Preface

Most modern set theory texts, even at the undergraduate level, introduce specific formal axiom systems such as ZFC relatively early, perhaps because of the (understandably real) fear of paradoxes. At the same time, most mathematicians and students of mathematics seem to care little about special formal systems, yet may still be interested in the part of set theory belonging to "mathematics proper," i.e., cardinals, order, ordinals, and the theory of the real continuum. There appears to be a gulf between texts of mainstream mathematics and those of set theory and logic.

This undergraduate set theory textbook regards the core material on cardinals, ordinals, and the continuum as a subject area of classical mathematics interesting in its own right. It separates and postpones all foundational issues (such as paradoxes and special axioms) into an optional part at the end. The main material is thus developed informally—not within any particular axiom system—to avoid getting bogged down in the details of formal development and its associated metamathematical baggage. I hope this will make this text suitable for a wide range of students interested in any field of mathematics and not just for those specializing in logic or foundations. At the same time, students with metamathematical interests will find an introduction to axiomatic ZF set theory in the last part, and some glimpses into key foundational topics in the postscript chapters at the end of each part.

Another feature of this book is that its coverage of the real continuum is confined exclusively to the real line **R**. All abstract or general concepts such as topological spaces, metric spaces, and even the Euclidean spaces of dimension 2 or higher are completely avoided. This may seem like a severe handicap, but even this highly restricted framework allows the introduction of many interesting topics in the theory of real point sets. In fact, not much substance in the theory is lost and a few deeper intuitions are gained. As evidenced by the teaching of undergraduate real analysis, the student who is first firmly grounded in the hard and concrete details of **R** will better enjoy and handle the abstraction found in later, more advanced studies.

The book grew out of an undergraduate course in introductory set theory that I taught at the University of Detroit Mercy. The prerequisite for the core material of the book is a post-calculus undergraduate US course in discrete mathematics or linear algebra, although precalculus and some exposure to proofs should technically suffice for Parts I and II.

The book starts with a “prerequisites” chapter on sets, relations, and functions, including equivalence relations and partitions, and the definition of linear order. The rest is divided into four relatively independent parts with quite distinct mathematical flavors. Certain basic techniques are emphasized across multiple parts, such as Cantor’s back-and-forth method, construction of perfect sets, Cantor–Bendixson analysis, and ordinal ranks.

Part I is a problem-based short course which, starting from Peano arithmetic, constructs the real numbers as Dedekind cuts of rationals in a routine way with two possible uses. A student of mathematics not going into formal ZF set theory will work out, once and for all, a detailed existence proof for a complete ordered field. And for a student who might later get into axiomatic ZF set theory, the redevelopment of Peano arithmetic and the theory of real numbers formally within ZF will become largely superfluous. One may also decide to skip Part I altogether and go directly to Part II.

Part II contains the core material of the book: The Cantor–Dedekind theory of the transfinite, especially order, the continuum, cardinals, ordinals, and the Axiom of Choice. The development is informal and naive (non-axiomatic), but mathematically rigorous. While the core material is intended to be interesting in its own right, it also forms the folklore set-theoretic prerequisite needed for graduate level topology, analysis, algebra, and logic. Useful forms of the Axiom of Choice, such as Zorn’s Lemma, are covered.

Part III of the book is about point sets of real numbers. It shows how the theory of sets and orders connects intimately to the continuum and its topology. In addition to the basic theory of $\mathbf{R}$ including measure and category, it presents more advanced topics such as Brouwer’s theorem, Cantor–Bendixson analysis, Sierpinski’s theorem, and an introduction to Borel and analytic sets—all in the context of the real line. Thus the reader gets access to significant higher results in a concrete manner via powerful techniques such as Cantor’s back-and-forth method. As mentioned earlier, all development is limited to the reals, but the apparent loss of generality is mostly illusory and the special case for real numbers captures much of the essential ideas and the central intuitions behind these theorems.

Parts II and III of the book focus on gaining intuition rather than on formal development. I have tried to start with specific and concrete cases of examples and theorems before proceeding to their more general and abstract versions. As a result, some important topics (e.g., the Cantor set) appear multiple times in the book, generally with increasing levels of sophistication. Thus, I have sacrificed compactness and conciseness in favor of intuition building and maintaining some independence between the four parts.

Part IV deals with foundational issues. The paradoxes are first introduced here, leading to formal set theory and the Zermelo–Fraenkel axiom system. Von Neumann ordinals are also first presented in this part.

Each part ends with a postscript chapter discussing topics beyond the scope of the main text, ranging from philosophical remarks to glimpses into modern landmark results of set theory such as the resolution of Lusin's problems on projective sets using determinacy of infinite games and large cardinals.

Problems form an integral and essential part of the book. While some of them are routine, they are generally meant to form an extension of the text. A harder problem will contain hints and sometimes an outline for a solution. Starred sections and problems may be regarded as optional.

The book has enough material for a one-year course for advanced undergraduates. The relative independence of the four parts allows various possibilities for covering topics. In a typical one-semester course, I usually briefly cover Part I, spend most of the time in Part II, and finish with a brief overview of Part IV. For students with more foundational interests, more time can be spent on the material of Part IV and the postscripts. On the other hand, for less foundationally inclined but more mathematically advanced students with prior exposure to advanced calculus or real analysis, only Parts II and III may be covered with Parts I and IV skipped altogether.

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