**Introduction**

The fundamental aim of this monograph is to present in a coherent and self-contained manner the main results on stability by linearization of Einstein’s equation, both in the case of the vacuum and in the case of matter. Before describing the contents of each chapter in detail, a brief explanation of the concept of stability is in order.

In classical mechanics, the potential \( V \) of the gravitational field created by a distribution of matter of density \( \rho \) fulfills Poisson’s equation

\[
\Delta V = 4\pi K \rho ,
\]

where \( \Delta \) is the Laplace operator in \( \mathbb{R}^3 \), \( \Delta = \sum_i \partial^2 / \partial x_i^2 \), and \( K \) is the gravitational constant. However, in general relativity, the equation analogous to the foregoing is Einstein’s equation, which relates the Lorentzian metric \( \tilde{g} \) (which is the concept that replaces the gravitational potential \( V \) of classical mechanics) and the stress-energy tensor \( T \) of the matter responsible for the gravitational field. Einstein’s equation is written

\[
G(\tilde{g}) = \chi T ,
\]

where \( \chi \) is a universal constant and \( G(\tilde{g}) \) is Einstein’s tensor of the metric \( \tilde{g} \) (see Chapter II for the precise definitions).

Since Einstein’s equation, written in coordinates, is equivalent to a nonlinear system of 10 second-order partial differential equations, it is no surprise that, except for problems with a high degree of symmetry, in most situations it is impossible to obtain exact solutions to this equation. It is for this reason that, since the introduction of the theory, the usual practice in many situations has been to replace the true equation with its linearization, in the belief that solutions to the latter depart very little from solutions to the true equation. However, it was not until the 1970s that certain researchers began to ask if this way of proceeding was in fact correct. Thus Y. Choquet-Bruhat and S. Deser [22] proved that in the case of metrics close to Minkowski’s metric \( \tilde{\eta} \) of \( \mathbb{R}^4 \), it made sense to linearize Einstein’s equation at the initial metric \( \tilde{\eta} \). Let us state here that when solutions of the linearized equation with respect to an initial metric differ little from solutions to the true equation (and, therefore, the linearization makes sense), Einstein’s equation
is said to be stable by linearization at the initial metric. This would be the case, for example, for Minkowski’s initial metric.

The foregoing case of the linearization of Einstein’s equation at Minkowski’s initial metric (which was addressed by Einstein) is particularly interesting from many points of view, and it is at the origin of the concept of the gravitational wave, as will be explained later. When there is no matter to create a gravitational field, we are dealing with the case of special relativity, and the corresponding metric is Minkowski’s metric $\tilde{\eta}$, which fulfils Einstein’s equation in the vacuum $G(\tilde{\eta}) = 0$. When we then address the case of a gravitational field created by matter with a small stress-energy tensor $T$, the Lorentzian metric $\tilde{g}$ associated with the gravitational field is assumed to be a small perturbation of Minkowski’s metric; that is, $\tilde{g} = \tilde{\eta} + \tilde{h}$, with $\tilde{h}$ being small. Then the linearization of Einstein’s equation at the initial metric $\tilde{\eta}$ can be written with appropriate coordinates (see Chapter III) in the form

$$\frac{1}{2} \left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} \right) \tilde{h}_{\alpha\beta} = \chi S_{\alpha\beta},$$

where $S$ is the tensor defined in terms of the stress-energy tensor $T$ by

$$S = T - \frac{1}{2} (\text{tr}_g T) \tilde{\eta}.$$ 

In regions where there is no matter (in the vacuum) we have $T = S = 0$, and then the linearization of Einstein’s equation in this region is none other than the wave equation

$$\left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} \right) \tilde{h}_{\alpha\beta} = 0.$$

Therefore, if it makes sense to substitute Einstein’s equation by its linearization, the metric $\tilde{g}$ corresponding to the gravitational field created is approximately the sum of Minkowski’s metric and a small perturbation $\tilde{h}$ whose components $\tilde{h}_{\alpha\beta}$ are propagated in the vacuum as if they were waves (since they fulfil the wave equation).

In the 1970s, stability by linearization of Einstein’s equation in the vacuum was extensively studied in the literature, and results of great beauty were obtained which we believe should be far better known than they are, both by mathematicians and by physicists. In addition to the already cited work by Y. Choquet-Bruhat and S. Deser, there is a series of important works in this regard, among which we may mention [23], [32], [34], [52], [2], [49]. An extensive bibliography can be found on the subject, for example, in [50]. For recent results we should cite [8], [26] and [27].

All the above-mentioned works refer to Einstein’s equation in the vacuum. However, very little literature exists on stability by linearization of Einstein’s
Introduction

xiii

equation with matter, a case that is particularly interesting from the point of view of physics. For instance, if we interpret the universe as a Robertson-Walker model of metric $\tilde{g}$ and stress-energy tensor $T$, the explosion of a supernova in a distant galaxy will produce a perturbation of the initial stress-energy tensor $T$, which will give rise to a perturbation of the initial metric $\tilde{g}$. The perturbed universe will cease to be of the Robertson-Walker type. In order to study the effects of such an explosion, is it admissible or not to work with the linearized Einstein equation? Note that this is not a question of linearizing Einstein’s equation in the vacuum, but rather of linearizing Einstein’s equation with matter (since Robertson-Walker models describe universes with matter). The authors of this monograph studied precisely this case in [14], [15] and [16].

We are now ready to describe the contents of the book in detail. The first two chapters cover preliminary material. Chapter I consists of a presentation (very concise and schematic) of the concepts on pseudo-Riemannian manifolds that are necessary for an understanding of the rest of this monograph. Its purpose is to clarify from the outset both the notation used and the definitions of basic concepts. While the theory is set out in an ordered manner in this chapter, together with references to most of the proofs, its content implicitly presupposes certain prior (very basic) knowledge on the part of the reader as regards Riemannian manifolds. Chapter II consists of an introduction to the theory of relativity, both special and general. Unlike the preceding chapter, the theory is set out in a much more intuitive way and is addressed to readers having no prior knowledge of the subject. Thus this chapter may be of use for mathematicians who have no background in physics.

Since the main aim of the book, as already stated, is to present the results on stability by linearization of Einstein’s equation, before dealing fully with the subject we consider it appropriate to present in Chapter III the oldest, most basic and most paradigmatic example of linearization of this equation (studied by Einstein himself in his works [30] and [31]): the linearization corresponding to the case where the initial metric is Minkowski’s metric and which relates Einstein’s equation with the wave equation.

The techniques normally employed in the study of stability by linearization of Einstein’s equation do not use the solutions to the equation directly, but rather Cauchy’s data of these solutions in a 3-dimensional hypersurface. This makes the Cauchy problem for Einstein’s equation an essential tool in all the results on stability by linearization. Nevertheless, Cauchy’s problem for Einstein’s equation with matter constitutes an enormous field of research that has given rise to a vast amount of literature. An extensive, up-to-date bibliography on the subject can be found, for example, in the article by H. Friedrich and A. Rendall [37]. Given such a plethora of literature, it is a far from easy task to choose a path to follow in order to make a self-contained, intelligible presentation which will be sufficient for the results on stability by linearization both in the vacuum and with matter. This is what we have attempted in Chapter IV of this monograph, which we trust will be
attractive for a broad range of readers (and not just those interested in stability by linearization of Einstein’s equation). The approach we adopt for proving the fundamental theorem in Chapter IV is inspired by Y. Choquet-Bruhat’s [21] paper, published in 1952, although in that work only Cauchy’s problem for Einstein’s equation in the vacuum is considered.

In Chapters V, VI, VII and VIII we deal with the body of the subject itself which gives its name to this monograph.

Chapter V covers the basic concepts of stability by linearization of Einstein’s equation: stability in the vacuum, used in most of the works published in the 1970s and early 1980s (refs: [22], [23], [32], [34], [2], [49], [52]), and the new concept of stability with matter ([14], [15] and [16]).

Chapter VI contains the main results on the linearization of Einstein’s equation in the case where the initial submanifold $M$ on which the Cauchy data are given is compact (results by A. Fischer, J. E. Marsden and V. Moncrief, [34], [52]).

Chapter VII is basically devoted to the proof of stability in the case where the initial metric is Minkowski’s metric. The result by Y. Choquet-Bruhat and S. Deser (in the vacuum), [22], is adapted to the case of weak gravitational fields created by a distribution of matter from the point of view of the new concept of stability with matter introduced in Chapter V. For the proof of this result, it is necessary to use a theorem by M. Cantor ([17], [18]), which establishes that the Euclidean Laplacian in $\mathbb{R}^n$ gives an isomorphism between certain Sobolev spaces with weights. The proof of this result provided in this monograph is different from that in M. Cantor’s original papers in order to underline the reason why Sobolev spaces with weights are necessary here rather than ordinary Sobolev spaces.

Y. Choquet-Bruhat, A. Fischer and J. E. Marsden, [23], extended the above-mentioned result on stability at the initial Minkowski metric to the case where the initial metric (instead of Minkowski’s metric) converges to Minkowski’s metric at infinity in any space-like direction. Section VII.5 in Chapter VII is devoted to this result, which we adapt to the case where matter is present. The version we give of it (Theorem VII.8) generalizes the result by the cited authors even in the case of the vacuum, which is the case they considered. The proof of Theorem VII.8 requires strong results on Fredholm operators in $\mathbb{R}^n$ due to L. A. Bagirov, V. A. Kondrat’ev, R. B. Lockhart and R. C. McOwen, [6], [45], [46], [47], [48]. We have attempted to summarize all the results by these authors that are required for the proof of our result in a single theorem (Theorem VII.14), in such a way that the reader may follow our proof simply on the basis of its statement.

Finally, Chapter VIII is devoted to the study of stability by linearization of Einstein’s equation when the initial metric and the initial stress-energy tensor correspond to a Robertson-Walker cosmological model ([14], [15] and [16]). The main theorems in this chapter require certain technical results related with the Laplacian in hyperbolic space ([13], [16] and [12]), for some of which a proof
different from that in the original papers is also given. We should also cite [1], [38] and [42] for analogous results on asymptotically hyperbolic manifolds involving weighted Sobolev spaces (while the results in [13], [16] and [12] concern usual Sobolev spaces).

Acknowledgements

The authors wish to express their gratitude for the invaluable help provided by different people in the writing of this monograph. First of all, to Joaquim Bruna (the second author’s brother) for providing us with some notes of his own which form the basis for Section VII.3, Chapter VII, and Subsection VIII.3.1, Chapter VIII.

We also wish to thank Xavier Cabré and Robert Lockhart for the bibliographic information they provided for us regarding elliptic operators and Fredholm operators in $\mathbb{R}^n$. We should also mention the many conversations that the first author held with Josep Maria Burguès concerning different subjects related with the monograph, especially those on Sobolev spaces and elliptic operators.
Stability by Linearization of Einstein's Field Equation
Bruna, L.; Girbau, J.
2010, XV, 208 p., Hardcover
ISBN: 978-3-0346-0303-4
A product of Birkhäuser Basel