2, 3.3 and 4.2) one could easily compose a semester course on Fields from Chap. 11.

ARITHMETIC: UFD’s, including PID’s with applications to Linear Algebra using Chaps. 7–10.

BASIC RING THEORY: leading to Wedderburn’s Theorem. Chapters 7, 8 and 12.

RINGS AND MODULES, TENSOR PRODUCTS AND MULTILINEAR ALGEBRA: Chaps. 7, 8 and 13.

References


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