## Contents

1. **Basic Terminology, Notation and Results** .................................. 1  
   1.1 Sets, Matrices and Vectors .............................................. 1  
   1.2 Digraphs, Subdigraphs, Neighbours, Degrees ...................... 2  
   1.3 Isomorphism and Basic Operations on Digraphs ...................... 6  
   1.4 Walks, Trails, Paths, Cycles and Path-Cycle Subdigraphs ....... 11  
   1.5 Strong and Unilateral Connectivity ................................. 15  
   1.6 Undirected Graphs, Biorientations and Orientations ............. 18  
   1.7 Trees and Euler Trails in Digraphs .................................. 21  
   1.8 Mixed Graphs, Orientations of Digraphs, and Hypergraphs ...... 24  
   1.9 Depth-First Search .................................................. 26  
   1.10 Exercises ..................................................................... 29  

2. **Classes of Digraphs** .......................................................... 31  
   2.1 Acyclic Digraphs ......................................................... 32  
   2.2 Multipartite Digraphs and Extended Digraphs ...................... 34  
   2.3 Transitive Digraphs, Transitive Closures and Reductions ....... 36  
   2.4 Line Digraphs ............................................................. 39  
   2.5 The de Bruijn and Kautz Digraphs ................................... 44  
   2.6 Series-Parallel Digraphs ................................................. 47  
   2.7 Quasi-Transitive Digraphs .............................................. 52  
   2.8 Path-Mergeable Digraphs ............................................... 55  
   2.9 Locally In/Out-Semicomplete Digraphs ............................... 57  
   2.10 Locally Semicomplete Digraphs ....................................... 59  
      2.10.1 Round Digraphs .................................................... 60  
      2.10.2 Non-Strong Locally Semicomplete Digraphs ................. 61  
      2.10.3 Strong Round Decomposable Locally Semicomplete  
          Digraphs ............................................................... 63  
      2.10.4 Classification of Locally Semicomplete Digraphs ......... 66  
   2.11 Totally \(\Phi\)-Decomposable Digraphs ................................ 69  
   2.12 Planar Digraphs ......................................................... 71  
   2.13 Digraphs of Bounded Width ......................................... 73  
      2.13.1 Digraphs of Bounded Tree-Width ............................... 74  
      2.13.2 Digraphs of Bounded Directed Widths ...................... 78  
   2.14 Other Families of Digraphs .......................................... 80
2.14.1 Circulant Digraphs ........................................... 80
2.14.2 Arc-Locally Semicomplete Digraphs ...................... 81
2.14.3 Intersection Digraphs ...................................... 82
2.15 Exercises ...................................................... 84

3. Distances .......................................................... 87
3.1 Terminology and Notation on Distances ....................... 87
3.2 Structure of Shortest Paths .................................. 89
3.3 Algorithms for Finding Distances in Digraphs ............. 91
  3.3.1 Breadth-First Search (BFS) ................................. 92
  3.3.2 Acyclic Digraphs ............................................. 93
  3.3.3 Dijkstra’s Algorithm ....................................... 94
  3.3.4 The Bellman-Ford-Moore Algorithm ....................... 97
  3.3.5 The Floyd-Warshall Algorithm ............................. 99
3.4 Inequalities on Diameter ...................................... 100
3.5 Minimum Diameter of Orientations of Multigraphs .......... 103
3.6 Minimum Diameter Orientations of Some Graphs and Digraphs108
  3.6.1 Generalizations of Tournaments ......................... 108
  3.6.2 Extended Digraphs ........................................ 111
  3.6.3 Cartesian Products of Graphs ............................. 113
  3.6.4 Chordal Graphs .......................................... 114
3.7 Kings in Digraphs .............................................. 115
  3.7.1 2-Kings in Tournaments ................................... 115
  3.7.2 Kings in Semicomplete Multipartite Digraphs ........... 116
  3.7.3 Kings in Generalizations of Tournaments ............... 118
3.8 \((k, l)\)-Kernels ............................................... 119
  3.8.1 Kernels .................................................... 119
  3.8.2 Quasi-Kernels ............................................. 122
3.9 Exercises ......................................................... 123

4. Flows in Networks ............................................... 127
4.1 Definitions and Basic Properties ............................ 127
  4.1.1 Flows and Their Balance Vectors ....................... 128
  4.1.2 The Residual Network .................................... 130
4.2 Reductions Among Different Flow Models .................... 131
  4.2.1 Eliminating Lower Bounds ................................. 131
  4.2.2 Flows with One Source and One Sink .................... 132
  4.2.3 Circulations ............................................. 133
  4.2.4 Networks with Bounds and Costs on the Vertices ....... 134
4.3 Flow Decompositions ......................................... 136
4.4 Working with the Residual Network ......................... 137
4.5 The Maximum Flow Problem .................................. 140
  4.5.1 The Ford-Fulkerson Algorithm .......................... 142
  4.5.2 Maximum Flows and Linear Programming ................ 145
4.6 Polynomial Algorithms for Finding a Maximum \((s, t)\)-Flow . 146
### 4.6.1 Augmenting Along Shortest Augmenting Paths 147
4.6.2 Maximal Flows in Layered Networks 148
4.6.3 The Push-Relabel Algorithm 149

4.7 Unit Capacity Networks and Simple Networks 153
4.7.1 Unit Capacity Networks 153
4.7.2 Simple Networks 155

4.8 Circulations and Feasible Flows 156
4.9 Minimum Value Feasible \((s, t)\)-Flows 158

4.10 Minimum Cost Flows 160
4.10.1 Characterizing Minimum Cost Flows 162
4.10.2 Building up an Optimal Solution 166
4.10.3 The Assignment and the Transportation Problem 169

4.11 Applications of Flows 170
4.11.1 Maximum Matchings in Bipartite Graphs 170
4.11.2 The Directed Chinese Postman Problem 174
4.11.3 Finding Subdigraphs with Prescribed Degrees 176
4.11.4 Path-Cycle Factors in Directed Multigraphs 177

4.12 Exercises 179

5. Connectivity of Digraphs 191
5.1 Additional Notation and Preliminaries 192
5.1.1 The Network Representation of a Directed Multigraph 194

5.2 Finding the Strong Components of a Digraph 195
5.3 Ear Decompositions 198
5.4 Menger’s Theorem 201

5.5 Determining Arc- and Vertex-Strong Connectivity 204
5.6 Minimally \(k\)-(Arc)-Strong Directed Multigraphs 207
5.6.1 Minimally \(k\)-Arc-Strong Directed Multigraphs 207
5.6.2 Minimally \(k\)-Strong Digraphs 213

5.7 Critically \(k\)-Strong Digraphs 218
5.8 Connectivity Properties of Special Classes of Digraphs 220
5.9 Disjoint X-Paths in Digraphs 223
5.10 Exercises 223

6. Hamiltonian, Longest and Vertex-Cheapest Paths and Cycles 227
6.1 Complexity 228
6.2 Hamilton Paths and Cycles in Path-Mergeable Digraphs 230
6.3 Hamilton Paths and Cycles in Locally In-Semicomplete Di-
graphs 231

6.4 Hamilton Cycles and Paths in Degree-Constrained Digraphs 233
6.4.1 Sufficient Conditions 233
6.4.2 The Multi-Insertion Technique 239
6.4.3 Proofs of Theorems 6.4.1 and 6.4.5 240
6.5 Longest Paths and Cycles in Degree-Constrained Oriented Graphs ................................................ 243
6.6 Longest Paths and Cycles in Semicomplete Multipartite Di- graphs ................................................ 244
6.6.1 Basic Results ........................................ 245
6.6.2 The Good Cycle Factor Theorem ................. 247
6.6.3 Consequences of Lemma 6.6.12 .................... 250
6.6.4 Yeo’s Irreducible Cycle Subdigraph Theorem and Its Applications ........................................ 253
6.7 Hamilton Paths and Cycles in Quasi-Transitive Digraphs .... 256
6.8 Vertex-Cheapest Paths and Cycles ...................... 260
6.8.1 Vertex-Cheapest Paths and Cycles in Quasi-Transitive Digraphs ........................................ 260
6.8.2 Minimum Cost $k$-Path-Cycle Subdigraphs ......... 261
6.8.3 Cheapest $i$-Path Subdigraphs in Quasi-Transitive Di- graphs .................................................. 263
6.8.4 Finding a Cheapest Cycle in a Quasi-Transitive Digraph .................................................. 265
6.9 Hamilton Paths and Cycles in Various Classes of Digraphs .... 265
6.10 Exercises .................................................. 271

7. Restricted Hamiltonian Paths and Cycles .................. 275
7.1 Hamiltonian Paths with a Prescribed End-Vertex .......... 275
7.2 Weakly Hamiltonian-Connected Digraphs ................. 277
7.2.1 Results for Extended Tournaments ................. 277
7.2.2 Results for Locally Semicomplete Digraphs ......... 283
7.3 Hamiltonian-Connected Digraphs ......................... 286
7.4 Hamiltonian Cycles Containing or Avoiding Prescribed Arcs ... 289
7.4.1 Hamiltonian Cycles Containing Prescribed Arcs ...... 290
7.4.2 Avoiding Prescribed Arcs with a Hamiltonian Cycle 292
7.4.3 Hamiltonian Cycles Avoiding Arcs in 2-Cycles ....... 295
7.5 Arc-Traceable Digraphs ................................... 296
7.6 Oriented Hamiltonian Paths and Cycles ................... 297
7.7 Exercises .................................................. 303

8. Paths and Cycles of Prescribed Lengths .................. 307
8.1 Pancyclicity of Digraphs .................................. 307
8.1.1 (Vertex-)Pancyclicity in Degree-Constrained Digraphs . 308
8.1.2 Pancyclicity in Extended Semicomplete and Quasi- Transitive Digraphs ...................................... 309
8.1.3 Panhypercyclic and Vertex-Panhypercyclic Semicomplete Digraphs .................................... 312
8.1.4 Further Pancyclicity Results ........................ 315
8.1.5 Cycle Extendability in Digraphs ..................... 317
8.1.6 arc-Pancyclicity .................................... 318
8.2 Colour Coding: Efficient Algorithms for Paths and Cycles .... 320
8.3 Cycles of Length \( k \) Modulo \( p \) ........................................... 324
8.3.1 Complexity of the Existence of Cycles of Length \( k \) Modulo \( p \) Problems ........................................... 324
8.3.2 Sufficient Conditions for the Existence of Cycles of Length \( k \) Modulo \( p \) ........................................... 326
8.4 Girth ........................................... 329
8.5 Short Cycles in Semicomplete Multipartite Digraphs ............. 332
8.6 Exercises ........................................... 336

9. Branchings ........................................... 339
9.1 Tutte’s Matrix Tree Theorem ........................................... 339
9.2 Optimum Branchings ........................................... 342
9.2.1 Matroid Intersection Formulation .............................. 343
9.2.2 A Simple Algorithm for Finding a Minimum Cost Out-Branching ........................................... 344
9.3 Arc-Disjoint Branchings ........................................... 345
9.4 Implications of Edmonds’ Branching Theorem .................... 348
9.5 Out-Branchings with Degree Bounds .............................. 351
9.6 Arc-Disjoint In- and Out-Branchings ........................................... 354
9.7 Out-Branchings with Extremal Number of Leaves ............... 358
9.7.1 Minimum Leaf Out-Branchings ........................................... 359
9.7.2 Maximum Leaf Out-Branchings ........................................... 361
9.8 The Source Location Problem ........................................... 363
9.9 Miscellaneous Topics ........................................... 365
9.9.1 Edge-Disjoint Mixed Branchings ........................................... 365
9.9.2 The Minimum Covering Out-Tree Problem .................... 366
9.9.3 Minimum Cost Arc-Disjoint Branchings with Bandwidth Constraints ........................................... 367
9.9.4 Out-Forests ........................................... 368
9.9.5 The Maximum Weight Out-Forest Problem .................... 368
9.9.6 Branchings and Edge-Disjoint Trees .............................. 370
9.10 Exercises ........................................... 370

10. Linkages in Digraphs ........................................... 373
10.1 Additional Definitions and Preliminaries ......................... 373
10.2 The Complexity of the \( k \)-Linkage Problem .................... 375
10.3 Sufficient Conditions for a Digraph to Be \( k \)-Linked ............ 379
10.4 The \( k \)-Linkage Problem for Acyclic Digraphs .................. 382
10.5 Linkages in (Generalizations of) Tournaments .................. 385
10.5.1 Sufficient Conditions in Terms of (Local-)Connectivity 385
10.5.2 The 2-Linkage Problem for Semicomplete Digraphs .......... 389
10.5.3 The 2-Linkage Problem for Generalizations of Tournaments ........................................... 391
10.6 Linkages in Planar Digraphs ........................................... 394
10.7 Weak Linkages ........................................... 398
10.7.1 Weak Linkages in Acyclic Directed Multigraphs ..... 400
10.7.2 Weak Linkages in Eulerian Directed Multigraphs ..... 401
10.7.3 Weak Linkages in Tournaments and Generalizations of
Tournaments ............................................. 407
10.8 Linkages in Digraphs with Large Minimum Out-Degree ..... 410
10.8.1 Subdivisions of Transitive Tournaments in Digraphs of
Large Out-Degree ....................................... 411
10.9 Miscellaneous Topics .................................. 412
10.9.1 Universal Arcs in 2-Cyclic Digraphs ............. 412
10.9.2 Integer Multicommodity Flows ................. 413
10.10 Exercises .................................................. 414

11. Orientations of Graphs and Digraphs .................. 417
11.1 Underlying Graphs of Various Classes of Digraphs .... 417
11.1.1 Underlying Graphs of Transitive and Quasi-Transitive
Digraphs ................................................. 418
11.1.2 Underlying Graphs of Locally Semicomplete Digraphs 421
11.1.3 Local Tournament Orientations of Proper Circular Arc
Graphs ..................................................... 423
11.1.4 Underlying Graphs of Locally
In-Semicomplete Digraphs ............................ 426
11.2 Orientations with No Even Cycles .................. 428
11.3 Colourings and Orientations of Graphs .............. 431
11.4 Orientations and Nowhere-Zero Integer Flows ....... 435
11.5 Orientations Achieving High Arc-Strong Connectivity .. 441
11.5.1 $k$-Arc-Strong Orientations ...................... 441
11.5.2 Well-Balanced and Best-Balanced Orientations ..... 443
11.5.3 Simultaneous Best-Balanced Orientations ......... 444
11.5.4 Best-Balanced Orientations of Eulerian Multigraphs 445
11.6 $k$-Strong Orientations ................................ 446
11.7 Orientations Respecting Degree Constraints .......... 448
11.7.1 Orientations with Prescribed Degree Sequences .... 448
11.7.2 Restrictions on Subsets of Vertices ............. 452
11.8 Submodular Flows ....................................... 453
11.8.1 Submodular Flow Models ......................... 454
11.8.2 Existence of Feasible Submodular Flows .......... 455
11.8.3 Minimum Cost Submodular Flows ............... 458
11.8.4 Applications of Submodular Flows .............. 459
11.9 Orientations of Mixed Multigraphs ................ 461
11.10 $k$-(Arc)-Strong Orientations of Digraphs ........ 466
11.11 Miscellaneous Topics ................................. 470
11.11.1 Another Measure of Well-Balancedness ......... 470
11.11.2 Orienting to Preserve Reachability
for Prescribed Pairs .................................... 470
11.12 Exercises ................................................. 472
12. Sparse Subdigraphs with Prescribed Connectivity .............. 479
  12.1 Minimum Strong Spanning Subdigraphs .................. 480
    12.1.1 Digraphs with High Minimum Degree .......... 482
  12.2 Polynomially Solvable Cases of the MSSS Problem ....... 483
    12.2.1 The MSSS Problem for Extended Semicomplete Digraphs .......... 484
    12.2.2 The MSSS Problem for Quasi-Transitive Digraphs .... 485
  12.3 Approximation Algorithms for the MSSS Problem .......... 487
    12.3.1 A Simple 7/4-Approximation Algorithm .... 487
    12.3.2 Better Approximation Algorithms .......... 488
  12.4 Small Certificates for k-(Arc)-Strong Connectivity ...... 489
    12.4.1 Small Certificates for k-Strong Connectivity ... 490
    12.4.2 Small Certificates for k-Arc-Strong Connectivity ... 491
    12.4.3 Certificates for Directed Multigraphs .......... 494
  12.5 Minimum Weight Strong Spanning Subdigraphs .......... 497
  12.6 Directed Steiner Problems ................................ 498
  12.7 Miscellaneous Topics .................................. 501
    12.7.1 The Directed Spanning Cactus Problem .... 501
    12.7.2 An FTP Algorithm for the MSSS Problem .... 501
    12.7.3 Minimum Cost Strong Subdigraphs .......... 502
  12.8 Exercises .................................................. 503

13. Packings, Coverings and Decompositions ...................... 505
  13.1 Packing Directed Cuts: The Lucchesi-Younger Theorem ...... 505
  13.2 Packing Dijoins: Woodall’s Conjecture ................. 511
  13.3 Packing Cycles .......................................... 512
  13.4 Arc-Disjoint Hamiltonian Paths and Cycles ............. 515
  13.5 Path Factors ............................................ 519
  13.6 Cycle Factors with the Minimum Number of Cycles ....... 521
  13.7 Cycle Factors with a Fixed Number of Cycles ........... 525
  13.8 Cycle Subdigraphs Covering Specified Vertices ....... 528
  13.9 Proof of Gallai’s Conjecture ............................ 529
  13.10 Decomposing a Tournament into Strong Spanning Subdigraphs .......... 536
  13.11 The Directed Path-Partition Conjecture ................ 542
  13.12 Miscellaneous Topics .................................. 546
    13.12.1 Maximum One-Way Cuts and Covering by One-Way Cuts ............ 546
    13.12.2 Acyclic Decompositions of Digraphs ............ 548
    13.12.3 Decomposing Tournaments into Strong Subtournaments .......... 548
    13.12.4 Decomposing Digraphs under Degree Constraints .... 549
  13.13 Exercises ................................................. 550
14. Increasing Connectivity ........................................... 553
  14.1 The Splitting Off Operation .................................. 553
  14.2 Increasing the Arc-Strong Connectivity Optimally .......... 557
  14.3 Increasing the Vertex-Strong Connectivity Optimally ...... 562
    14.3.1 One-Way Pairs ........................................ 563
    14.3.2 Optimal $k$-Strong Augmentation ....................... 565
    14.3.3 Special Classes of Digraphs ............................ 566
  14.4 Arc Reversals and Vertex-Strong Connectivity .............. 568
  14.5 Arc-Reversals and Arc-Strong Connectivity ................. 570
    14.5.1 Determining $r_{deg}^k(D)$ Efficiently ................ 571
    14.5.2 Reversals of Arcs to Achieve High Arc-Strong Connectivity in Tournaments ....................... 572
  14.6 Increasing Connectivity by Deorienting Arcs ............... 573
  14.7 Miscellaneous Topics ........................................ 576
    14.7.1 Increasing Arc-Strong Connectivity of a Bipartite Di- graph ........................................ 576
    14.7.2 Augmenting Arc-Strong Connectivity in Directed Hypergraphs .................................. 577
    14.7.3 Weighted Versions of Local Arc-Connectivity Problems ........................................ 578
  14.8 Exercises ..................................................... 580

15. Feedback Sets and Vertex Orderings ......................... 583
  15.1 Two Conjectures on Feedback Arc Sets ...................... 584
  15.2 Optimal Orderings in Tournaments .......................... 585
  15.3 Complexity of the Feedback Set Problems .................. 586
    15.3.1 $NP$-Completeness Results ............................. 587
    15.3.2 FAS for Planar Digraphs ............................... 590
    15.3.3 Approximation Algorithms ................................ 591
    15.3.4 Fixed-Parameter Tractability Results ................ 593
  15.4 Disjoint Cycles Versus Feedback Sets ....................... 596
    15.4.1 Relations Between Parameters $\nu_i$ and $\tau_i$ .......... 596
    15.4.2 Solution of Younger's Conjecture ....................... 598
  15.5 Optimal Orderings and Seymour's Second Neighbourhood Conjecture ........................................ 600
  15.6 Ádám's Conjecture ............................................ 603
  15.7 Exercises ..................................................... 605

  16.1 Terminology, Notation and Initial Observations .......... 608
  16.2 Properly Coloured Euler Trails .............................. 610
  16.3 Properly Coloured Cycles ..................................... 613
  16.4 Gadget Graphs and Shortest PC Cycles and $(s, t)$-Paths .... 617
    16.4.1 P-Gadgets ............................................... 617
    16.4.2 P-Gadget Graphs ......................................... 618
  16.5 Long PC Cycles and Paths .................................... 621
16.6 Connectivity of Edge-Coloured Multigraphs .......................... 622
16.7 Alternating Cycles in 2-Edge-Coloured Bipartite Multigraphs 625
16.8 Paths and Cycles in 2-Edge-Coloured Complete Multigraphs . 628
16.9 PC Paths and Cycles in $c$-Edge-Coloured Complete Graphs, $c \geq 3$ ....................................................... 635
16.10 Exercises .......................................................... 640

17. Applications of Digraphs and Edge-Coloured Graphs ............ 643
17.1 A Digraph Model in Quantum Mechanics .......................... 643
  17.1.1 Lower Bound for $\mu(n)$ ................................... 644
  17.1.2 Families of Sets and $\mu(n)$ .............................. 644
  17.1.3 Upper Bounds for $\mu(n)$ .................................. 646
  17.1.4 When $\mu(n) > f(n)$ ....................................... 647
  17.1.5 Mediated Digraphs in Quantum Mechanics ................. 647
17.2 Embedded Computing and Convex Sets in Acyclic Digraphs . 649
  17.2.1 Embedded Computing Systems and Convex Sets ......... 649
  17.2.2 Bounds for the Number of Convex Sets .................. 650
  17.2.3 Algorithms for Generating Convex and Connected
      Convex Sets .................................................. 652
17.3 When Greedy-Like Algorithms Fail ................................. 655
  17.3.1 Greedy Algorithm ........................................... 656
  17.3.2 Max-Regret Algorithms .................................... 659
17.4 Domination Analysis of ATSP Heuristics .......................... 660
  17.4.1 ATSP Heuristics with Factorial Domination Numbers .. 662
  17.4.2 Upper Bounds on Domination Numbers .................... 664
17.5 Solving the 2-Satisfiability Problem .............................. 666
17.6 Alternating Hamilton Cycles in Genetics ........................ 670
  17.6.1 Proof of Theorem 17.6.1 ................................. 672
  17.6.2 Proof of Theorem 17.6.2 ................................. 673
17.7 Gaussian Elimination ............................................. 674
17.8 Markov Chains ................................................... 677
17.9 List Edge-Colourings ............................................. 679
17.10 Digraph Models of Bartering ..................................... 683
17.11 PERT/CPM in Project Scheduling ................................ 685
17.12 Finite Automata .................................................. 687
17.13 Puzzles and Digraphs ............................................. 689
17.14 Gossip Problems .................................................. 690
17.15 Deadlocks of Computer Processes ................................ 692
17.16 Exercises .......................................................... 694

18. Algorithms and Their Complexity ................................. 695
18.1 Combinatorial Algorithms ......................................... 696
18.2 $\mathcal{NP}$-Complete and $\mathcal{NP}$-Hard Problems .......... 700
18.3 The Satisfiability Problem ....................................... 702
18.4 Fixed-Parameter Tractability and Intractability ............... 703
Digraphs
Theory, Algorithms and Applications
Bang-Jensen, J.; Gutin, G.Z.
2009, XXII, 798 p. 175 illus., Hardcover
ISBN: 978-1-84800-997-4