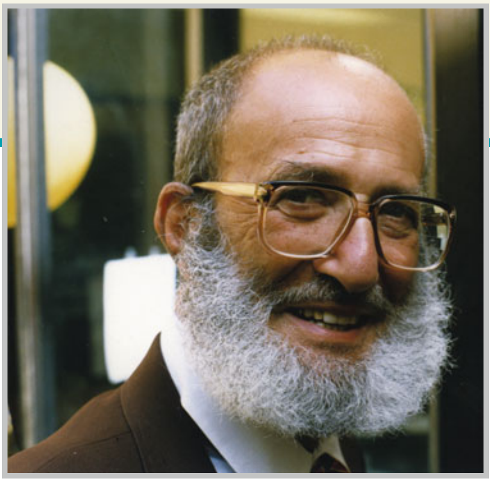
CHAPTER

2



American mathematician Paul Halmos (1916–2006), who in 1942 published the first modern linear algebra book. The title of Halmos's book was the same as the title of this chapter.

Finite-Dimensional Vector Spaces

Let's review our standing assumptions:

2.1 Notation F, V

- **F** denotes **R** or **C**.
- V denotes a vector space over **F**.

In the last chapter we learned about vector spaces. Linear algebra focuses not on arbitrary vector spaces, but on finite-dimensional vector spaces, which we introduce in this chapter.

LEARNING OBJECTIVES FOR THIS CHAPTER

- span
- linear independence
- bases
- dimension

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2.A *Span and Linear Independence*

We have been writing lists of numbers surrounded by parentheses, and we will continue to do so for elements of $\mathbf{F}^n$; for example, $(2, -7, 8) \in \mathbf{F}^3$. However, now we need to consider lists of vectors (which may be elements of $\mathbf{F}^n$ or of other vector spaces). To avoid confusion, we will usually write lists of vectors without surrounding parentheses. For example, $(4, 1, 6), (9, 5, 7)$ is a list of length 2 of vectors in $\mathbf{R}^3$.

2.2 Notation ***list of vectors***

We will usually write lists of vectors without surrounding parentheses.

Linear Combinations and Span

Adding up scalar multiples of vectors in a list gives what is called a linear combination of the list. Here is the formal definition:

2.3 Definition ***linear combination***

A ***linear combination*** of a list $v_1, \dots, v_m$ of vectors in V is a vector of the form

$$a_1 v_1 + \cdots + a_m v_m,$$

where $a_1, \dots, a_m \in \mathbf{F}$.

2.4 Example In $\mathbf{F}^3$,

- $(17, -4, 2)$ is a linear combination of $(2, 1, -3), (1, -2, 4)$ because

$$(17, -4, 2) = 6(2, 1, -3) + 5(1, -2, 4).$$

- $(17, -4, 5)$ is not a linear combination of $(2, 1, -3), (1, -2, 4)$ because there do not exist numbers $a_1, a_2 \in \mathbf{F}$ such that

$$(17, -4, 5) = a_1(2, 1, -3) + a_2(1, -2, 4).$$

  In other words, the system of equations

$$\begin{aligned} 17 &= 2a_1 + a_2 \\ -4 &= a_1 - 2a_2 \\ 5 &= -3a_1 + 4a_2 \end{aligned}$$

  has no solutions (as you should verify).

2.5 Definition ***span***

The set of all linear combinations of a list of vectors $v_1, \dots, v_m$ in V is called the ***span*** of $v_1, \dots, v_m$, denoted $\operatorname{span}(v_1, \dots, v_m)$. In other words,

$$\operatorname{span}(v_1, \dots, v_m) = \{a_1 v_1 + \cdots + a_m v_m : a_1, \dots, a_m \in \mathbf{F}\}.$$

The span of the empty list $(\,)$ is defined to be $\{0\}$.

2.6 Example The previous example shows that in $\mathbf{F}^3$,

- $(17, -4, 2) \in \operatorname{span}\big((2, 1, -3), (1, -2, 4)\big)$;
- $(17, -4, 5) \notin \operatorname{span}\big((2, 1, -3), (1, -2, 4)\big)$.

Some mathematicians use the term ***linear span***, which means the same as span.

2.7 Span is the smallest containing subspace

The span of a list of vectors in V is the smallest subspace of V containing all the vectors in the list.

Proof Suppose $v_1, \dots, v_m$ is a list of vectors in V.

First we show that $\operatorname{span}(v_1, \dots, v_m)$ is a subspace of V. The additive identity is in $\operatorname{span}(v_1, \dots, v_m)$, because

$$0 = 0v_1 + \cdots + 0v_m.$$

Also, $\operatorname{span}(v_1, \dots, v_m)$ is closed under addition, because

$$(a_1 v_1 + \cdots + a_m v_m) + (c_1 v_1 + \cdots + c_m v_m) = (a_1 + c_1)v_1 + \cdots + (a_m + c_m)v_m.$$

Furthermore, $\operatorname{span}(v_1, \dots, v_m)$ is closed under scalar multiplication, because

$$\lambda(a_1 v_1 + \cdots + a_m v_m) = \lambda a_1 v_1 + \cdots + \lambda a_m v_m.$$

Thus $\operatorname{span}(v_1, \dots, v_m)$ is a subspace of V (by 1.34).

Each v_j is a linear combination of $v_1, \dots, v_m$ (to show this, set $a_j = 1$ and let the other a's in 2.3 equal 0). Thus $\operatorname{span}(v_1, \dots, v_m)$ contains each v_j. Conversely, because subspaces are closed under scalar multiplication and addition, every subspace of V containing each v_j contains $\operatorname{span}(v_1, \dots, v_m)$. Thus $\operatorname{span}(v_1, \dots, v_m)$ is the smallest subspace of V containing all the vectors $v_1, \dots, v_m$. ■

2.8 Definition ***spans***

If $\operatorname{span}(v_1, \dots, v_m)$ equals V, we say that $v_1, \dots, v_m$ ***spans*** V.

2.9 Example Suppose n is a positive integer. Show that

$$(1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$$

spans $\mathbf{F}^n$. Here the j^{th} vector in the list above is the n-tuple with 1 in the j^{th} slot and 0 in all other slots.

Solution Suppose $(x_1, \dots, x_n) \in \mathbf{F}^n$. Then

$$(x_1, \dots, x_n) = x_1(1, 0, \dots, 0) + x_2(0, 1, 0, \dots, 0) + \cdots + x_n(0, \dots, 0, 1).$$

Thus $(x_1, \dots, x_n) \in \operatorname{span}\big((1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, \dots, 0, 1)\big)$, as desired.

Now we can make one of the key definitions in linear algebra.

2.10 Definition ***finite-dimensional vector space***

A vector space is called ***finite-dimensional*** if some list of vectors in it spans the space.

Recall that by definition every list has finite length.

Example 2.9 above shows that $\mathbf{F}^n$ is a finite-dimensional vector space for every positive integer n.

The definition of a polynomial is no doubt already familiar to you.

2.11 Definition ***polynomial,*** $\mathcal{P}(\mathbf{F})$

- A function $p\colon \mathbf{F} \to \mathbf{F}$ is called a ***polynomial*** with coefficients in $\mathbf{F}$ if there exist $a_0, \dots, a_m \in \mathbf{F}$ such that

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$$

for all $z \in \mathbf{F}$.

- $\mathcal{P}(\mathbf{F})$ is the set of all polynomials with coefficients in $\mathbf{F}$.

With the usual operations of addition and scalar multiplication, $\mathcal{P}(\mathbf{F})$ is a vector space over $\mathbf{F}$, as you should verify. In other words, $\mathcal{P}(\mathbf{F})$ is a subspace of $\mathbf{F}^{\mathbf{F}}$, the vector space of functions from $\mathbf{F}$ to $\mathbf{F}$.

If a polynomial (thought of as a function from $\mathbf{F}$ to $\mathbf{F}$) is represented by two sets of coefficients, then subtracting one representation of the polynomial from the other produces a polynomial that is identically zero as a function on $\mathbf{F}$ and hence has all zero coefficients (if you are unfamiliar with this fact, just believe it for now; we will prove it later—see 4.7). **Conclusion:** the coefficients of a polynomial are uniquely determined by the polynomial. Thus the next definition uniquely defines the degree of a polynomial.

2.12 Definition ***degree of a polynomial,*** deg p

- A polynomial $p \in \mathcal{P}(\mathbf{F})$ is said to have ***degree*** m if there exist scalars $a_0, a_1, \ldots, a_m \in \mathbf{F}$ with $a_m \neq 0$ such that

$$p(z) = a_0 + a_1 z + \cdots + a_m z^m$$

for all $z \in \mathbf{F}$. If p has degree m, we write $\deg p = m$.

- The polynomial that is identically 0 is said to have degree $-\infty$.

In the next definition, we use the convention that $-\infty < m$, which means that the polynomial 0 is in $\mathcal{P}_m(\mathbf{F})$.

2.13 Definition $\mathcal{P}_m(\mathbf{F})$

For m a nonnegative integer, $\mathcal{P}_m(\mathbf{F})$ denotes the set of all polynomials with coefficients in $\mathbf{F}$ and degree at most m.

To verify the next example, note that $\mathcal{P}_m(\mathbf{F}) = \operatorname{span}(1, z, \ldots, z^m)$; here we are slightly abusing notation by letting z^k denote a function.

2.14 Example $\mathcal{P}_m(\mathbf{F})$ is a finite-dimensional vector space for each nonnegative integer m.

2.15 Definition ***infinite-dimensional vector space***

A vector space is called ***infinite-dimensional*** if it is not finite-dimensional.

2.16 Example Show that $\mathcal{P}(\mathbf{F})$ is infinite-dimensional.

Solution Consider any list of elements of $\mathcal{P}(\mathbf{F})$. Let m denote the highest degree of the polynomials in this list. Then every polynomial in the span of this list has degree at most m. Thus z^{m+1} is not in the span of our list. Hence no list spans $\mathcal{P}(\mathbf{F})$. Thus $\mathcal{P}(\mathbf{F})$ is infinite-dimensional.

Linear Independence

Suppose $v_1, \dots, v_m \in V$ and $v \in \operatorname{span}(v_1, \dots, v_m)$. By the definition of span, there exist $a_1, \dots, a_m \in \mathbf{F}$ such that

$$v = a_1 v_1 + \cdots + a_m v_m.$$

Consider the question of whether the choice of scalars in the equation above is unique. Suppose $c_1, \dots, c_m$ is another set of scalars such that

$$v = c_1 v_1 + \cdots + c_m v_m.$$

Subtracting the last two equations, we have

$$0 = (a_1 - c_1)v_1 + \cdots + (a_m - c_m)v_m.$$

Thus we have written 0 as a linear combination of $(v_1, \dots, v_m)$. If the only way to do this is the obvious way (using 0 for all scalars), then each $a_j - c_j$ equals 0, which means that each a_j equals c_j (and thus the choice of scalars was indeed unique). This situation is so important that we give it a special name—linear independence—which we now define.

2.17 Definition ***linearly independent***

- A list $v_1, \dots, v_m$ of vectors in V is called ***linearly independent*** if the only choice of $a_1, \dots, a_m \in \mathbf{F}$ that makes $a_1 v_1 + \cdots + a_m v_m$ equal 0 is $a_1 = \cdots = a_m = 0$.
- The empty list ( ) is also declared to be linearly independent.

The reasoning above shows that $v_1, \dots, v_m$ is linearly independent if and only if each vector in $\operatorname{span}(v_1, \dots, v_m)$ has only one representation as a linear combination of $v_1, \dots, v_m$.

2.18 Example ***linearly independent lists***

(a) A list v of one vector $v \in V$ is linearly independent if and only if $v \neq 0$.

(b) A list of two vectors in V is linearly independent if and only if neither vector is a scalar multiple of the other.

(c) $(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0)$ is linearly independent in $\mathbf{F}^4$.

(d) The list $1, z, \ldots, z^m$ is linearly independent in $\mathcal{P}(\mathbf{F})$ for each nonnegative integer m.

If some vectors are removed from a linearly independent list, the remaining list is also linearly independent, as you should verify.

2.19 Definition ***linearly dependent***

- A list of vectors in V is called ***linearly dependent*** if it is not linearly independent.
- In other words, a list $v_1, \ldots, v_m$ of vectors in V is linearly dependent if there exist $a_1, \ldots, a_m \in \mathbf{F}$, not all 0, such that $a_1 v_1 + \cdots + a_m v_m = 0$.

2.20 Example ***linearly dependent lists***

- $(2, 3, 1), (1, -1, 2), (7, 3, 8)$ is linearly dependent in $\mathbf{F}^3$ because

$$2(2, 3, 1) + 3(1, -1, 2) + (-1)(7, 3, 8) = (0, 0, 0).$$

- The list $(2, 3, 1), (1, -1, 2), (7, 3, c)$ is linearly dependent in $\mathbf{F}^3$ if and only if $c = 8$, as you should verify.
- If some vector in a list of vectors in V is a linear combination of the other vectors, then the list is linearly dependent. (Proof: After writing one vector in the list as equal to a linear combination of the other vectors, move that vector to the other side of the equation, where it will be multiplied by -1.)
- Every list of vectors in V containing the 0 vector is linearly dependent. (This is a special case of the previous bullet point.)

The lemma below will often be useful. It states that given a linearly dependent list of vectors, one of the vectors is in the span of the previous ones and furthermore we can throw out that vector without changing the span of the original list.

2.21 Linear Dependence Lemma

Suppose $v_1, \ldots, v_m$ is a linearly dependent list in V. Then there exists $j \in \{1, 2, \ldots, m\}$ such that the following hold:

(a) $v_j \in \operatorname{span}(v_1, \ldots, v_{j-1})$;

(b) if the j^{th} term is removed from $v_1, \ldots, v_m$, the span of the remaining list equals $\operatorname{span}(v_1, \ldots, v_m)$.

Proof Because the list $v_1, \ldots, v_m$ is linearly dependent, there exist numbers $a_1, \ldots, a_m \in \mathbf{F}$, not all 0, such that

$$a_1 v_1 + \cdots + a_m v_m = 0.$$

Let j be the largest element of $\{1, \ldots, m\}$ such that $a_j \neq 0$. Then

2.22
$$v_j = -\frac{a_1}{a_j} v_1 - \cdots - \frac{a_{j-1}}{a_j} v_{j-1},$$

proving (a).

To prove (b), suppose $u \in \operatorname{span}(v_1, \ldots, v_m)$. Then there exist numbers $c_1, \ldots, c_m \in \mathbf{F}$ such that

$$u = c_1 v_1 + \cdots + c_m v_m.$$

In the equation above, we can replace v_j with the right side of 2.22, which shows that u is in the span of the list obtained by removing the j^{th} term from $v_1, \ldots, v_m$. Thus (b) holds. ■

Choosing $j = 1$ in the Linear Dependence Lemma above means that $v_1 = 0$, because if $j = 1$ then condition (a) above is interpreted to mean that $v_1 \in \operatorname{span}(\,)$; recall that $\operatorname{span}(\,) = \{0\}$. Note also that the proof of part (b) above needs to be modified in an obvious way if $v_1 = 0$ and $j = 1$.

In general, the proofs in the rest of the book will not call attention to special cases that must be considered involving empty lists, lists of length 1, the subspace $\{0\}$, or other trivial cases for which the result is clearly true but needs a slightly different proof. Be sure to check these special cases yourself.

Now we come to a key result. It says that no linearly independent list in V is longer than a spanning list in V.

2.23 Length of linearly independent list $\leq$ length of spanning list

In a finite-dimensional vector space, the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.

Proof Suppose $u_1, \dots, u_m$ is linearly independent in V. Suppose also that $w_1, \dots, w_n$ spans V. We need to prove that $m \leq n$. We do so through the multi-step process described below; note that in each step we add one of the u's and remove one of the w's.

Step 1

Let B be the list $w_1, \dots, w_n$, which spans V. Thus adjoining any vector in V to this list produces a linearly dependent list (because the newly adjoined vector can be written as a linear combination of the other vectors). In particular, the list

$$u_1, w_1, \dots, w_n$$

is linearly dependent. Thus by the Linear Dependence Lemma (2.21), we can remove one of the w's so that the new list B (of length n) consisting of u_1 and the remaining w's spans V.

Step j

The list B (of length n) from step $j-1$ spans V. Thus adjoining any vector to this list produces a linearly dependent list. In particular, the list of length $(n+1)$ obtained by adjoining u_j to B, placing it just after $u_1, \dots, u_{j-1}$, is linearly dependent. By the Linear Dependence Lemma (2.21), one of the vectors in this list is in the span of the previous ones, and because $u_1, \dots, u_j$ is linearly independent, this vector is one of the w's, not one of the u's. We can remove that w from B so that the new list B (of length n) consisting of $u_1, \dots, u_j$ and the remaining w's spans V.

After step m, we have added all the u's and the process stops. At each step as we add a u to B, the Linear Dependence Lemma implies that there is some w to remove. Thus there are at least as many w's as u's. ■

The next two examples show how the result above can be used to show, without any computations, that certain lists are not linearly independent and that certain lists do not span a given vector space.

2.24 Example Show that the list $(1, 2, 3), (4, 5, 8), (9, 6, 7), (-3, 2, 8)$ is not linearly independent in $\mathbf{R}^3$.

Solution The list $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ spans $\mathbf{R}^3$. Thus no list of length larger than 3 is linearly independent in $\mathbf{R}^3$.

2.25 Example Show that the list $(1, 2, 3, -5), (4, 5, 8, 3), (9, 6, 7, -1)$ does not span $\mathbf{R}^4$.

Solution The list $(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)$ is linearly independent in $\mathbf{R}^4$. Thus no list of length less than 4 spans $\mathbf{R}^4$.

Our intuition suggests that every subspace of a finite-dimensional vector space should also be finite-dimensional. We now prove that this intuition is correct.

2.26 Finite-dimensional subspaces

Every subspace of a finite-dimensional vector space is finite-dimensional.

Proof Suppose V is finite-dimensional and U is a subspace of V. We need to prove that U is finite-dimensional. We do this through the following multi-step construction.

Step 1

If $U = \{0\}$, then U is finite-dimensional and we are done. If $U \neq \{0\}$, then choose a nonzero vector $v_1 \in U$.

Step j

If $U = \operatorname{span}(v_1, \dots, v_{j-1})$, then U is finite-dimensional and we are done. If $U \neq \operatorname{span}(v_1, \dots, v_{j-1})$, then choose a vector $v_j \in U$ such that

$$v_j \notin \operatorname{span}(v_1, \dots, v_{j-1}).$$

After each step, as long as the process continues, we have constructed a list of vectors such that no vector in this list is in the span of the previous vectors. Thus after each step we have constructed a linearly independent list, by the Linear Dependence Lemma (2.21). This linearly independent list cannot be longer than any spanning list of V (by 2.23). Thus the process eventually terminates, which means that U is finite-dimensional. ■

EXERCISES 2.A

1 Suppose v_1, v_2, v_3, v_4 spans V. Prove that the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

also spans V.

2 Verify the assertions in Example 2.18.

3 Find a number t such that

$$(3, 1, 4), (2, -3, 5), (5, 9, t)$$

is not linearly independent in $\mathbf{R}^3$.

4 Verify the assertion in the second bullet point in Example 2.20.

5 (a) Show that if we think of $\mathbf{C}$ as a vector space over $\mathbf{R}$, then the list $(1 + i, 1 - i)$ is linearly independent.

(b) Show that if we think of $\mathbf{C}$ as a vector space over $\mathbf{C}$, then the list $(1 + i, 1 - i)$ is linearly dependent.

6 Suppose v_1, v_2, v_3, v_4 is linearly independent in V. Prove that the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

is also linearly independent.

7 Prove or give a counterexample: If $v_1, v_2, \ldots, v_m$ is a linearly independent list of vectors in V, then

$$5v_1 - 4v_2, v_2, v_3, \ldots, v_m$$

is linearly independent.

8 Prove or give a counterexample: If $v_1, v_2, \ldots, v_m$ is a linearly independent list of vectors in V and $\lambda \in \mathbf{F}$ with $\lambda \neq 0$, then $\lambda v_1, \lambda v_2, \ldots, \lambda v_m$ is linearly independent.

9 Prove or give a counterexample: If $v_1, \ldots, v_m$ and $w_1, \ldots, w_m$ are linearly independent lists of vectors in V, then $v_1 + w_1, \ldots, v_m + w_m$ is linearly independent.

10 Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Prove that if $v_1 + w, \ldots, v_m + w$ is linearly dependent, then $w \in \operatorname{span}(v_1, \ldots, v_m)$.

11 Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Show that $v_1, \ldots, v_m, w$ is linearly independent if and only if

$$w \notin \operatorname{span}(v_1, \ldots, v_m).$$

12 Explain why there does not exist a list of six polynomials that is linearly independent in $\mathcal{P}_4(\mathbf{F})$.

13 Explain why no list of four polynomials spans $\mathcal{P}_4(\mathbf{F})$.

14 Prove that V is infinite-dimensional if and only if there is a sequence $v_1, v_2, \ldots$ of vectors in V such that $v_1, \ldots, v_m$ is linearly independent for every positive integer m.

15 Prove that $\mathbf{F}^\infty$ is infinite-dimensional.

16 Prove that the real vector space of all continuous real-valued functions on the interval [0, 1] is infinite-dimensional.

17 Suppose $p_0, p_1, \ldots, p_m$ are polynomials in $\mathcal{P}_m(\mathbf{F})$ such that $p_j(2) = 0$ for each j. Prove that $p_0, p_1, \ldots, p_m$ is not linearly independent in $\mathcal{P}_m(\mathbf{F})$.

2.B Bases

In the last section, we discussed linearly independent lists and spanning lists. Now we bring these concepts together.

2.27 Definition ***basis***

A ***basis*** of V is a list of vectors in V that is linearly independent and spans V.

2.28 Example ***bases***

(a) The list $(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$ is a basis of $\mathbf{F}^n$, called the ***standard basis*** of $\mathbf{F}^n$.

(b) The list $(1, 2), (3, 5)$ is a basis of $\mathbf{F}^2$.

(c) The list $(1, 2, -4), (7, -5, 6)$ is linearly independent in $\mathbf{F}^3$ but is not a basis of $\mathbf{F}^3$ because it does not span $\mathbf{F}^3$.

(d) The list $(1, 2), (3, 5), (4, 13)$ spans $\mathbf{F}^2$ but is not a basis of $\mathbf{F}^2$ because it is not linearly independent.

(e) The list $(1, 1, 0), (0, 0, 1)$ is a basis of $\{(x, x, y) \in \mathbf{F}^3 : x, y \in \mathbf{F}\}$.

(f) The list $(1, -1, 0), (1, 0, -1)$ is a basis of

$$\{(x, y, z) \in \mathbf{F}^3 : x + y + z = 0\}.$$

(g) The list $1, z, \ldots, z^m$ is a basis of $\mathcal{P}_m(\mathbf{F})$.

In addition to the standard basis, $\mathbf{F}^n$ has many other bases. For example, $(7, 5), (-4, 9)$ and $(1, 2), (3, 5)$ are both bases of $\mathbf{F}^2$.

The next result helps explain why bases are useful. Recall that "uniquely" means "in only one way".

2.29 Criterion for basis

A list $v_1, \ldots, v_n$ of vectors in V is a basis of V if and only if every $v \in V$ can be written uniquely in the form

2.30
$$v = a_1 v_1 + \cdots + a_n v_n,$$

where $a_1, \ldots, a_n \in \mathbf{F}$.

Proof First suppose that $v_1, \dots, v_n$ is a basis of V. Let $v \in V$. Because $v_1, \dots, v_n$ spans V, there exist $a_1, \dots, a_n \in \mathbf{F}$ such that 2.30 holds. To show that the representation in 2.30 is unique, suppose $c_1, \dots, c_n$ are scalars such that we also have

This proof is essentially a repetition of the ideas that led us to the definition of linear independence.

$$v = c_1 v_1 + \cdots + c_n v_n.$$

Subtracting the last equation from 2.30, we get

$$0 = (a_1 - c_1)v_1 + \cdots + (a_n - c_n)v_n.$$

This implies that each $a_j - c_j$ equals 0 (because $v_1, \dots, v_n$ is linearly independent). Hence $a_1 = c_1, \dots, a_n = c_n$. We have the desired uniqueness, completing the proof in one direction.

For the other direction, suppose every $v \in V$ can be written uniquely in the form given by 2.30. Clearly this implies that $v_1, \dots, v_n$ spans V. To show that $v_1, \dots, v_n$ is linearly independent, suppose $a_1, \dots, a_n \in \mathbf{F}$ are such that

$$0 = a_1 v_1 + \cdots + a_n v_n.$$

The uniqueness of the representation 2.30 (taking $v = 0$) now implies that $a_1 = \cdots = a_n = 0$. Thus $v_1, \dots, v_n$ is linearly independent and hence is a basis of V. ■

A spanning list in a vector space may not be a basis because it is not linearly independent. Our next result says that given any spanning list, some (possibly none) of the vectors in it can be discarded so that the remaining list is linearly independent and still spans the vector space.

As an example in the vector space $\mathbf{F}^2$, if the procedure in the proof below is applied to the list $(1,2), (3,6), (4,7), (5,9)$, then the second and fourth vectors will be removed. This leaves $(1,2), (4,7)$, which is a basis of $\mathbf{F}^2$.

2.31 Spanning list contains a basis

Every spanning list in a vector space can be reduced to a basis of the vector space.

Proof Suppose $v_1, \dots, v_n$ spans V. We want to remove some of the vectors from $v_1, \dots, v_n$ so that the remaining vectors form a basis of V. We do this through the multi-step process described below.

Start with B equal to the list $v_1, \dots, v_n$.

Step 1

If $v_1 = 0$, delete v_1 from B. If $v_1 \neq 0$, leave B unchanged.

Step j

If v_j is in $\operatorname{span}(v_1, \ldots, v_{j-1})$, delete v_j from B. If v_j is not in $\operatorname{span}(v_1, \ldots, v_{j-1})$, leave B unchanged.

Stop the process after step n, getting a list B. This list B spans V because our original list spanned V and we have discarded only vectors that were already in the span of the previous vectors. The process ensures that no vector in B is in the span of the previous ones. Thus B is linearly independent, by the Linear Dependence Lemma (2.21). Hence B is a basis of V. ■

Our next result, an easy corollary of the previous result, tells us that every finite-dimensional vector space has a basis.

2.32 Basis of finite-dimensional vector space

Every finite-dimensional vector space has a basis.

Proof By definition, a finite-dimensional vector space has a spanning list. The previous result tells us that each spanning list can be reduced to a basis. ■

Our next result is in some sense a dual of 2.31, which said that every spanning list can be reduced to a basis. Now we show that given any linearly independent list, we can adjoin some additional vectors (this includes the possibility of adjoining no additional vectors) so that the extended list is still linearly independent but also spans the space.

2.33 Linearly independent list extends to a basis

Every linearly independent list of vectors in a finite-dimensional vector space can be extended to a basis of the vector space.

Proof Suppose $u_1, \ldots, u_m$ is linearly independent in a finite-dimensional vector space V. Let $w_1, \ldots, w_n$ be a basis of V. Thus the list

$$u_1, \ldots, u_m, w_1, \ldots, w_n$$

spans V. Applying the procedure of the proof of 2.31 to reduce this list to a basis of V produces a basis consisting of the vectors $u_1, \ldots, u_m$ (none of the u's get deleted in this procedure because $u_1, \ldots, u_m$ is linearly independent) and some of the w's. ■

As an example in $\mathbf{F}^3$, suppose we start with the linearly independent list $(2,3,4), (9,6,8)$. If we take w_1, w_2, w_3 in the proof above to be the standard basis of $\mathbf{F}^3$, then the procedure in the proof above produces the list $(2,3,4), (9,6,8), (0,1,0)$, which is a basis of $\mathbf{F}^3$.

Using the same basic ideas but considerably more advanced tools, the next result can be proved without the hypothesis that V is finite-dimensional.

As an application of the result above, we now show that every subspace of a finite-dimensional vector space can be paired with another subspace to form a direct sum of the whole space.

2.34 Every subspace of V is part of a direct sum equal to V

Suppose V is finite-dimensional and U is a subspace of V. Then there is a subspace W of V such that $V = U \oplus W$.

Proof Because V is finite-dimensional, so is U (see 2.26). Thus there is a basis $u_1, \dots, u_m$ of U (see 2.32). Of course $u_1, \dots, u_m$ is a linearly independent list of vectors in V. Hence this list can be extended to a basis $u_1, \dots, u_m, w_1, \dots, w_n$ of V (see 2.33). Let $W = \operatorname{span}(w_1, \dots, w_n)$.

To prove that $V = U \oplus W$, by 1.45 we need only show that

$$V = U + W \quad \text{and} \quad U \cap W = \{0\}.$$

To prove the first equation above, suppose $v \in V$. Then, because the list $u_1, \dots, u_m, w_1, \dots, w_n$ spans V, there exist $a_1, \dots, a_m, b_1, \dots, b_n \in \mathbf{F}$ such that

$$v = \underbrace{a_1 u_1 + \cdots + a_m u_m}_{u} + \underbrace{b_1 w_1 + \cdots + b_n w_n}_{w}.$$

In other words, we have $v = u + w$, where $u \in U$ and $w \in W$ are defined as above. Thus $v \in U + W$, completing the proof that $V = U + W$.

To show that $U \cap W = \{0\}$, suppose $v \in U \cap W$. Then there exist scalars $a_1, \dots, a_m, b_1, \dots, b_n \in \mathbf{F}$ such that

$$v = a_1 u_1 + \cdots + a_m u_m = b_1 w_1 + \cdots + b_n w_n.$$

Thus

$$a_1 u_1 + \cdots + a_m u_m - b_1 w_1 - \cdots - b_n w_n = 0.$$

Because $u_1, \dots, u_m, w_1, \dots, w_n$ is linearly independent, this implies that $a_1 = \cdots = a_m = b_1 = \cdots = b_n = 0$. Thus $v = 0$, completing the proof that $U \cap W = \{0\}$. ∎

EXERCISES 2.B

1 Find all vector spaces that have exactly one basis.

2 Verify all the assertions in Example 2.28.

3 (a) Let U be the subspace of $\mathbf{R}^5$ defined by

$$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{R}^5 : x_1 = 3x_2 \text{ and } x_3 = 7x_4\}.$$

Find a basis of U.

(b) Extend the basis in part (a) to a basis of $\mathbf{R}^5$.

(c) Find a subspace W of $\mathbf{R}^5$ such that $\mathbf{R}^5 = U \oplus W$.

4 (a) Let U be the subspace of $\mathbf{C}^5$ defined by

$$U = \{(z_1, z_2, z_3, z_4, z_5) \in \mathbf{C}^5 : 6z_1 = z_2 \text{ and } z_3 + 2z_4 + 3z_5 = 0\}.$$

Find a basis of U.

(b) Extend the basis in part (a) to a basis of $\mathbf{C}^5$.

(c) Find a subspace W of $\mathbf{C}^5$ such that $\mathbf{C}^5 = U \oplus W$.

5 Prove or disprove: there exists a basis p_0, p_1, p_2, p_3 of $\mathcal{P}_3(\mathbf{F})$ such that none of the polynomials p_0, p_1, p_2, p_3 has degree 2.

6 Suppose v_1, v_2, v_3, v_4 is a basis of V. Prove that

$$v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$$

is also a basis of V.

7 Prove or give a counterexample: If v_1, v_2, v_3, v_4 is a basis of V and U is a subspace of V such that $v_1, v_2 \in U$ and $v_3 \notin U$ and $v_4 \notin U$, then v_1, v_2 is a basis of U.

8 Suppose U and W are subspaces of V such that $V = U \oplus W$. Suppose also that $u_1, \dots, u_m$ is a basis of U and $w_1, \dots, w_n$ is a basis of W. Prove that

$$u_1, \dots, u_m, w_1, \dots, w_n$$

is a basis of V.

2.C Dimension

Although we have been discussing finite-dimensional vector spaces, we have not yet defined the dimension of such an object. How should dimension be defined? A reasonable definition should force the dimension of $\mathbf{F}^n$ to equal n. Notice that the standard basis

$$(1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$$

of $\mathbf{F}^n$ has length n. Thus we are tempted to define the dimension as the length of a basis. However, a finite-dimensional vector space in general has many different bases, and our attempted definition makes sense only if all bases in a given vector space have the same length. Fortunately that turns out to be the case, as we now show.

2.35 Basis length does not depend on basis

Any two bases of a finite-dimensional vector space have the same length.

Proof Suppose V is finite-dimensional. Let B_1 and B_2 be two bases of V. Then B_1 is linearly independent in V and B_2 spans V, so the length of B_1 is at most the length of B_2 (by 2.23). Interchanging the roles of B_1 and B_2, we also see that the length of B_2 is at most the length of B_1. Thus the length of B_1 equals the length of B_2, as desired. ■

Now that we know that any two bases of a finite-dimensional vector space have the same length, we can formally define the dimension of such spaces.

2.36 Definition ***dimension*, dim V**

- The ***dimension*** of a finite-dimensional vector space is the length of any basis of the vector space.
- The dimension of V (if V is finite-dimensional) is denoted by $\dim V$.

2.37 Example ***dimensions***

- $\dim \mathbf{F}^n = n$ because the standard basis of $\mathbf{F}^n$ has length n.
- $\dim \mathcal{P}_m(\mathbf{F}) = m + 1$ because the basis $1, z, \dots, z^m$ of $\mathcal{P}_m(\mathbf{F})$ has length $m + 1$.

Every subspace of a finite-dimensional vector space is finite-dimensional (by 2.26) and so has a dimension. The next result gives the expected inequality about the dimension of a subspace.

2.38 Dimension of a subspace

If V is finite-dimensional and U is a subspace of V, then $\dim U \le \dim V$.

Proof Suppose V is finite-dimensional and U is a subspace of V. Think of a basis of U as a linearly independent list in V, and think of a basis of V as a spanning list in V. Now use 2.23 to conclude that $\dim U \le \dim V$. ∎

To check that a list of vectors in V is a basis of V, we must, according to the definition, show that the list in question satisfies two properties: it must be linearly independent and it must span V. The next two results show that if the list in question has the right length, then we need only check that it satisfies one of the two required properties. First we prove that every linearly independent list with the right length is a basis.

The real vector space $\mathbf{R}^2$ has dimension 2; the complex vector space $\mathbf{C}$ has dimension 1. As sets, $\mathbf{R}^2$ can be identified with $\mathbf{C}$ (and addition is the same on both spaces, as is scalar multiplication by real numbers). Thus when we talk about the dimension of a vector space, the role played by the choice of $\mathbf{F}$ cannot be neglected.

2.39 Linearly independent list of the right length is a basis

Suppose V is finite-dimensional. Then every linearly independent list of vectors in V with length $\dim V$ is a basis of V.

Proof Suppose $\dim V = n$ and $v_1, \dots, v_n$ is linearly independent in V. The list $v_1, \dots, v_n$ can be extended to a basis of V (by 2.33). However, every basis of V has length n, so in this case the extension is the trivial one, meaning that no elements are adjoined to $v_1, \dots, v_n$. In other words, $v_1, \dots, v_n$ is a basis of V, as desired. ∎

2.40 **Example** Show that the list $(5, 7), (4, 3)$ is a basis of $\mathbf{F}^2$.

Solution This list of two vectors in $\mathbf{F}^2$ is obviously linearly independent (because neither vector is a scalar multiple of the other). Note that $\mathbf{F}^2$ has dimension 2. Thus 2.39 implies that the linearly independent list $(5, 7), (4, 3)$ of length 2 is a basis of $\mathbf{F}^2$ (we do not need to bother checking that it spans $\mathbf{F}^2$).

2.41 Example Show that $1, (x-5)^2, (x-5)^3$ is a basis of the subspace U of $\mathcal{P}_3(\mathbf{R})$ defined by

$$U = \{p \in \mathcal{P}_3(\mathbf{R}) : p'(5) = 0\}.$$

Solution Clearly each of the polynomials 1, $(x-5)^2$, and $(x-5)^3$ is in U.

Suppose $a, b, c \in \mathbf{R}$ and

$$a + b(x-5)^2 + c(x-5)^3 = 0$$

for every $x \in \mathbf{R}$. Without explicitly expanding the left side of the equation above, we can see that the left side has a cx^3 term. Because the right side has no x^3 term, this implies that $c = 0$. Because $c = 0$, we see that the left side has a bx^2 term, which implies that $b = 0$. Because $b = c = 0$, we can also conclude that $a = 0$.

Thus the equation above implies that $a = b = c = 0$. Hence the list $1, (x-5)^2, (x-5)^3$ is linearly independent in U.

Thus $\dim U \geq 3$. Because U is a subspace of $\mathcal{P}_3(\mathbf{R})$, we know that $\dim U \leq \dim \mathcal{P}_3(\mathbf{R}) = 4$ (by 2.38). However, $\dim U$ cannot equal 4, because otherwise when we extend a basis of U to a basis of $\mathcal{P}_3(\mathbf{R})$ we would get a list with length greater than 4. Hence $\dim U = 3$. Thus 2.39 implies that the linearly independent list $1, (x-5)^2, (x-5)^3$ is a basis of U.

Now we prove that a spanning list with the right length is a basis.

2.42 Spanning list of the right length is a basis

Suppose V is finite-dimensional. Then every spanning list of vectors in V with length $\dim V$ is a basis of V.

Proof Suppose $\dim V = n$ and $v_1, \dots, v_n$ spans V. The list $v_1, \dots, v_n$ can be reduced to a basis of V (by 2.31). However, every basis of V has length n, so in this case the reduction is the trivial one, meaning that no elements are deleted from $v_1, \dots, v_n$. In other words, $v_1, \dots, v_n$ is a basis of V, as desired. ■

The next result gives a formula for the dimension of the sum of two subspaces of a finite-dimensional vector space. This formula is analogous to a familiar counting formula: the number of elements in the union of two finite sets equals the number of elements in the first set, plus the number of elements in the second set, minus the number of elements in the intersection of the two sets.

2.43 Dimension of a sum

If U_1 and U_2 are subspaces of a finite-dimensional vector space, then

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2).$$

Proof Let $u_1, \dots, u_m$ be a basis of $U_1 \cap U_2$; thus $\dim(U_1 \cap U_2) = m$. Because $u_1, \dots, u_m$ is a basis of $U_1 \cap U_2$, it is linearly independent in U_1. Hence this list can be extended to a basis $u_1, \dots, u_m, v_1, \dots, v_j$ of U_1 (by 2.33). Thus $\dim U_1 = m + j$. Also extend $u_1, \dots, u_m$ to a basis $u_1, \dots, u_m, w_1, \dots, w_k$ of U_2; thus $\dim U_2 = m + k$.

We will show that

$$u_1, \dots, u_m, v_1, \dots, v_j, w_1, \dots, w_k$$

is a basis of $U_1 + U_2$. This will complete the proof, because then we will have

$$\begin{aligned} \dim(U_1 + U_2) &= m + j + k \\ &= (m + j) + (m + k) - m \\ &= \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2). \end{aligned}$$

Clearly $\operatorname{span}(u_1, \dots, u_m, v_1, \dots, v_j, w_1, \dots, w_k)$ contains U_1 and U_2 and hence equals $U_1 + U_2$. So to show that this list is a basis of $U_1 + U_2$ we need only show that it is linearly independent. To prove this, suppose

$$a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_j v_j + c_1 w_1 + \cdots + c_k w_k = 0,$$

where all the a's, b's, and c's are scalars. We need to prove that all the a's, b's, and c's equal 0. The equation above can be rewritten as

$$c_1 w_1 + \cdots + c_k w_k = -a_1 u_1 - \cdots - a_m u_m - b_1 v_1 - \cdots - b_j v_j,$$

which shows that $c_1 w_1 + \cdots + c_k w_k \in U_1$. All the w's are in U_2, so this implies that $c_1 w_1 + \cdots + c_k w_k \in U_1 \cap U_2$. Because $u_1, \dots, u_m$ is a basis of $U_1 \cap U_2$, we can write

$$c_1 w_1 + \cdots + c_k w_k = d_1 u_1 + \cdots + d_m u_m$$

for some choice of scalars $d_1, \dots, d_m$. But $u_1, \dots, u_m, w_1, \dots, w_k$ is linearly independent, so the last equation implies that all the c's (and d's) equal 0. Thus our original equation involving the a's, b's, and c's becomes

$$a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_j v_j = 0.$$

Because the list $u_1, \dots, u_m, v_1, \dots, v_j$ is linearly independent, this equation implies that all the a's and b's are 0. We now know that all the a's, b's, and c's equal 0, as desired. ■

EXERCISES 2.C

1 Suppose V is finite-dimensional and U is a subspace of V such that $\dim U = \dim V$. Prove that $U = V$.

2 Show that the subspaces of $\mathbf{R}^2$ are precisely $\{0\}$, $\mathbf{R}^2$, and all lines in $\mathbf{R}^2$ through the origin.

3 Show that the subspaces of $\mathbf{R}^3$ are precisely $\{0\}$, $\mathbf{R}^3$, all lines in $\mathbf{R}^3$ through the origin, and all planes in $\mathbf{R}^3$ through the origin.

4 (a) Let $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(6) = 0\}$. Find a basis of U.

(b) Extend the basis in part (a) to a basis of $\mathcal{P}_4(\mathbf{F})$.

(c) Find a subspace W of $\mathcal{P}_4(\mathbf{F})$ such that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

5 (a) Let $U = \{p \in \mathcal{P}_4(\mathbf{R}) : p''(6) = 0\}$. Find a basis of U.

(b) Extend the basis in part (a) to a basis of $\mathcal{P}_4(\mathbf{R})$.

(c) Find a subspace W of $\mathcal{P}_4(\mathbf{R})$ such that $\mathcal{P}_4(\mathbf{R}) = U \oplus W$.

6 (a) Let $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(2) = p(5)\}$. Find a basis of U.

(b) Extend the basis in part (a) to a basis of $\mathcal{P}_4(\mathbf{F})$.

(c) Find a subspace W of $\mathcal{P}_4(\mathbf{F})$ such that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

7 (a) Let $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(2) = p(5) = p(6)\}$. Find a basis of U.

(b) Extend the basis in part (a) to a basis of $\mathcal{P}_4(\mathbf{F})$.

(c) Find a subspace W of $\mathcal{P}_4(\mathbf{F})$ such that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

8 (a) Let $U = \{p \in \mathcal{P}_4(\mathbf{R}) : \int_{-1}^{1} p = 0\}$. Find a basis of U.

(b) Extend the basis in part (a) to a basis of $\mathcal{P}_4(\mathbf{R})$.

(c) Find a subspace W of $\mathcal{P}_4(\mathbf{R})$ such that $\mathcal{P}_4(\mathbf{R}) = U \oplus W$.

9 Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Prove that

$$\dim \operatorname{span}(v_1 + w, \ldots, v_m + w) \geq m - 1.$$

10 Suppose $p_0, p_1, \ldots, p_m \in \mathcal{P}(\mathbf{F})$ are such that each p_j has degree j. Prove that $p_0, p_1, \ldots, p_m$ is a basis of $\mathcal{P}_m(\mathbf{F})$.

11 Suppose that U and W are subspaces of $\mathbf{R}^8$ such that $\dim U = 3$, $\dim W = 5$, and $U + W = \mathbf{R}^8$. Prove that $\mathbf{R}^8 = U \oplus W$.

12 Suppose U and W are both five-dimensional subspaces of $\mathbf{R}^9$. Prove that $U \cap W \neq \{0\}$.

13 Suppose U and W are both 4-dimensional subspaces of $\mathbf{C}^6$. Prove that there exist two vectors in $U \cap W$ such that neither of these vectors is a scalar multiple of the other.

14 Suppose $U_1, \dots, U_m$ are finite-dimensional subspaces of V. Prove that $U_1 + \cdots + U_m$ is finite-dimensional and

$$\dim(U_1 + \cdots + U_m) \leq \dim U_1 + \cdots + \dim U_m.$$

15 Suppose V is finite-dimensional, with $\dim V = n \geq 1$. Prove that there exist 1-dimensional subspaces $U_1, \dots, U_n$ of V such that

$$V = U_1 \oplus \cdots \oplus U_n.$$

16 Suppose $U_1, \dots, U_m$ are finite-dimensional subspaces of V such that $U_1 + \cdots + U_m$ is a direct sum. Prove that $U_1 \oplus \cdots \oplus U_m$ is finite-dimensional and

$$\dim U_1 \oplus \cdots \oplus U_m = \dim U_1 + \cdots + \dim U_m.$$

[*The exercise above deepens the analogy between direct sums of subspaces and disjoint unions of subsets. Specifically, compare this exercise to the following obvious statement: if a set is written as a disjoint union of finite subsets, then the number of elements in the set equals the sum of the numbers of elements in the disjoint subsets.*]

17 You might guess, by analogy with the formula for the number of elements in the union of three subsets of a finite set, that if U_1, U_2, U_3 are subspaces of a finite-dimensional vector space, then

$$\begin{aligned}\dim(U_1 + U_2 + U_3)&\\ &= \dim U_1 + \dim U_2 + \dim U_3\\ &\quad - \dim(U_1 \cap U_2) - \dim(U_1 \cap U_3) - \dim(U_2 \cap U_3)\\ &\quad + \dim(U_1 \cap U_2 \cap U_3).\end{aligned}$$

Prove this or give a counterexample.

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