Preface

This book offers a systematic and thorough examination of theoretical and computational aspects of the modern *mimetic finite difference* (MFD) method. The MFD method preserves or mimics underlying properties of physical and mathematical models, thereby improving the fidelity and predictive capability of computer simulations. We focus here on the numerical solution of elliptic partial differential equation (PDEs) on unstructured polygonal and polyhedral meshes for which the MFD method has proven to be very successful in the last five decades.

The book covers advanced research topics and issues. Most of the presented material is the result of our research work that has been published in the last decade. Our intention is to offer a deep introduction to the major aspects of the MFD method such as the design principles for the development of new schemes, tools for the convergence analysis, and matrix formulas ready for a code implementation, to the widest possible audience. Nonetheless, to appreciate our effort a minimum background is required in the linear algebra, functional analysis, and numerical analysis of PDEs. It will be helpful for the reader to have some familiarity with the classical lowest-order finite element schemes, such as the primal linear and mixed Raviart-Thomas methods, the classical finite volume and finite difference schemes.

The book is structured in three parts with four chapters each.

The MFD method has a strong theoretical foundation, which is reviewed in Part I, entitled *Foundation*. In Chap. 1, after a short motivation for using the MFD method in applications, we give an historical introduction to the development of the mimetic technology and an overview of all mathematical models considered in the book. We present their strong and weak formulations and summarize results concerning the existence and regularity of weak solutions. We also introduce the notion of shape-regular polyhedral and polygonal meshes that are extensively used throughout the book. In Sect.1.3 we illustrate a few basic design principles of the mimetic discretization method on the simplest one-dimensional Poisson equation. This section is particularly suitable for readers not familiar with the mimetic technology.

The theoretical foundation of the existing compatible discretization methods dates back to the fundamental work of Withney on geometric integration. No surprise that the MFD method is related to some of the most basic concepts of discrete differential forms such as the chain-cochain duality and discrete Stokes theorems. The mimetic schemes are derived in part by mimicking the Stokes theorems in a discrete setting. The further development of this concept is in Chap. 2, where a discrete vector and tensor calculus (DVTC), the core of the MFD method that separates it from finite volume methods, is introduced. Using fundamental physical principles, we formulate natural discrete analogues of the first-order differential operators *divergence*, *gradient*, and *curl*. Compatible adjoint discrete operators are defined via duality relationships, more precisely, via discrete integration by parts formulas. The derivation of these operators uses the notion of mimetic inner products that approximate L^2 products of scalar or vector functions.

The practical construction of accurate inner products on unstructured polygonal and polyhedral meshes requires a set of new theoretical tools that are introduced in Chap. 3. We introduce the stability and consistency conditions that play the fundamental role in proving well-posedness and accuracy of the mimetic discretizations. We also connect the mimetic inner products with reconstruction operators that make a useful theoretical tool but are never built in practice. This chapter highlights a unique feature of the MFD method. On polyhedral (including hexahedral and sometimes simplicial) meshes, it produces a family of schemes with equivalent properties such as the stencil size and convergence rate.

In Chap. 4 we extend the mimetic discretization technology to general bilinear forms, which allows us to apply the MFD method to a wider range of problems. We moreover present a different approach to mimetic discretizations that takes the steps from the weak formulation of the problem, rather than the strong one. Although this approach turns out to be often equivalent to the construction presented in the previous chapters, this is not always the case and it is very useful to have a clear picture of both methodologies. Furthermore, in Chap. 4 we focus on the detailed analysis of the stability and consistency conditions. We show again that the MFD method provides a family of schemes that share some important properties, e.g., accuracy and stability, so that the convergence analysis can be carried out simultaneously for the entire family.

Part II is entitled *Mimetic Discretization of Basic PDEs*. It explains how the MFD method can be applied for solving the steady-state diffusion equation in the primal and mixed formulations, Maxwell's equations, and the steady Stokes equations. We extended the construction of mimetic inner products (in three discrete spaces) to the case of tensorial coefficients. We also provide theoretical construction of various reconstruction operators, prove stability results, and derive a priori and a posteriori error estimates in mesh-dependent norms.

A useful but also limited viewpoint is to consider the MFD method as an extension of some classical discretization methods to polygonal and polyhedral meshes. Indeed, the family of low-order mimetic schemes contains many well-known finite volume, finite difference and finite elements schemes. On special regular grids (orthogonal Cartesian grids or logically rectangular grids), we recover such schemes as the particular members of the mimetic family. However, the mimetic schemes work perfectly on unstructured polygonal and polyhedral meshes, with arbitrarily-shaped cells that may be even non-convex and degenerate.

In Chap. 5, we apply the MFD method for solving the steady-state diffusion equation in a *mixed* form and show how this method generalizes the lowest-order Raviart-Thomas and BDM finite element methods on simplicial meshes to unstructured polygonal and polyhedral meshes in two and three spatial dimensions. We also investigate additional important issues such as the super convergence, solution post-processing, a-posteriori error estimation and adaptivity.

In Chap. 6, we apply the MFD method for solving the steady-state diffusion equation in a *primal* form and show how this method generalizes the linear Galerkin method on simplicial meshes to unstructured polygonal and polyhedral meshes in two and three spatial dimensions. On meshes of simplices, the nodal mimetic formulation of coincides with the linear Galerkin finite element method. On rectangular meshes, particular members of the mimetic family coincide with a number of classical finite difference schemes (5-point Laplacian, 9-point Laplacian, Q_1 finite element method). For two-dimensional problems, we also describe and analyze arbitrary-order mimetic schemes. Finally, we consider the a-posteriori error estimation and adaptivity for the low order mimetic schemes.

In Chap. 7, we apply the MFD for two time-dependent problems governed by Maxwell's equations and the magnetostatic problem. We discuss the conservation of energy in the mimetic discretizations and provide formulas ready for the code implementation. The convergence analysis of the MFD method for the magnetostatic problem is a work in progress and therefore is incomplete.

In Chap. 8, we derive and analyze mimetic schemes for the steady-state Stokes equations. Analysis of the inf-sup stability condition imposes constraints on the discrete spaces for the velocity and pressure. We first develop a mimetic method that takes inspiration from classical finite elements and show the good behavior of such scheme. Afterwards, we use the flexibility of the mimetic technology to build a more computationally efficient method, that makes use of much less degrees of freedom and still satisfies the constraints above.

We were asked frequently by our colleagues about the applicability of the MFD method to a wider class of problem. The Part III entitled *Further Developments* describes how the mimetic technology contributes to solving challenging problems emerging in modeling complex physical processes. This includes solution of nonlinear PDEs, preservation of maximum principles, and stability and accuracy of discretizations on deforming (e.g., Lagrangian) meshes.

Chapter 9 is devoted to the problems of structural mechanics. We first present an MFD method for the linear elasticity problem, considering both the displacement-pressure and stress-displacement formulations. Afterwards, we present a mimetic scheme for the Reissner-Mindlin plate bending problem, which uses deflection and rotation as unknown variables. Our additional interest of these problems is related to the fact that, in order to derive and analyze the numerical schemes, a large number of mimetic operators and discrete spaces must be considered at once.

In Chap. 10, we present the MFD method for the convection-diffusion equation, and the obstacle problem. We also consider the case of high Peclet numbers characterizing a convection-dominated regime where the continuum solution may display strong parabolic and exponential boundary layers.

In Chap. 11, we consider a new emerging research direction dubbed *m-adaptation*, which stands for *mimetic adaptation*. The m-adaptation allows us to select an optimal scheme from the family of mimetic schemes in accordance with some problem-dependent criteria that may include a discrete maximum principle (DMP), reduction of a numerical dispersion, and boosting performance of algebraic solvers. Even if the m-adaptation is still under development, a few interesting results are already available for the derivation of positive schemes or schemes satisfying a DMP. In this chapter, we analyze the family of the lowest-order mimetic schemes for the diffusion equation in the mixed and primal forms. We formulate the constructive sufficient conditions for the existence of a subfamily of mimetic scheme that satisfy the DMP.

In Chap. 12, we extend the MFD method to generalized polyhedral meshes with cells featuring non-planar faces. Such cells appear in Lagrangian simulations where the computational mesh is moved and deformed with the fluid. We use again the flexibility of the mimetic construction to add velocity unknowns only on strongly curved mesh faces in order to recover the optimal convergence rate.

Our final note is about the *computational aspects* of the mimetic technology. In each of the Chaps. 5–12, one or more sections are dedicated to the implementation details. The reader will find explicit formulas for the local mass and stiffness matrices. Additional interesting implementation details can be found in Chap. 4. Once the local matrices are coded, building the global mass or stiffness matrix can be done using the conventional assembly process, like in the finite element method.

Los Alamos, New Mexico and Milano-Pavia, Italy
September, 2013

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http://www.springer.com/978-3-319-02662-6

The Mimetic Finite Difference Method for Elliptic Problems
Beirao da Veiga, L.; Lipnikov, K.; Manzini, G.
2014, XVI, 394 p., Hardcover
ISBN: 978-3-319-02662-6