Chapter 2

Hermite’s Theorem

We will begin with the proof that $e$ is transcendental, a result first proved by Charles Hermite in 1873.

**Theorem 2.1** $e$ is transcendental.

**Proof.** We make the observation that for a polynomial $f$ and a complex number $t$,

$$\int_0^t e^{-u} f(u) du = [-e^{-u} f(u)]_0^t + \int_0^t e^{-u} f'(u) du$$

which is easily seen on integrating by parts. Here the integral is taken over the line joining 0 and $t$. If we let

$$I(t, f) := \int_0^t e^{t-u} f(u) du,$$

then we see that

$$I(t, f) = e^t f(0) - f(t) + I(t, f').$$

If $f$ is a polynomial of degree $m$, then iterating this relation gives

$$I(t, f) = e^t \sum_{j=0}^m f^{(j)}(0) - \sum_{j=0}^m f^{(j)}(t). \quad (2.1)$$

If $F$ is the polynomial obtained from $f$ by replacing each coefficient of $f$ by its absolute value, then it is easy to see from the definition of $I(t, f)$ that

$$|I(t, f)| \leq |t| e^{|t|} F(|t|). \quad (2.2)$$
With these observations, we are now ready to prove the theorem. Suppose \( e \) is algebraic of degree \( n \). Then
\[
a_n e^n + a_{n-1} e^{n-1} + \cdots + a_1 e + a_0 = 0
\] (2.3)
for some integers \( a_i \) and \( a_0 a_n \neq 0 \). We will consider the combination
\[
J := \sum_{k=0}^{n} a_k I(k, f)
\]
with
\[
f(x) = x^{p-1} (x-1)^p \cdots (x-n)^p
\]
where \( p > |a_0| \) is a large prime. Using (2.3), we see that
\[
J = - \sum_{j=0}^{m} \sum_{k=0}^{n} a_k f^{(j)}(k)
\]
where \( m = (n+1)p - 1 \). Since \( f \) has a zero of order \( p \) at \( 1, 2, \ldots, n \) and a zero of order \( p-1 \) at 0, we have that the summation actually starts from \( j = p - 1 \). For \( j = p - 1 \), the contribution from \( f \) is
\[
f^{(p-1)}(0) = (p-1)! (1)^{n} n!^p.
\]
Thus if \( n < p \), then \( f^{(p-1)}(0) \) is divisible by \( (p-1)! \) but not by \( p \). If \( j \geq p \), we see that \( f^{(j)}(0) \) and \( f^{(j)}(k) \) are divisible by \( p! \) for \( 1 \leq k \leq n \). Hence \( J \) is a non-zero integer divisible by \( (p-1)! \) and consequently
\[
(p-1)! \leq |J|.
\]
On the other hand, our estimate (2.2) shows that
\[
|J| \leq \sum_{k=0}^{n} |a_k| e^k F(k) k \leq A n e^n (2n)!^p
\]
where \( A \) is the maximum of the absolute values of the \( a_k \)'s. The elementary observation
\[
e^p \geq \frac{p^{p-1}}{(p-1)!}
\]
gives
\[
p^{p-1} e^{-p} \leq (p-1)! \leq |J| \leq A n e^n (2n)!^p.
\]
For \( p \) sufficiently large, this is a contradiction. \( \square \)
Exercises

1. Show that for any polynomial \( f \), we have
\[
\int_0^\pi f(x) \sin x \, dx = f(\pi) + f(0) - \int_0^\pi f''(x) \sin x \, dx.
\]

2. Utilise the identity in the previous exercise to show \( \pi \) is irrational as follows. Suppose \( \pi = a/b \) with \( a, b \) coprime integers. Let
\[
f(x) = \frac{x^n(a - bx)^n}{n!}.
\]
Prove that
\[
\int_0^\pi f(x) \sin x \, dx
\]
is a non-zero integer and derive a contradiction from this.

3. Use Euler’s identity
\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}
\]
to prove that there are infinitely many primes.

4. Use the series \( \sum_{n=0}^{\infty} 1/n! \) to show that \( e \) is irrational.

5. Show that \( e \) is not algebraic of degree 2 by considering the relation
\[
Ae + Be^{-1} + C = 0, \quad A, B, C \in \mathbb{Z},
\]
and using the infinite series for \( e \) and \( e^{-1} \) and arguing as in the previous exercise.

6. Prove that \( e^{\sqrt{2}} \) is irrational (Hint: Consider the series expansion for \( \alpha = e^{\sqrt{2}} + e^{-\sqrt{2}} \)).
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