Chapter 2
Existence of Spikes for the Gierer-Meinhardt System in One Dimension

In this chapter we give a full account of the existence of multiple spikes for the Gierer-Meinhardt system in an interval on the real line. Without loss of generality, we assume that this interval is $(-1, 1)$. We will construct the solution rigorously by (i) reducing the problem to finite dimensions by applying the Liapunov-Schmidt reduction and (ii) employing a fixed-point argument (e.g. using the mapping degree) to solve this finite-dimensional problem. Here we present a simplification in the case of two spikes. The time-dependent problem consists of a two-component reaction-diffusion system of activator-inhibitor type with no-flux (Neumann) boundary conditions and can be stated as follows:

$$\begin{cases} A_t = \epsilon^2 A'' - A + \frac{A^2}{H}, & x \in (-1,1), t > 0, \\ \tau H_t = DH'' - H + A^2, & x \in (-1,1), t > 0, \\ A'(\pm 1, t) = H'(\pm 1, t) = 0, & t > 0. \end{cases} \tag{2.1}$$

This system has three parameters, namely two diffusivities and one time relaxation constant which satisfy

$$0 < \epsilon \ll 1, \qquad 0 < D < \infty, \qquad \tau \geq 0.$$

Throughout this chapter, we assume that

$$D \text{ and } \tau \text{ are real constants which are independent of } \epsilon.$$

The corresponding stationary problem is given by

$$\begin{cases} \epsilon^2 A'' - A + \frac{A^2}{H} = 0, & x \in (-1,1), \\ DH'' - H + A^2 = 0, & x \in (-1,1), \\ A'(\pm 1) = H'(\pm 1) = 0. \end{cases} \tag{2.2}$$

Before giving a full discussion of the problem, let us summarise what we mean by a spike. It is a pattern which is narrowly concentrated near a point in the domain

J. Wei, M. Winter, *Mathematical Aspects of Pattern Formation in Biological Systems*, Applied Mathematical Sciences 189, DOI 10.1007/978-1-4471-5526-3_2,

which is characterised by its profile, amplitude and position. The pattern is observed in the activator component and the inhibitor component plays a stabilising role. A multi-spike steady state consisting of N spikes satisfies

$$a_\epsilon \sim \xi_{i,\epsilon} w\left(\frac{x - t_i^\epsilon}{\epsilon}\right) \quad \text{as } \epsilon \to 0, i = 1, \ldots, N,$$

where the profile function w is the unique solution of the problem

$$\begin{cases} w'' - w + w^2 = 0 & \text{in } \mathbb{R}, \\ w > 0, \quad w(0) = \max_{y \in \mathbb{R}} w(y), \\ w(y) \to 0 & \text{as } |y| \to \infty \end{cases} \tag{2.3}$$

and the notation $A(\epsilon) \sim B(\epsilon)$ means that $\lim_{\epsilon \to 0} \frac{A(\epsilon)}{B(\epsilon)} = c_0 > 0$, for some positive number c_0. Existence and uniqueness have been proved in Sect. 13.2. Below we will first compute the amplitudes $\xi_{i,\epsilon}$. We will also determine the positions t_i^ϵ of the spike centres which will approach certain limiting locations t_i^0 as $\epsilon \to 0$.

Problem (2.2) has been studied by numerous authors. Let us first recall several important results on the formation of spiky patterns.

2.1 Symmetric Multi-spike Solutions: A Rigorous Proof of Existence

I. Takagi [226] first established the existence of N-spike steady-state solutions with maxima located exactly at

$$t_j^0 = -1 + \frac{2j-1}{N}, \quad j = 1, \ldots, N, \tag{2.4}$$

for $\epsilon \ll 1$ and $\epsilon^2 \ll D$. These solutions are periodic and they are obtained by first constructing a single spike on a smaller interval and then using periodic continuation to extend the solution to multiple spikes on the original domain. We call them *symmetric* N-spike solutions since all the spikes have the same amplitudes, and in this special case they are exact copies of each other. Takagi's proof uses the implicit function theorem to construct single spikes. The implicit function theorem is applied in a suitable functional space of even functions. The argument of the existence proof is very elegant since the linearised operator around a suitable approximate solution restricted to this space of even functions is invertible, see Sect. 3.1. These solutions are close to the approximate solution

$$A \sim \xi_\epsilon w\left(\frac{x}{\epsilon}\right), \qquad H(0) = \xi_\epsilon$$

for suitable amplitude ξ_ϵ and given shape function w solving (2.3). Note that w is given by

$$w(y) = \frac{3}{2\cosh^2(y/2)}. \tag{2.5}$$

Therefore single-spike solutions are even functions defined on the interval $(-\frac{1}{N}, \frac{1}{N})$ with the boundary conditions

$$A'\left(-\frac{1}{N}\right) = A'\left(\frac{1}{N}\right) = H'\left(-\frac{1}{N}\right) = H'\left(\frac{1}{N}\right) = 0.$$

Because of these properties, the periodic continuation of the solution from the interval $(-\frac{1}{N}, \frac{1}{N})$ to the whole domain $(-1, 1)$ is a symmetric multi-spike solution.

2.2 Asymmetric Multi-spike Solutions: A Formal Derivation

Using matched asymptotic expansions, Ward and Wei in [240] showed that for any given positive integer N and under the condition $D < D_N$, where the sequence $D_1 > D_2 > \cdots > D_N > \cdots > 0$ has been stated explicitly, problem (2.2) has *asymmetric N-spike solutions* if ϵ is small enough. These asymmetric solutions are generated by two different types of spikes which we call type **A** or type **B**, respectively. For any given order, e.g.

ABAABBB...**ABBBA**...**B**,

there is a corresponding asymmetric N-spike solution such that the two types of spikes follow this order. In each of the resulting subintervals the two types of spikes are given asymptotically by the formula

$$A \sim \xi_{i,\epsilon} w\left(\frac{x - t_i^\epsilon}{\epsilon}\right), \qquad H\left(t_i^\epsilon\right) = \xi_{i,\epsilon}, \tag{2.6}$$

where t_i^ϵ is the centre of the spike, $\xi_{i,\epsilon}$ its amplitude and w, given by (2.5), is its shape function. Here ξ_i can either be small (for type **A** spikes) or large (for type **B** spikes). It is also seen that for the small spikes the subinterval is small and for the large spikes the subinterval is large. Further, the small spikes are exact copies of each other as are the large spikes.

First we start from (2.2) in a small interval $(-l, l)$: Let A and H be even functions in the set (compare Sect. 13.1)

$$H_N^2(-l, l) := \left\{v \in H^2(-1, 1) : v'(-1) = v'(1) = 0\right\}$$

satisfying

$$\begin{cases} \epsilon^2 A'' - A + \frac{A^2}{H} = 0 & \text{in } (-l, l), \\ DH'' - H + A^2 = 0 & \text{in } (-l, l), \\ A(x) > 0, \qquad H(x) > 0 & \text{in } (-l, l). \end{cases} \tag{2.7}$$

Consider the single-spike solution which was constructed by I. Takagi [226]. By some simple computations based on (2.13) below, it follows that we have

$$H(l) = c(D) b\left(\frac{l}{\sqrt{D}}\right) + o(1) \quad \text{as } \epsilon \to 0, \tag{2.8}$$

where $c(D)$ is some positive constant depending on D only and the function $b(z)$ is given by

$$b(z) := \frac{\sinh(z)}{\cosh^2(z)}. \tag{2.9}$$

We note that $b(0) = \lim_{z\to\infty} b(z) = 0$ and b has a unique maximum for which $\cosh^2 z_{\max} = 2$ and $b(z_{\max}) = \frac{1}{2}$.

The approach now is to fix l and find a positive number $\bar{l}$ such that

$$b\left(\frac{l}{\sqrt{D}}\right) = b\left(\frac{\bar{l}}{\sqrt{D}}\right), \quad 0 < l < \bar{l} < 1, \tag{2.10}$$

which will imply that $H(l) = H(\bar{l}) + o(1)$. This shows that if there exists a solution to (2.10), then $H(l)$ and $H(-\bar{l})$ can be matched. Using the fact that w decays exponentially, solutions of (2.7) in different subintervals can be connected.

It turns out that for D small enough (2.10) is always solvable. Now (2.10) has to be solved together with the following constraint to fit N spikes into the interval $(-1, 1)$:

$$N_1 l + N_2 \bar{l} = 1, \qquad N_1 + N_2 = N. \tag{2.11}$$

For a solution l of (2.10) and (2.11) and $j = 1, \ldots, N$ we set

$$l_j = l \quad \text{or} \quad l_j = \bar{l}, \tag{2.12}$$

where the order of the l's and $\bar{l}$'s can be chosen arbitrarily under the constraint that the number of j's such that $l_j = l$ is N_1 and the number of j's such that $l_j = \bar{l}$ is N_2.

Finally, we compute $t_j^0 = \lim_{\epsilon\to 0} t_j^\epsilon$, $j = 1, \ldots, N$, so that

$$t_{j+1}^0 - t_j^0 = l_j + l_{j+1}, \quad j = 1, \ldots, N-1$$

and $t_1^0 = -1 + l_1$, $t_N^0 = 1 - l_N$. This concludes the formal construction of asymmetric multi-spike solutions.

2.3 Existence of Symmetric and Asymmetric Multiple Spikes: A Unified Rigorous Approach

In this section and Chap. 4, we present a unified approach to a rigorous theoretical treatment of the existence and stability of general N-spike (symmetric or asymmetric) solutions for the Gierer-Meinhardt system (2.2) on the interval $(-1, 1)$. The existence proof firstly uses Liapunov-Schmidt reduction to deduce a finite-dimensional problem and secondly a fixed-point argument (e.g. using the mapping degree) to solve the finite-dimensional problem.

The stability is shown by first separating the problem into the case of large eigenvalues which tend to a nonzero limit and the case of small eigenvalues which tend to zero as $\epsilon \to 0$. Large eigenvalues are then explored by studying nonlocal eigenvalue problems. Small eigenvalues are calculated explicitly by an asymptotic analysis with rigorous error estimates. It turns out that for the case of symmetric N-spike solutions the instability always arises first from the small eigenvalues.

Finally, in Sect. 2.4 we state results on the existence of multiple clusters for (2.2) for which different spikes may approach the same point.

We note also that in [50] an alternative dynamical systems approach is used to study the stability of symmetric spikes.

Before stating our main results, we introduce some notation. Let $L^2(-1, 1)$ and $H^2(-1, 1)$ be the usual Lebesgue and Sobolev spaces (see Sect. 13.1). Let $\Omega = (-1, 1)$ and $G_D(x, z)$ be the following Green's function:

$$\begin{cases} DG''_D(x,z) - G_D(x,z) + \delta_z(x) = 0 & \text{in } (-1,1), \\ G'_D(-1,z) = G'_D(1,z) = 0. \end{cases} \tag{2.13}$$

We can calculate explicitly

$$G_D(x,z) = \begin{cases} \frac{\theta}{\sinh(2\theta)} \cosh[\theta(1+x)] \cosh[\theta(1-z)], & -1 < x < z < 1, \\ \frac{\theta}{\sinh(2\theta)} \cosh[\theta(1-x)] \cosh[\theta(1+z)], & -1 < z < x < 1, \end{cases} \tag{2.14}$$

where

$$\theta = \frac{1}{\sqrt{D}}. \tag{2.15}$$

We set

$$K_D(|x-z|) = \frac{1}{2\sqrt{D}} e^{-(1/\sqrt{D})|x-z|} \tag{2.16}$$

to be the non-smooth part of $G_D(x, z)$, and we define the regular part H_D of G_D by $H_D = K_D - G_D$. Note that G_D is C^∞ for $(x, z) \in \Omega \times \Omega \setminus \{x = z\}$ and H_D is C^∞ for all $(x, z) \in \Omega \times \Omega$.

We use the abbreviation e.s.t. to denote an exponentially small term of order $O(e^{-d/\epsilon})$ for some $d > 0$ in a suitable norm. By C we denote a generic constant which may change from line to line.

For simplicity of presentation, we will assume that the number of spikes is given by

$$N = 2. \tag{2.17}$$

Let $\mathbf{t}^0 = (t_1^0, t_2^0)$ be fixed, where $-1 < t_1^0 < t_2 < 1$ are two different points in $(-1, 1)$. Using the unique solution w of (2.3), we set

$$\xi_\epsilon := \left(\epsilon \int_{\mathbb{R}} w^2(z)dz\right)^{-1} \tag{2.18}$$

and define

$$\hat{\xi}_i = \xi_{i,\epsilon}(\xi_\epsilon)^{-1}. \tag{2.19}$$

Next we introduce several matrices for later use: Let

$$\mathcal{G}_D(\mathbf{t}) = \big(G_D(t_i, t_j)\big), \tag{2.20}$$

where the Green's function G_D has been defined in (2.14) and $\mathbf{t} = (t_1, t_2) \in (-1, 1)^2$ with $-1 < t_1 < t_2 < 1$ is arbitrary. Then we introduce the matrix

$$\mathcal{B} = (b_{ij}), \quad \text{where } b_{ij} = G_D\big(t_i^0, t_j^0\big)\hat{\xi}_j^0. \tag{2.21}$$

Remark 2.1 Since the matrix $\mathcal{B}$ is of the form $\mathcal{G}_D\mathcal{D}$, where $\mathcal{G}_D$ is a symmetric and $\mathcal{D}$ is a diagonal matrix, the eigenvalues of $\mathcal{B}$ are real.

Next we compute the partial derivatives of $G_D(t_i, t_j)$. Recall that

$$G_D(t_i, t_j) = K_D\big(|t_i - t_j|\big) - H_D(t_i, t_j).$$

For $i \neq j$, we define

$$\nabla_{t_i} G(t_i, t_j) := \frac{\partial}{\partial x} G(x, t_j)\bigg|_{x=t_i^0}.$$

For $i = j$, we have that $K_D(|t_i - t_j|) = K_D(0) = \frac{1}{2\sqrt{D}}$ is a constant and we set

$$\nabla_{t_i} G_D(t_i, t_i) = -\nabla_{t_i} H_D(t_i, t_i) := -\frac{\partial}{\partial x} H_D(x, t_i)\bigg|_{x=t_i}.$$

Similarly, we define

$$\nabla_{t_i}\nabla_{t_j} G_D(t_i, t_j) = \begin{cases} -\frac{\partial}{\partial x}\big|_{x=t_i}\frac{\partial}{\partial y}\big|_{y=t_i} H_D(x, y) & \text{if } i = j, \\ \frac{\partial}{\partial x}\big|_{x=t_i}\frac{\partial}{\partial y}\big|_{y=t_j} G_D(x, y) & \text{if } i \neq j. \end{cases} \tag{2.22}$$

Then the following two matrices of first and second order derivatives of $\mathcal{G}_D$ can be introduced:

$$\nabla\mathcal{G}_D(\mathbf{t}) = \big(\nabla_{t_i} G_D(t_i, t_j)\big), \qquad \nabla^2\mathcal{G}_D(\mathbf{t}) = \big(\nabla_{t_i}\nabla_{t_j} G_D(t_i, t_j)\big). \tag{2.23}$$

In order to guarantee the existence of two-spike solutions, we make the following three assumptions. The first two assumptions will ensure that we can find suitable amplitudes for the spikes. This can be seen by the following leading-order computation: Substituting (2.6) into (2.2) and using (2.19), we compute

$$\hat{\xi}_i \sim \sum_{j=1}^{2} G\big(t_i^\epsilon, t_j^\epsilon\big)(\hat{\xi}_j)^2$$

which gives condition (H1). Then the nondegeneracy with respect to the amplitudes $(\hat{\xi}_1, \hat{\xi}_2)$ is given by

$$\det\left(\nabla_{(\hat{\xi}_1,\hat{\xi}_2)}\left(\sum_{j=1}^{2} G\big(t_i^\epsilon, t_j^\epsilon\big)(\hat{\xi}_j)^2 - \hat{\xi}_i\right)_{i=1,2}\right) \neq 0$$

which implies condition (H2). We now state these two conditions.

(H1) *For given* $\mathbf{t}^0 \in (-1,1)^2$ *with* $-1 < t_1^0 < t_2^0 < 1$, *there exists a solution* $(\hat{\xi}_1^0(\mathbf{t}), \hat{\xi}_2^0(\mathbf{t}))$ *of the equation*

$$\sum_{j=1}^{2} G_D\big(t_i^0, t_j^0\big)\big(\hat{\xi}_j^0\big)^2 = \hat{\xi}_i^0, \quad i = 1, 2. \tag{2.24}$$

(H2) *We have*

$$\frac{1}{2} \notin \sigma(\mathcal{B}), \tag{2.25}$$

where $\sigma(\mathcal{B})$ *is the set of eigenvalues of* $\mathcal{B}$.

Applying the implicit function theorem to equation (2.24) for $\mathbf{t} = (t_1, t_2)$ in a small neighbourhood of $\mathbf{t}^0 = (t_1^0, t_2^0)$, the linearised operator is invertible by assumption (H2). Thus there exists a unique solution $\hat{\boldsymbol{\xi}}(\mathbf{t}) = (\hat{\xi}_1(\mathbf{t}), \hat{\xi}_2(\mathbf{t}))$ of the equation

$$\sum_{j=1}^{2} G_D(t_i, t_j)\hat{\xi}_j^2 = \hat{\xi}_i, \quad i = 1, 2, \tag{2.26}$$

if $\mathbf{t}$ is sufficiently close to $\mathbf{t}^0$.

Our final assumption assures us that we will be able to choose suitable positions for the spikes. It is stated in terms of a vector field $F(\mathbf{t})$ which is defined by

$$F(\mathbf{t}) = \big(F_1(\mathbf{t}), F_2(\mathbf{t})\big),$$

where

$$F_i(\mathbf{t}) = \sum_{l=1}^{2} \big(\nabla_{t_i} G_D(t_i, t_l)\big)\hat{\xi}_l^2$$
$$= -\big(\nabla_{t_i} H_D(t_i, t_i)\big)\hat{\xi}_i^2 + \big(\nabla_{t_i} G_D(t_i, t_{3-i})\big)\hat{\xi}_{3-i}^2, \quad i = 1, 2, \tag{2.27}$$

and a matrix $\mathcal{M}(\mathbf{t})$ which is given by

$$\mathcal{M}(\mathbf{t}) = \big(\hat{\xi}_i^{-1} \nabla_{t_j} F_i(\mathbf{t})\big). \tag{2.28}$$

In Sect. 3.4 we will see how these expressions for $F(\mathbf{t})$ and $\mathcal{M}(\mathbf{t})$ will naturally appear in an explicit calculation. We now state our third condition:

(H3) *We assume that at the point* $\mathbf{t}^0 = (t_1^0, t_2^0)$ *given in* (H1) we have

$$F\big(\mathbf{t}^0\big) = 0, \tag{2.29}$$

$$\det\big(\mathcal{M}\big(\mathbf{t}^0\big)\big) \neq 0. \tag{2.30}$$

Let us now calculate the matrix $\mathcal{M}(\mathbf{t})$. We first compute the derivatives of $\hat{\boldsymbol{\xi}}(\mathbf{t})$. Using (2.26), it follows that locally around $\mathbf{t}^0$ the function $\hat{\boldsymbol{\xi}}(\mathbf{t})$ is C^1 and we can calculate

$$\nabla_{t_j} \hat{\xi}_i = 2\sum_{l=1}^{2} G_D(t_i, t_l)\hat{\xi}_l \nabla_{t_j} \hat{\xi}_l + \sum_{l=1}^{2} \left(\frac{\partial}{\partial t_j} G_D(t_i, t_l)\right)\hat{\xi}_l^2.$$

For $i \neq j$, we have

$$\nabla_{t_j} \hat{\xi}_i = 2\sum_{l=1}^{2} G_D(t_i, t_l)\hat{\xi}_l \nabla_{t_j} \hat{\xi}_l + \nabla_{t_j} G_D(t_i, t_j)\hat{\xi}_j^2,$$

where

$$\nabla_{t_j} G_D(t_i, t_j) = \frac{\partial}{\partial t_j} G_D(t_i, t_j).$$

For $i = j$, we get

$$\nabla_{t_i} \hat{\xi}_i = 2\sum_{l=1}^{2} G_D(t_i, t_l)\hat{\xi}_l \nabla_{t_i} \hat{\xi}_l + \sum_{l=1}^{2} \frac{\partial}{\partial t_i}\big(G_D(t_i, t_l)\big)\hat{\xi}_l^2$$
$$= 2\sum_{l=1}^{2} G_D(t_i, t_l)\hat{\xi}_l \nabla_{t_i} \hat{\xi}_l + \nabla_{t_i} G_D(t_i, t_i)\hat{\xi}_i^2 + \sum_{l=1}^{2} \nabla_{t_i} G_D(t_i, t_l)\hat{\xi}_l^2.$$

Note that by (H3) we have

$$\sum_{l=1}^{2} \nabla_{t_i} G_D(t_i, t_l)\hat{\xi}_l^2 = 0.$$

Then, recalling that

$$\big(\nabla_{t_j} G_D(t_i, t_j)\big) = \big(\nabla \mathcal{G}_D(\mathbf{t})\big)^T,$$

setting

$$\mathcal{H}(\mathbf{t}) = \big(\hat{\xi}_i(\mathbf{t})\delta_{ij}\big) \tag{2.31}$$

and using matrix notation

$$\nabla\xi(\mathbf{t}) = \big(\nabla_{t_j}\hat{\xi}_i(\mathbf{t})\big), \tag{2.32}$$

we have

$$\nabla\xi(\mathbf{t}) = \big(I - 2\mathcal{G}_D(\mathbf{t})\mathcal{H}(\mathbf{t})\big)^{-1}\big(\nabla\mathcal{G}_D(\mathbf{t})\big)^T\big(\mathcal{H}(\mathbf{t})\big)^2. \tag{2.33}$$

Let

$$\mathcal{Q} = (q_{ij}) = \left(\left(-\frac{\theta^2}{\hat{\xi}_i} + \frac{\theta^3}{2}\right)\delta_{ij}\right). \tag{2.34}$$

Using (2.33), we now compute $\mathcal{M}(\mathbf{t})$. First note that for $i \neq j$ we have

$$\left(\sum_{l=1}^{2} \frac{\partial}{\partial t_j}\nabla_{t_i} G_D(t_i, t_l)\right)\hat{\xi}_l^2 = \big(\nabla_{t_i}\nabla_{t_j} G_D(t_i, t_j)\big)\hat{\xi}_j^2$$

and for $i = j$ we get

$$\begin{aligned}
&\left(\sum_{l=1}^{2} \frac{\partial}{\partial t_i}\nabla_{t_i} G_D(t_i, t_l)\right)\hat{\xi}_l^2 \\
&= \left(\frac{\partial^2}{\partial t_i^2} G_D(t_i, t_{3-i})\right)\hat{\xi}_{3-i}^2 - \frac{\partial^2}{\partial x^2}\bigg|_{x=t_i} H_D(x, t_i)\hat{\xi}_i^2 + \big(\nabla_{t_i}\nabla_{t_i} G_D(t_i, t_i)\big)\hat{\xi}_i^2 \\
&= \theta^2 \sum_{l=1}^{2} G_D(t_i, t_l)\hat{\xi}_l^2 - \theta^2 K_D(0)\hat{\xi}_i^2 + \nabla_{t_i}\nabla_{t_i} G_D(t_i, t_i)\hat{\xi}_i^2 \\
&= \theta^2\hat{\xi}_i - \frac{\theta^3}{2}\hat{\xi}_i^2 + \nabla_{t_i}\nabla_{t_i} G_D(t_i, t_i)\hat{\xi}_i^2
\end{aligned}$$

since

$$\frac{\partial^2}{\partial t_i^2} G_D(t_i, t_{3-i}) - G_D(t_i, t_{3-i}) = 0, \qquad \frac{\partial^2}{\partial x^2}\bigg|_{x=t_i} H_D(t_i, t_i) - H_D(t_i, t_i) = 0.$$

In vector notation we get

$$\begin{aligned}\mathcal{M}(\mathbf{t}) &= \big(\mathcal{H}(\mathbf{t})\big)^{-1}\big(\nabla^2\mathcal{G}_D(\mathbf{t}) - \mathcal{Q}\big)\big(\mathcal{H}(\mathbf{t})\big)^2 \\ &\quad + 2\big(\mathcal{H}(\mathbf{t})\big)^{-1}\nabla\mathcal{G}_D(\mathbf{t})\mathcal{H}(\mathbf{t})\big(I - 2\mathcal{G}_D(\mathbf{t})\mathcal{H}(\mathbf{t})\big)^{-1}\big(\nabla\mathcal{G}_D(\mathbf{t})\big)^T\big(\mathcal{H}(\mathbf{t})\big)^2 \\ &= \big(\mathcal{H}(\mathbf{t})\big)^{-1}\big[\nabla^2\mathcal{G}_D(\mathbf{t}) - \mathcal{Q} + 2\nabla\mathcal{G}_D(\mathbf{t})\mathcal{H}(\mathbf{t})\big(I - 2\mathcal{G}_D(\mathbf{t})\mathcal{H}(\mathbf{t})\big)^{-1}\big(\nabla\mathcal{G}_D(\mathbf{t})\big)^T\big] \\ &\quad \times \big(\mathcal{H}(\mathbf{t})\big)^2. \end{aligned} \tag{2.35}$$

Our first result can be stated as follows:

Theorem 2.2 *Assume that* (H1), (H2) *and* (H3) *are satisfied. Then for* $\epsilon \ll 1$, *problem* (2.2) *has an* N*-spike solution which satisfies in the limit* $\epsilon \to 0$:

$$A_\epsilon(x) = \sum_{j=1}^{N} \xi_\epsilon \hat{\xi}_j^0 w\left(\frac{x - t_j^\epsilon}{\epsilon}\right) + o(1) \quad \textit{in } H_N^2(-1,1), \tag{2.36}$$

$$H_\epsilon\left(t_i^\epsilon\right) \sim \xi_\epsilon \hat{\xi}_i^0 + o(1), \quad i = 1, \ldots, N, \tag{2.37}$$

$$t_i^\epsilon \to t_i^0, \quad i = 1, \ldots, N. \tag{2.38}$$

Theorem 2.2 for the case $N = 2$ will be proved in the following subsections.

Remark 2.3 In the case of symmetric N-spike solutions, conditions (H2) and (H3) are not needed for the existence proof since in the construction of solutions one can restrict the function space to the class of symmetric functions (see for example [226]). Note that then for all ϵ small enough the spikes are exact copies of each other and thus they are placed equidistantly. For use in the stability proof the three assumptions (H1), (H2) and (H3) will be computed in Sect. 4.1.3 in the case of symmetric spikes.

Remark 2.4 Our results will provide a *rigorous proof* for the existence and stability of asymmetric N-spike solutions which consist of spikes with different amplitudes after the three assumptions (H1), (H2) and (H3) have been verified.

2.3.1 Some Preliminaries

In this subsection, we consider the following vectorial linear operator:

$$L\Phi := \Phi'' - \Phi + 2w\Phi - 2\mathcal{B}\frac{\int_{\mathbb{R}} w\Phi\, dy}{\int_{\mathbb{R}} w^2 dy} w^2, \tag{2.39}$$

where $\mathcal{B}$ is given by (2.21) and

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{pmatrix} \in \left(H^2(\mathbb{R})\right)^N.$$

In this subsection, we consider a general positive integer N for later use. Set

$$L_0 u := u'' - u + 2wu, \quad \text{where } u \in H^2(\mathbb{R}). \tag{2.40}$$

Then, using Remark 2.1, the conjugate operator of L under the scalar product in $L^2(\mathbb{R})$ is given by

$$L^*\Psi = \Psi'' - \Psi + 2w\Psi - 2\mathcal{B}^T \frac{\int_{\mathbb{R}} w^2 \Psi \, dy}{\int_{\mathbb{R}} w^2 \, dy} w, \tag{2.41}$$

where

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix} \in \left(H^2(\mathbb{R})\right)^N.$$

We obtain the following result.

Lemma 2.5 *Assume that* (H2) *holds. Then*

$$\operatorname{Ker}(L) = X_0 \oplus X_0 \oplus \cdots \oplus X_0, \tag{2.42}$$

where

$$X_0 = \operatorname{span}\left\{w'(y)\right\}$$

and

$$\operatorname{Ker}\left(L^*\right) = X_0 \oplus X_0 \oplus \cdots \oplus X_0. \tag{2.43}$$

Proof Let us first prove (2.42). Suppose that

$$L\Phi = 0.$$

We diagonalise $\mathcal{B}$ so that

$$P^{-1}\mathcal{B}P = J,$$

where P is an orthogonal matrix. Note that by Remark 2.1 J has diagonal form, i.e.,

$$J = \begin{pmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ & & \ddots & \\ 0 & & & \sigma_N \end{pmatrix}$$

with suitable real numbers σ_j for $j = 1, \ldots, N$. Defining

$$\Phi = P\tilde{\Phi},$$

we have

$$\tilde{\Phi}'' - \tilde{\Phi} + 2w\tilde{\Phi} - 2\frac{\int_{\mathbb{R}} w(J\tilde{\Phi})dy}{\int_{\mathbb{R}} w^2 dy} w^2 = 0. \tag{2.44}$$

For $l = 1, \ldots, N$ we consider the l-th equation of system (2.44):

$$\tilde{\phi}_l'' - \tilde{\phi}_l + 2w\tilde{\phi}_l - 2\sigma_l \frac{\int_{\mathbb{R}} w\tilde{\phi}_l dy}{\int_{\mathbb{R}} w^2 dy} w^2 = 0. \tag{2.45}$$

By Theorem 3.1(3) below, the last equation (2.45) implies that

$$\tilde{\phi}_l \in X_0, \quad l = 1, \ldots, N. \tag{2.46}$$

Here we have used condition (H2) which gives $2\sigma_l \neq 1$. Thus (2.42) is proved.

To prove (2.43), we proceed in the same way for L^*. Using $\sigma(\mathcal{B}) = \sigma(\mathcal{B}^T)$, the l-th equation of the diagonalised system is given by

$$\tilde{\psi}_l'' - \tilde{\psi}_l + 2w\tilde{\psi}_l - 2\sigma_l \frac{\int_{\mathbb{R}} w^p \tilde{\psi}_l dy}{\int_{\mathbb{R}} w^r dy} w = 0, \quad l = 1, \ldots, N. \tag{2.47}$$

Multiplying (2.47) by w and integrating over the real line, we obtain

$$(1 - 2\sigma_l)\int_{\mathbb{R}} w^2 \tilde{\psi}_l dy = 0,$$

which implies that

$$\int_{\mathbb{R}} w^2 \tilde{\psi}_l dy = 0,$$

since by (H2) we know that $2\sigma_l \neq 1$. Thus all the nonlocal terms vanish and we have

$$L_0 \tilde{\psi}_l = 0, \quad l = 1, \ldots, N. \tag{2.48}$$

This implies that $\tilde{\psi}_l \in X_0$ for $l = 1, \ldots, N$. Now (2.43) follows. □

As a consequence of Lemma 2.5, we have

Lemma 2.6 *The operator*

$$L : \left(H^2(\mathbb{R})\right)^N \to \left(L^2(\mathbb{R})\right)^N$$

is invertible if it is restricted as follows

$$L : (X_0 \oplus \cdots \oplus X_0)^\perp \cap \left(H^2(\mathbb{R})\right)^N \to (X_0 \oplus \cdots \oplus X_0)^\perp \cap \left(L^2(\mathbb{R})\right)^N.$$

Moreover, L^{-1} is bounded.

Proof This follows from the Fredholm Alternative (see Theorem 13.2) and Lemma 2.5. □

2.3.2 Study of the Approximate Solutions

Let $-1 < t_1^0 < t_2^0 < 1$ be two points satisfying the assumptions (H1)–(H3) and again we will use the notation $\mathbf{t}^0 = (t_1^0, t_2^0)$. Let $\hat{\xi}^0 = (\hat{\xi}_1^0, \hat{\xi}_2^0)$ be the unique solution of (2.24).

We first construct an approximate two-spike solution to (2.2) which concentrates near these prescribed two points.

Let $-1 < t_1 < t_2 < 1$ be such that $\mathbf{t} = (t_1, t_2) \in B_{c_0\epsilon}(\mathbf{t}^0)$, where the constant c_0 will be chosen below. Set

$$w_j(x) = w\left(\frac{x - t_j}{\epsilon}\right) \tag{2.49}$$

and

$$r_0 = \frac{1}{10}\left(\min\left(t_1^0 + 1, 1 - t_2^0, \frac{1}{2}|t_2^0 - t_1^0|\right)\right). \tag{2.50}$$

Let $\chi : \mathbb{R} \to [0, 1]$ be a smooth cut-off function such that

$$\chi(x) = 1 \quad \text{for } |x| < 1 \quad \text{and} \quad \chi(x) = 0 \quad \text{for } |x| > 2. \tag{2.51}$$

We now define our approximate solution. Firstly, we set

$$\tilde{w}_j(x) = w_j(x)\chi\left(\frac{x - t_j}{r_0}\right). \tag{2.52}$$

Then it is easy to see that $\tilde{w}_j(x) \in H_N^2(-1, 1)$ satisfies

$$\epsilon^2 \tilde{w}_j'' - \tilde{w}_j + \tilde{w}_j^2 = \text{e.s.t.} \tag{2.53}$$

in $L^2(-1, 1)$ where e.s.t. denotes an exponentially small term.

Secondly, we let $\hat{\boldsymbol{\xi}}(\mathbf{t}) = (\hat{\xi}_1, \hat{\xi}_2)$ be the unique solution of (2.26) and put

$$w_{\epsilon,\mathbf{t}}(x) = \sum_{j=1}^{2} \hat{\xi}_j \tilde{w}_j(x). \tag{2.54}$$

For any function $A \in H^2(-1, 1)$ we define $T[A]$ to be the unique solution of the linear problem

$$\begin{cases} DT[A]'' - T[A] + \xi_\epsilon A^2 = 0, & -1 < x < 1, \\ T[A]'(-1) = T[A]'(1) = 0, \end{cases} \tag{2.55}$$

where ξ_ϵ was defined in (2.18). Then the solution $T[A]$ is unique and positive.

For $A = w_{\epsilon,\mathbf{t}}$, where $\mathbf{t} \in B_{c_0\epsilon}(\mathbf{t}^0)$, we now compute

$$\tau_i := T[A](t_i). \tag{2.56}$$

From (2.55), we have

$$\begin{aligned} \tau_i &= \xi_\epsilon \int_{-1}^{1} G_D(t_i, z) A^2(z) dz \\ &= \xi_\epsilon \epsilon \sum_{j=1}^{2} \hat{\xi}_j^2 \int_{-1}^{1} G_D(t_i, z) \tilde{w}_j^2(z) dz \big(1 + O(\epsilon)\big) \\ &= \xi_\epsilon \epsilon \sum_{j=1}^{2} \hat{\xi}_j^2 \left[G_D(t_i, t_j) \int_{-\infty}^{+\infty} w_j^2(y) dy + O(\epsilon) \right] = \sum_{j=1}^{2} G_D(t_i, t_j) \hat{\xi}_j^2 + O(\epsilon), \end{aligned}$$

where we have used (2.18). Thus we have derived the following system of algebraic equations:

$$\tau_i = \sum_{j=1}^{2} G_D(t_i, t_j) \hat{\xi}_j^2 + O(\epsilon). \tag{2.57}$$

By the implicit function theorem and assumptions (H1), (H2), the system (2.57) has a unique solution

$$\tau_i = \hat{\xi}_i + O(\epsilon), \quad i = 1, 2.$$

Hence

$$T[A](t_i) = \hat{\xi}_i + O(\epsilon). \tag{2.58}$$

Next for $x = t_i + \epsilon y$ and $A = w_{\epsilon,\mathbf{t}}$ we calculate

$$\begin{aligned} & T[A](x) - T[A](t_i) \\ &= \xi_\epsilon \int_{-1}^{1} \big[G_D(x, z) - G_D(t_i, z) \big] A^2(z) dz \end{aligned}$$

$$
\begin{aligned}
&= \xi_\epsilon \hat{\xi}_i^2 \int_{-1}^{1} \big[G_D(x,z) - G_D(t_i,z)\big] \tilde{w}_i^2(z)\,dz \\
&\quad + \xi_\epsilon \hat{\xi}_{3-i}^2 \int_{-1}^{1} \big[G_D(x,z) - G_D(t_i,z)\big] \tilde{w}_{3-i}^2(z)\,dz \\
&= \xi_\epsilon \hat{\xi}_i^2 \int_{-1}^{1} \big[K_D\big(|x-z|\big) - K_D\big(|t_i - z|\big)\big] \tilde{w}_i^2(z)\,dz \\
&\quad - \xi_\epsilon \hat{\xi}_i^2 \int_{-1}^{1} \big[H_D(x,z) - H_D(t_i,z)\big] \tilde{w}_i^2(z)\,dz \\
&\quad + \xi_\epsilon \hat{\xi}_{3-i}^2 \int_{-1}^{1} \big[G_D(x,z) - G_D(t_i,z)\big] \tilde{w}_{3-i}^2(z)\,dz \\
&= \epsilon^2 \xi_\epsilon \hat{\xi}_i^2 \int_{-\infty}^{+\infty} \left[\frac{1}{2D}|z| - \frac{1}{2D}|y-z|\right] w^2\big(|z|\big)\,dz\big(1 + O\big(\epsilon|y|\big)\big) \\
&\quad + \epsilon \hat{\xi}_i^2 \big[-y \nabla_{t_i} H_D(t_i,t_i) + O\big(\epsilon y^2\big)\big] \\
&\quad + \epsilon \big[y \nabla_{t_i} G_D(t_i, t_{3-i}) \hat{\xi}_{3-i}^2 + O\big(\epsilon y^2\big)\big] \\
&= \epsilon \big[\hat{\xi}_i^2 P^i\big(|y|\big) - \hat{\xi}_i^2 y \nabla_{t_i} H_D(t_i,t_i) \\
&\quad + y \nabla_{t_i} G_D(t_i, t_{3-i}) \hat{\xi}_{3-i}^2 + O\big(\epsilon y^2\big)\big],
\end{aligned} \tag{2.59}
$$

where

$$
\begin{aligned}
P^i\big(|y|\big) = \left(\int_{-\infty}^{+\infty} w^2\right)^{-1} \\
\times \int_{-\infty}^{+\infty} \left[\frac{1}{2D}|z| - \frac{1}{2D}|y-z|\right] w^2\big(|z|\big)\,dz.
\end{aligned} \tag{2.60}
$$

Note that P^i is an even function.

Let us now define the rescaled domain $\Omega_\epsilon = (-\frac{1}{\epsilon}, \frac{1}{\epsilon})$ and the operator

$$
S : H_N^2(\Omega_\epsilon) \to L^2(\Omega_\epsilon), \quad S[A] := A'' - A + \frac{A^2}{(T[A])}, \tag{2.61}
$$

where $T[A]$ has been introduced in (2.55). Then, choosing $A = w_{\epsilon,\mathbf{t}}$, we compute $S[w_{\epsilon,\mathbf{t}}]$ as follows:

$$
\begin{aligned}
S[w_{\epsilon,\mathbf{t}}] &= w_{\epsilon,\mathbf{t}}'' - w_{\epsilon,\mathbf{t}} + \frac{w_{\epsilon,\mathbf{t}}^2}{T[w_{\epsilon,\mathbf{t}}]} \\
&= \sum_{j=1}^{2} \hat{\xi}_j \big(\tilde{w}_j'' - \tilde{w}_j\big) + \frac{w_{\epsilon,\mathbf{t}}^2}{T[w_{\epsilon,\mathbf{t}}]} + \text{e.s.t.}
\end{aligned}
$$

$$= \left[\frac{(\sum_{j=1}^{2} \hat{\xi}_j \tilde{w}_j)^2}{T[w_{\epsilon,\mathbf{t}}]} - \sum_{j=1}^{2} \hat{\xi}_j \tilde{w}_j^2 \right] + \text{e.s.t.}$$

$$= E_1 + E_2 + \text{e.s.t.} \quad \text{in } L^2\left(-\frac{1}{\epsilon}, \frac{1}{\epsilon}\right), \tag{2.62}$$

where

$$E_1 = \left[\frac{(\sum_{j=1}^{2} \hat{\xi}_j \tilde{w}_j)^2}{T[w_{\epsilon,\mathbf{t}}](t_j)} - \sum_{j=1}^{2} \hat{\xi} \tilde{w}_j^2 \right] \tag{2.63}$$

and

$$E_2 = \left[\frac{(\sum_{j=1}^{2} \hat{\xi}_j \tilde{w}_j)^2}{T[w_{\epsilon,\mathbf{t}}](x)} - \frac{(\sum_{j=1}^{2} \hat{\xi}_j \tilde{w}_j)^2}{T[w_{\epsilon,\mathbf{t}}](t_j)} \right]. \tag{2.64}$$

For E_1, we calculate using (2.58)

$$E_1 = \frac{(\sum_{j=1}^{2} \hat{\xi}_j \tilde{w}_j)^2}{T[w_{\epsilon,\mathbf{t}}](t_j)} - \sum_{j=1}^{2} \hat{\xi}_j \tilde{w}_j^2$$

$$= \sum_{j=1}^{2} \left(\frac{\hat{\xi}_j^2}{\hat{\xi}_j + O(\epsilon)} - \hat{\xi}_j \right) \tilde{w}_j^2 = O(\epsilon) \sum_{j=1}^{2} \hat{\xi}_j \tilde{w}_j^2. \tag{2.65}$$

Thus we have

$$\|E_1\|_{L^2(-1/\epsilon, 1/\epsilon)} = O(\epsilon). \tag{2.66}$$

For E_2, we calculate

$$E_2 = -\sum_{j=1}^{2} \frac{(\hat{\xi}_j \tilde{w}_j)^2}{(T[w_{\epsilon,\mathbf{t}}](t_j))^2} \big(T[w_{\epsilon,\mathbf{t}}](x) - T[w_{\epsilon,\mathbf{t}}](t_j) \big)$$

$$+ O\left(\sum_{j=1}^{2} \big| T[w_{\epsilon,\mathbf{t}}](x) - T[w_{\epsilon,\mathbf{t}}](t_j) \big|^2 \tilde{w}_j^2 \right)$$

$$= -\sum_{j=1}^{2} \hat{\xi}_j \tilde{w}_j^2 \frac{T[w_{\epsilon,\mathbf{t}}](x) - T[w_{\epsilon,\mathbf{t}}](t_j)}{T[w_{\epsilon,\mathbf{t}}](t_j)} + O\left(\epsilon^2 y^2 \sum_{j=1}^{2} \tilde{w}_j^2 \right)$$

$$= -\epsilon \sum_{j=1}^{2} \tilde{w}_j^2 \left\{ \hat{\xi}_j^2 P^j\big(|y|\big) - \hat{\xi}_j^2 y \nabla_{t_j} H_D(t_j, t_j) + y \nabla_{t_j} G_D(t_j, t_{3-j}) \hat{\xi}_l^2 \right\}$$

$$+ O\left(\epsilon^2 y^2 \sum_{j=1}^{2} \tilde{w}_j^2 \right). \tag{2.67}$$

This implies that

$$\|E_2\|_{L^2(-1/\epsilon,1/\epsilon)} = O(\epsilon). \tag{2.68}$$

Combining (2.66) and (2.68), we conclude that

$$\left\|S[w_{\epsilon,\mathbf{t}}]\right\|_{L^2(-1/\epsilon,1/\epsilon)} = O(\epsilon). \tag{2.69}$$

The estimates in this subsection show that the approximate solution solves the system up to a small error.

2.3.3 The Liapunov-Schmidt Reduction Method

In this subsection, we use the Liapunov-Schmidt reduction method to solve the problem

$$S[w_{\epsilon,\mathbf{t}} + v] = \sum_{j=1}^{2} \beta_j \frac{d\tilde{w}_j}{dx} \tag{2.70}$$

for real constants β_j and a function $v \in H_N^2(-\frac{1}{\epsilon}, \frac{1}{\epsilon})$ which is small in the H^2-norm, where $\tilde{w}_j$ and $w_{\epsilon,\mathbf{t}}$ are given by (2.52) and (2.54), respectively.

To this end, we need to study the linearised operator

$$\tilde{L}_{\epsilon,\mathbf{t}} : H^2(\Omega_\epsilon) \to L^2(\Omega_\epsilon)$$

defined by

$$\tilde{L}_{\epsilon,\mathbf{t}} := S_\epsilon'[A]\phi = \phi'' - \phi + \frac{2A\phi}{T[A]} - \frac{A^2}{(T[A])^2}\big(T'[A]\phi\big),$$

where $A = w_{\epsilon,\mathbf{t}}$, and, for given $\phi \in L^2(\Omega)$, we denote by $T'[A]\phi$ the unique solution of the linear problem

$$\begin{cases} D(T'[A]\phi)'' - (T'[A]\phi) + 2\xi_\epsilon A\phi = 0, & -1 < x < 1, \\ (T'[A]\phi)'(-1) = (T'[A]\phi)'(1) = 0. \end{cases} \tag{2.71}$$

We define the approximate kernel and cokernel, respectively, as follows:

$$\mathcal{K}_{\epsilon,\mathbf{t}} := \operatorname{span}\left\{\frac{d\tilde{w}_i}{dx} : i = 1, 2\right\} \subset H_N^2(\Omega_\epsilon),$$

$$\mathcal{C}_{\epsilon,\mathbf{t}} := \operatorname{span}\left\{\frac{d\tilde{w}_i}{dx} : i = 1, 2\right\} \subset L^2(\Omega_\epsilon).$$

Recall the definition of the following vectorial linear operator introduced in (2.39):

$$L\Phi := \Phi'' - \Phi + 2w\Phi - 2\mathcal{B}\frac{\int_{\mathbb{R}} w\Phi}{\int_{\mathbb{R}} w^2} w^2,$$

where

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in \left(H^2(\mathbb{R})\right)^2.$$

By Lemma 2.5, we know that

$$L : (X_0 \oplus X_0)^{\perp} \cap \left(H^2(\mathbb{R})\right)^2 \to (X_0 \oplus X_0)^{\perp} \cap \left(L^2(\mathbb{R})\right)^2$$

is invertible with a bounded inverse.

We will see that this system is the limit of the linear operator $\tilde{L}_{\epsilon,\mathbf{t}}$ as $\epsilon \to 0$. To this end, we introduce the projection $\pi^{\perp}_{\epsilon,\mathbf{t}} : L^2(\Omega_\epsilon) \to \mathcal{C}^{\perp}_{\epsilon,\mathbf{t}}$ and study the linear operator $L_{\epsilon,\mathbf{t}} := \pi^{\perp}_{\epsilon,\mathbf{t}} \circ \tilde{L}_{\epsilon,\mathbf{t}}$. By letting $\epsilon \to 0$, we will show that $L_{\epsilon,\mathbf{t}} : \mathcal{K}^{\perp}_{\epsilon,\mathbf{t}} \to \mathcal{C}^{\perp}_{\epsilon,\mathbf{t}}$ is invertible with a bounded inverse provided ϵ is small enough. This statement is contained in the following proposition.

Proposition 2.7 *There exist positive constants* $\bar{\epsilon}$, $\bar{\delta}$, λ *such that for all* $\epsilon \in (0, \bar{\epsilon})$ *and all* $\mathbf{t} \in \Omega^2$ *satisfying* $\min(|1 + t_1|, |1 - t_2|, \frac{1}{2}|t_1 - t_2|) > \bar{\delta}$ *we have*

$$\|L_{\epsilon,\mathbf{t}}\phi\|_{L^2(\Omega_\epsilon)} \geq \lambda \|\phi\|_{H^2(\Omega_\epsilon)} \quad \textit{for all } \phi \in \mathcal{K}^{\perp}_{\epsilon,\mathbf{t}}. \tag{2.72}$$

Further, the linear operator

$$L_{\epsilon,\mathbf{t}} = \pi^{\perp}_{\epsilon,\mathbf{t}} \circ \tilde{L}_{\epsilon,\mathbf{t}} : \mathcal{K}^{\perp}_{\epsilon,\mathbf{t}} \to \mathcal{C}^{\perp}_{\epsilon,\mathbf{t}}$$

is surjective.

Proof This proof follows the method of Liapunov-Schmidt reduction.

Suppose (2.72) is false. Then there are sequences $\{\epsilon_k\}$, $\{\mathbf{t}^k\}$, $\{\phi^k\}$ such that $\epsilon_k \to 0$, $\mathbf{t}^k \in \Omega^2$ with $\min(|1 + t_1^k|, |1 - t_2^k|, \frac{1}{2}|t_1^k - t_2^k|) > \bar{\delta}$ and $\phi^k = \phi_{\epsilon_k} \in \mathcal{K}^{\perp}_{\epsilon_k,\mathbf{t}^k}$, $k = 1, 2, \ldots$ such that

$$\left\| L_{\epsilon_k,\mathbf{t}^k}\phi^k \right\|_{L^2(\Omega_{\epsilon_k})} \to 0 \quad \text{as } k \to \infty, \tag{2.73}$$

$$\left\| \phi^k \right\|_{H^2(\Omega_{\epsilon_k})} = 1, \quad k = 1, 2, \ldots. \tag{2.74}$$

By using the cut-off function introduced in (2.51), we define $\phi_{\epsilon,i}$, $i = 1, 2, 3$ for $\epsilon > 0$ small enough as follows:

$$\phi_{\epsilon,i}(x) = \phi_\epsilon(x)\chi\left(\frac{x - t_i}{r_0}\right), \quad x \in \Omega, i = 1, 2,$$
$$\phi_{\epsilon,3}(x) = \phi_\epsilon(x) - \sum_{i=1}^{2} \phi_{\epsilon,i}(x), \quad x \in \Omega. \tag{2.75}$$

After rescaling, first the functions $\phi_{\epsilon,i}$ are defined only on Ω_ϵ. Then, by a standard result (see Sect. 7.12 in [74]), they can be extended to $\mathbb{R}$ so that for ϵ small enough their norms in $H^2(\mathbb{R})$ are bounded by a constant independent of ϵ and $\mathbf{t}$. In the following we will study this extension, where for simplicity we use the same notation for the original functions and its extension. Since for $i = 1, 2$ both sequences $\{\phi_i^k\} := \{\phi_{\epsilon_k,i}\}$ $(k = 1, 2, \ldots)$ are bounded in $H^2_{\mathrm{loc}}(\mathbb{R})$, they have weak limits in $H^2_{\mathrm{loc}}(\mathbb{R})$ and thus also strong limits in $L^2_{\mathrm{loc}}(\mathbb{R})$ and $L^\infty_{\mathrm{loc}}(\mathbb{R})$. We denote these limits by ϕ_i. Further, by a barrier argument, these functions have uniform exponential decay, and the limits are also strong in the $H^2(\mathbb{R})$ and $L^\infty(\mathbb{R})$ sense. Thus $\phi = (\phi_1, \phi_2)^T$ solves the system $L\phi = 0$ with the operator L introduced in (2.39). By Lemma 2.5, we know that $\phi \in \mathrm{Ker}(L) = X_0 \oplus X_0$. Since $\phi^k \in \mathcal{K}^\perp_{\epsilon_k,\mathbf{t}^k}$, by taking the limit $k \to \infty$ we get $\phi \in \mathrm{Ker}(L)^\perp$ and so $\phi = 0$.

By elliptic estimates, we derive $\|\phi_i^k\|_{H^2(\mathbb{R})} \to 0$ as $k \to \infty$ for $i = 1, 2$ and $\phi_3^k \to \phi_3$ in $H^2(\mathbb{R})$, where ϕ_3 satisfies

$$\Delta\phi_3 - \phi_3 = 0 \quad \text{in } \mathbb{R}.$$

Therefore we conclude $\phi_3 = 0$ and $\|\phi_3^k\|_{H^2(\mathbb{R})} \to 0$ as $k \to \infty$. This contradicts $\|\phi^k\|_{H^2(\Omega_{\epsilon_k})} = 1$.

To complete the proof of Proposition 2.7, we just need to show that the conjugate operator of $L_{\epsilon,\mathbf{t}}$ (denoted by $L^*_{\epsilon,\mathbf{t}}$) is injective from $\mathcal{K}^\perp_{\epsilon,\mathbf{t}}$ to $\mathcal{C}^\perp_{\epsilon,\mathbf{t}}$. Note that $L^*_{\epsilon,\mathbf{t}}\psi = \pi_{\epsilon,\mathbf{t}} \circ \tilde{L}^*_{\epsilon,\mathbf{t}}$ with

$$\tilde{L}^*_{\epsilon,\mathbf{t}}\psi = \epsilon^2\Delta\psi - \psi + \frac{2A\psi}{(T[A])} - T'[A]\frac{A^2\psi}{(T[A])^2}.$$

The proof for $L^*_{\epsilon,\mathbf{t}}$ follows exactly along the same lines as the proof for $L_{\epsilon,\mathbf{t}}$ and is therefore omitted. □

Now we have derived all the technical tools needed to solve the equation

$$\pi^\perp_{\epsilon,\mathbf{t}} \circ S_\epsilon(w_{\epsilon,\mathbf{t}} + \phi) = 0. \tag{2.76}$$

Since the restriction of the linear operator $L_{\epsilon,\mathbf{t}}$ to $\mathcal{K}^\perp_{\epsilon,\mathbf{t}}$ is invertible we can write (2.76) in equivalent form as

$$\phi = -\left(L^{-1}_{\epsilon,\mathbf{t}} \circ \pi^\perp_{\epsilon,\mathbf{t}} \circ S_\epsilon(w_{\epsilon,\mathbf{t}})\right) - \left(L^{-1}_{\epsilon,\mathbf{t}} \circ \pi^\perp_{\epsilon,\mathbf{t}} \circ N_{\epsilon,\mathbf{t}}(\phi)\right) =: M_{\epsilon,\mathbf{t}}(\phi), \tag{2.77}$$

where $L_{\epsilon,\mathbf{t}}^{-1}$ is the inverse of $L_{\epsilon,\mathbf{t}}$, and the nonlinear operators

$$N_{\epsilon,\mathbf{t}}(\phi) = S_\epsilon(w_{\epsilon,\mathbf{t}} + \phi) - S_\epsilon(w_{\epsilon,\mathbf{t}}) - S'_\epsilon(w_{\epsilon,\mathbf{t}})\phi \tag{2.78}$$

and $M_{\epsilon,\mathbf{t}}(\phi)$ introduced in (2.77) are both defined for $\phi \in H_N^2(\Omega_\epsilon)$.

Finally, we show that the operator $M_{\epsilon,\mathbf{t}}$ is a contraction on

$$B_{\epsilon,\delta} \equiv \left\{\phi \in H_N^2(\Omega_\epsilon) : \|\phi\|_{H^2(\Omega_\epsilon)} < \delta\right\}$$

if δ and ϵ are suitably chosen. First, from (2.69) and Proposition 2.7 we know that

$$\begin{aligned} \left\|M_{\epsilon,\mathbf{t}}(\phi)\right\|_{H^2(\Omega_\epsilon)} &\le \lambda^{-1}\left(\left\|\pi_{\epsilon,\mathbf{t}}^{\perp} \circ N_{\epsilon,\mathbf{t}}(\phi)\right\|_{L^2(\Omega_\epsilon)} + \left\|\pi_{\epsilon,\mathbf{t}}^{\perp} \circ S_\epsilon(w_{\epsilon,\mathbf{t}})\right\|_{L^2(\Omega_\epsilon)}\right) \\ &\le \lambda^{-1} C_0\big(c(\delta)\delta + \epsilon\big), \end{aligned}$$

where $\lambda > 0$ is independent of $\delta > 0$, $\epsilon > 0$ and $c(\delta) \to 0$ as $\delta \to 0$. Similarly, we have

$$\left\|M_{\epsilon,\mathbf{t}}(\phi) - M_{\epsilon,\mathbf{t}}(\phi')\right\|_{H^2(\Omega_\epsilon)} \le \lambda^{-1} C_0\big(c(\delta)\delta\big)\left\|\phi - \phi'\right\|_{H^2(\Omega_\epsilon)},$$

where $c(\delta) \to 0$ as $\delta \to 0$. If we choose

$$\delta = C_1\epsilon, \quad \text{where } \lambda^{-1}C_0 < C_1 \text{ and } \epsilon \text{ small enough}, \tag{2.79}$$

then $M_{\epsilon,\mathbf{t}}$ maps from $B_{\epsilon,\delta}$ into $B_{\epsilon,\delta}$ and is a contraction mapping in $B_{\epsilon,\delta}$. Now the existence of a fixed point $\phi_{\epsilon,\mathbf{t}} \in B_{\epsilon,\delta}$ follows from the standard contraction mapping principle. Thus we have rigorously constructed a solution $\phi_{\epsilon,\mathbf{t}} \in H_N^2(\Omega_\epsilon)$ of (2.77).

We summarise our result as follows:

Lemma 2.8 *There exist $\overline{\epsilon} > 0$ and $\overline{\delta} > 0$ such that for every pair $\epsilon, \mathbf{t}$ with $0 < \epsilon < \overline{\epsilon}$ and $\mathbf{t} \in \Omega^2$, $1 + t_1 > \overline{\delta}$, $1 - t_2 > \overline{\delta}$, $\frac{1}{2}|t_2 - t_1| > \overline{\delta}$ there is a unique $\phi_{\epsilon,\mathbf{t}} \in \mathcal{K}_{\epsilon,\mathbf{t}}^{\perp}$ satisfying $S_\epsilon(w_{\epsilon,\mathbf{t}} + \phi_{\epsilon,\mathbf{t}}) \in \mathcal{C}_{\epsilon,\mathbf{t}}$. Further, we have the estimate*

$$\|\phi_{\epsilon,\mathbf{t}}\|_{H^2(\Omega_\epsilon)} \le C_1\epsilon, \tag{2.80}$$

where C_1 has been defined in (2.79).

2.3.4 The Reduced Problem

In this subsection, we solve the reduced problem and conclude the proof of our main existence result given in Theorem 2.2.

By Lemma 2.8, for every $\mathbf{t} \in B_{c_0\epsilon}(\mathbf{t}^0)$, there exists a unique solution $\phi_{\epsilon,\mathbf{t}} \in \mathcal{K}_{\epsilon,\mathbf{t}}^{\perp}$ such that

$$S[w_{\epsilon,\mathbf{t}} + \phi_{\epsilon,\mathbf{t}}] = v_{\epsilon,\mathbf{t}} \in \mathcal{C}_{\epsilon,\mathbf{t}}. \tag{2.81}$$

Now we are going to determine the position $\mathbf{t}^\epsilon = (t_1^\epsilon, t_2^\epsilon) \in B_{c_0\epsilon}(\mathbf{t}^0)$ such that also

$$S[w_{\epsilon,\mathbf{t}^\epsilon} + \phi_{\epsilon,\mathbf{t}^\epsilon}] \perp \mathcal{C}_{\epsilon,\mathbf{t}^\epsilon}. \tag{2.82}$$

Then, by combining (2.81) and (2.82), we have found $\mathbf{t}^\epsilon = (t_1^\epsilon, t_2^\epsilon) \in B_{c_0\epsilon}(\mathbf{t}^0)$ and $\phi_{\epsilon,\mathbf{t}^\epsilon} \in \mathcal{K}^\perp_{\epsilon,\mathbf{t}^\epsilon}$ such that $S[w_{\epsilon,\mathbf{t}^\epsilon} + \phi_{\epsilon,\mathbf{t}^\epsilon}] = 0$. This means that we have found a solution of (2.2) and Theorem 2.2 follows in the case of two spikes.

To this end, we introduce the vector field

$$W_{\epsilon,i}(\mathbf{t}) := \epsilon^{-1} \int_{-1}^{1} S[w_{\epsilon,\mathbf{t}} + \phi_{\epsilon,\mathbf{t}}] \frac{d\tilde{w}_i}{dx} dx,$$
$$W_\epsilon(\mathbf{t}) := \big(W_{\epsilon,1}(\mathbf{t}), W_{\epsilon,2}(\mathbf{t})\big) : B_{c_0\epsilon}\big(\mathbf{t}^0\big) \to \mathbb{R}^2.$$

Then $W_\epsilon(\mathbf{t})$ is a continuous map in $\mathbf{t}$ and our problem is reduced to finding a zero of the vector field $W_\epsilon(\mathbf{t})$.

Next we explicitly calculate $W_\epsilon(\mathbf{t})$:

$$\begin{aligned}
W_{\epsilon,i}(\mathbf{t}) &= \epsilon^{-1} \int_{-1}^{1} S[w_{\epsilon,\mathbf{t}} + \phi_{\epsilon,\mathbf{t}}] \frac{d\tilde{w}_i}{dx} dx \\
&= \epsilon^{-1} \int_{-1}^{1} S[w_{\epsilon,\mathbf{t}}] \frac{d\tilde{w}_i}{dx} dx + \epsilon^{-1} \int_{-1}^{1} S'_\epsilon[w_{\epsilon,\mathbf{t}}] \phi_{\epsilon,\mathbf{t}} \frac{d\tilde{w}_i}{dx} dx \\
&\quad + \epsilon^{-1} \int_{-1}^{1} N_\epsilon(\phi_{\epsilon,\mathbf{t}}) \frac{d\tilde{w}_i}{dx} dx \\
&= I_1 + I_2 + I_3,
\end{aligned}$$

where I_1, I_2 and I_3 are defined by the last equality.

The computation of I_3 is the easiest: Note that by Taylor expansion for (2.78), the first term in the expansion of N_ϵ is quadratic in $\phi_{\epsilon,\mathbf{t}}$ and so

$$I_3 = O(\epsilon). \tag{2.83}$$

We will now compute I_1 and I_2. The result will be that I_1 is the leading term and $I_2 = O(\epsilon)$.

For I_1, we have

$$\begin{aligned}
I_1 &= \epsilon^{-1} \int_{-1}^{1} (E_1 + E_2) \frac{d\tilde{w}_i}{dx} dx \\
&= \epsilon^{-1} \int_{-1}^{1} E_2 \frac{d\tilde{w}_i}{dx} dx + O(\epsilon),
\end{aligned}$$

where E_1 and E_2 were defined in (2.63) and (2.64), respectively. Here we have used that E_1 is an even function.

We calculate by (2.67)

$$\epsilon^{-1}\int_{-1}^{1} E_2 \frac{d\tilde{w}_i}{dx}dx$$
$$=\sum_{j=1}^{2}\nabla_{t_i} G_D(t_i,t_j)\hat{\xi}_j^2\int_{\mathbb{R}} yw^2(y)w'(y)dy+O(\epsilon)$$
$$=\sum_{j=1}^{2}\nabla_{t_i} G_D(t_i,t_j)\hat{\xi}_j^2\frac{1}{3}\int_{\mathbb{R}} w^3dy+O(\epsilon).$$

Thus we have

$$I_1=\sum_{j=1}^{2}\nabla_{t_i} G_D(t_i,t_j)\hat{\xi}_j^2\frac{1}{3}\int_{\mathbb{R}} w^3(y)dy+O(\epsilon). \tag{2.84}$$

For I_2, we calculate

$$\begin{aligned}
\epsilon I_2 &= \int_{-1}^{1} S'[w_{\epsilon,\mathbf{t}}](\phi_{\epsilon,\mathbf{t}})\frac{d\tilde{w}_i}{dx}dx\\
&= \int_{-1}^{1}\left[\epsilon^2\Delta\phi_{\epsilon,\mathbf{t}}-\phi_{\epsilon,\mathbf{t}}+\frac{2w_{\epsilon,\mathbf{t}}\phi_{\epsilon,\mathbf{t}}}{T[w_{\epsilon,\mathbf{t}}]}-\frac{w_{\epsilon,\mathbf{t}}^2}{(T[w_{\epsilon,\mathbf{t}}])^2}\big(T'[w_{\epsilon,\mathbf{t}}]\phi_{\epsilon,\mathbf{t}}\big)\right]\frac{d\tilde{w}_i}{dx}dx\\
&= \int_{-1}^{1}\left[\epsilon^2\Delta\frac{d\tilde{w}_i}{dx}-\frac{d\tilde{w}_i}{dx}+\frac{d\tilde{w}_i}{dx}\frac{2w_{\epsilon,\mathbf{t}}}{T[w_{\epsilon,\mathbf{t}}]}\right]\phi_{\epsilon,\mathbf{t}}dx\\
&\quad -\int_{-1}^{1}\frac{w_{\epsilon,\mathbf{t}}^2}{(T[w_{\epsilon,\mathbf{t}}])^2}\big(T'[w_{\epsilon,\mathbf{t}}]\phi_{\epsilon,\mathbf{t}}\big)\frac{d\tilde{w}_i}{dx}dx\\
&= \int_{-1}^{1}\left(2\frac{\hat{\xi}_i\tilde{w}_i}{T[w_{\epsilon,\mathbf{t}}]}-2\tilde{w}_i\right)\phi_{\epsilon,\mathbf{t}}\frac{d\tilde{w}_i}{dx}dx-\int_{-1}^{1}\frac{w_{\epsilon,\mathbf{t}}^2}{(T[w_{\epsilon,\mathbf{t}}])^2}\big(T'[w_{\epsilon,\mathbf{t}}]\phi_{\epsilon,\mathbf{t}}\big)\frac{d\tilde{w}_i}{dx}dx\\
&= O\big(\epsilon^2\big),
\end{aligned}$$

since

$$\begin{aligned}
\left\|\left(\frac{2\hat{\xi}_i\tilde{w}_i}{T[w_{\epsilon,\mathbf{t}}]}-2\tilde{w}_i\right)\phi_{\epsilon,\mathbf{t}}\right\|_{L^2(\Omega_\epsilon)} &= O(\epsilon),\\
\|\phi_{\epsilon,\mathbf{t}}\|_{H^2(\Omega_\epsilon)} &= O(\epsilon),\\
\big|T'[w_{\epsilon,\mathbf{t}}](\phi_{\epsilon,\mathbf{t}})(t_i)\big| &= O(\epsilon),\\
\big|T'[w_{\epsilon,\mathbf{t}}](\phi_{\epsilon,\mathbf{t}})(t_i+\epsilon y)-T'[w_{\epsilon,\mathbf{t}}](\phi_{\epsilon,\mathbf{t}})(t_i)\big| &= O\big(\epsilon^2|y|\big).
\end{aligned}$$

Combining I_1 and I_2, we have

$$W_{\epsilon,i}(\mathbf{t}) = \sum_{j=1}^{2} \nabla_{t_i} G_D(t_i, t_j)\hat{\xi}_j^2 \frac{1}{3}\int_{\mathbb{R}} w^3 dy + O(\epsilon)$$
$$= F_i(\mathbf{t})\frac{1}{3}\int_{\mathbb{R}} w^3 dy + O(\epsilon),$$

where $F_i(\mathbf{t})$ was defined in (2.27). By assumption (H3), we have $F(\mathbf{t}^0) = 0$ and

$$\det\big(\nabla_{\mathbf{t}^0} F\big(\mathbf{t}^0\big)\big) \neq 0.$$

This implies

$$\begin{aligned} W_\epsilon(\mathbf{t}) &= -c_1 \mathcal{H}\big(\mathbf{t}^0\big)\mathcal{M}\big(\mathbf{t}^0\big)\big(\mathbf{t}-\mathbf{t}^0\big) + O\big(\big|\mathbf{t}-\mathbf{t}^0\big|^2 + \epsilon\big), \\ c_1 &= -\frac{1}{3}\int_{\mathbb{R}} w^3 dy = -2.4. \end{aligned} \tag{2.85}$$

Then by standard degree theory (see Sect. 13.1) we conclude that for ϵ small enough there exists a $\mathbf{t}^\epsilon \in B_{c_0\epsilon}(\mathbf{t}^0)$ such that $W_\epsilon(\mathbf{t}^\epsilon) = 0$. Here it is important to note that, by choosing c_0 large enough (independent of ϵ), the leading term in (2.85) dominates the error terms for all $\mathbf{t} \in B_{c_0\epsilon}(\mathbf{t}^0)$ and ϵ small enough. We have thus proved the following proposition:

Proposition 2.9 *For ϵ small enough there exist points $\mathbf{t}^\epsilon$ with $\mathbf{t}^\epsilon \to \mathbf{t}^0$ such that $W_\epsilon(\mathbf{t}^\epsilon) = 0$.*

Finally, we conclude the proof of Theorem 2.2.

Proof of Theorem 2.2 By Proposition 2.9, there exist $\mathbf{t}^\epsilon \to \mathbf{t}^0$ such that $W_\epsilon(\mathbf{t}^\epsilon) = 0$. This implies $S[w_{\epsilon,\mathbf{t}^\epsilon} + \phi_{\epsilon,\mathbf{t}^\epsilon}] = 0$. Let $A_\epsilon = \xi_\epsilon(w_{\epsilon,\mathbf{t}^\epsilon} + \phi_{\epsilon,\mathbf{t}^\epsilon})$ and $H_\epsilon = \xi_\epsilon T[w_{\epsilon,\mathbf{t}^\epsilon} + \phi_{\epsilon,\mathbf{t}^\epsilon}]$. By the maximum principle, it follows that $A_\epsilon > 0$ and $H_\epsilon > 0$. Then (A_ϵ, H_ϵ) satisfies all the properties of Theorem 2.2. □

2.4 Clustered Multiple Spikes

We first study n-spike clusters on the real line for any positive integer n, i.e. there are n spikes which all approximate the same point on the real line. Secondly, taking multiple spike clusters and placing them near different points in a bounded interval, we will derive results on multiple clusters. This construction is analogous to that of multiple spikes in Sect. 2.3, where each spike is replaced by a cluster.

The result on an n-spike cluster on the real line can be stated as follows:

Theorem 2.10 (n-spike cluster on the real line) *Consider the stationary Gierer-Meinhardt system on the real line*:

$$\begin{cases} 0 = \epsilon^2 \Delta A - A + \frac{A^2}{H}, & x \in \mathbb{R}, \\ 0 = \Delta H - H + A^2, & x \in \mathbb{R}. \end{cases} \tag{2.86}$$

Then (2.86) *has a solution with which consists of n spikes which all approach the same point*, *where*

$$A_\epsilon(x) \sim \sum_{k=1}^{n} \xi_\epsilon \hat{\xi}^0 w\left(\frac{x - t_k^\epsilon}{\epsilon}\right),$$

$$H_\epsilon\left(t_k^\epsilon\right) \sim \xi_\epsilon \hat{\xi}^0, \quad k = 1, \dots, n,$$

$$t_k^\epsilon \to t^0 \in \mathbb{R}, \quad k = 1, \dots, n,$$

where ξ_ϵ *has been defined in* (2.18). *Further, we have*

$$t_s^\epsilon - t_{s-1}^\epsilon = \epsilon \log \frac{1}{\epsilon} - \epsilon \log\left[\frac{\hat{\xi}^0}{2D}(s-1)(n+1-s)\right] + o(\epsilon), \tag{2.87}$$

$s = 2, \dots, n$, *and*

$$\hat{\xi}^0 = \frac{2}{n\theta}.$$

Remark 2.11

1. The spikes are able to stabilise each other even without the presence of a boundary. There is a balance between the mutual attraction of the activators and the mutual repulsion of the inhibitors which leads to the existence of an n-spike ground state. Here the activator part of each spike interacts only with its neighbour(s) but the inhibitor parts interact with all other spikes.
2. Note that the distance between neighbouring spikes scales like $O(\epsilon \log \frac{1}{\epsilon})$ as $\epsilon \to 0$, i.e. it tends to 0 and all spikes approach the same point.
3. Note that unlike multiple spikes on the interval which are exact copies of each other this is not the case here. In contrast, for these ground states the distance between neighbouring spikes changes from spike to spike.
4. Due to translation invariance, the point $t_0 \in \mathbb{R}$ can be chosen arbitrarily.

Proof of Theorem 2.10 The proof uses Liapunov-Schmidt reduction similarly to Sect. 2.3.3. Solving the reduced problem consists of finding a zero of the vector field

$$W_{\epsilon,k} = \sum_{l \neq k} \left(\frac{1}{2D}\hat{\xi}^0 - \frac{1}{\epsilon} w\left(\frac{x_k - x_l}{\epsilon}\right)\right) \frac{x_k - x_l}{|x_k - x_l|}$$
$$+ O\left(\epsilon^{3/4}\right), \quad k = 1, \dots, n. \tag{2.88}$$

Here the positive and negative terms can be balanced for $x_{k+1} - x_k \sim C_k \epsilon \log \frac{1}{\epsilon}$, where C_k are positive constants which depend on k. □

Next we consider the existence of multiple clusters in an interval, where each of the clusters consists of multiple spikes approaching the same point. Again, the proof is based on Liapunov-Schmidt reduction.

We need to make three assumptions which are extensions of those given in Sect. 2.3 for multiple spikes. If the clusters are indexed by j, then n_j denotes the number of spikes at cluster j.

Our first assumption is as follows:

(H1a) There exists a solution $(\hat{\xi}_1^0, \ldots, \hat{\xi}_N^0)$ of the equation

$$\sum_{j=1}^{N} G_D\big(t_i^0, t_j^0\big) n_j \big(\hat{\xi}_j^0\big)^2 = \hat{\xi}_i^0, \quad i = 1, \ldots, N.$$

Next we introduce the matrix

$$b_{ij} = G_D\big(t_i^0, t_j^0\big) n_j \big(\hat{\xi}_j^0\big), \quad \mathcal{B} = (b_{ij}).$$

Our second assumption is the following:

(H2a) It holds that

$$\frac{1}{2} \notin \sigma(\mathcal{B}),$$

where $\sigma(\mathcal{B})$ is the set of eigenvalues of $\mathcal{B}$.

We define the following vector field:

$$F(\mathbf{t}) = \big(F_1(\mathbf{t}), \ldots, F_N(\mathbf{t})\big),$$

where

$$\begin{aligned} F_i(\mathbf{t}) &= \sum_{l=1}^{N} \nabla_{t_i} G_D(t_i, t_l) n_l \hat{\xi}_l^2 \\ &= -\nabla_{x_i} H_D(t_i, t_i) n_i \hat{\xi}_i^2 + \sum_{l \neq i} \nabla_{t_i} G_D(t_i, t_l) n_l \hat{\xi}_l^2, \quad i = 1, \ldots, N. \end{aligned}$$

Set

$$\mathcal{M}(\mathbf{t}) = \big(\nabla_{t_j} F_i(\mathbf{t})\big).$$

Our third assumption concerns the vector field $F(\mathbf{t})$:

(H3a) We assume that at $\mathbf{t}^0 = (x_1^0, \ldots, x_N^0)$ we have

$$F\big(\mathbf{t}^0\big) = 0,$$
$$\det\big(\mathcal{M}\big(\mathbf{t}^0\big)\big) \neq 0.$$

Our first result is about the existence of *symmetric* multiple cluster solution.

Theorem 2.12 (Existence of symmetric multiple clusters) *Let N and n be two positive integers and*

$$t_j^0 = -1 + \frac{2j-1}{N}, \quad j = 1, \ldots, N.$$

Then, for $\epsilon \ll 1$, problem (2.2) has a solution with N equidistant clusters which concentrate at $t_1^0, \ldots, t_N^0$ and each of which consists of n spikes. We have

$$A_\epsilon(x) \sim \sum_{j=1}^{N} \sum_{k=1}^{n} \xi_\epsilon \hat{\xi}^0 w\left(\frac{x - t_{j,k}^\epsilon}{\epsilon}\right),$$

$$H_\epsilon\left(t_{j,k}^\epsilon\right) \sim \xi_\epsilon \hat{\xi}^0, \quad j = 1, \ldots, N, k = 1, \ldots, n,$$

$$t_{j,k}^\epsilon \to t_j^0, \quad j = 1, \ldots, N, k = 1, \ldots, n,$$

where ξ_ϵ has been defined in (2.18). Further,

$$t_{j,s}^\epsilon - t_{j,s-1}^\epsilon = \epsilon \log \frac{1}{\epsilon} - \epsilon \log\left[\frac{\hat{\xi}_0}{2D}(s-1)(n+1-s)\right] + o(\epsilon), \tag{2.89}$$

$j = 1, \ldots, N$, $s = 2, \ldots, n$, *and*

$$\hat{\xi}^0 = \frac{2\tanh(\theta/N)}{n\theta}.$$

Our final result concerns the existence of *asymmetric* multiple clusters.

Theorem 2.13 (Existence of asymmetric multiple clusters) *Let $N, n_1, \ldots, n_N$ be $N+1$ positive integers.*

Assume that for $(t_1^0, \ldots, t_N^0) \in (-1, 1)^N$ with $t_1^0 < \cdots < t_N^0$ assumptions (H1a), (H2a) *and* (H3a) *are satisfied. Let $(\xi_1^0, \ldots, \xi_N^0)$ be given by* (H1). *Then for $\epsilon \ll 1$, problem (2.2) has a solution with N clusters which concentrate at $t_1^\epsilon, \ldots, t_N^\epsilon$. Namely, it holds that*

$$A_\epsilon(x) \sim \sum_{j=1}^{N} \sum_{k=1}^{n_j} \xi_\epsilon \hat{\xi}_j^0 w\left(\frac{x - t_{j,k}^\epsilon}{\epsilon}\right),$$

$$H_\epsilon\left(t_{j,k}^\epsilon\right) \sim \xi_\epsilon \hat{\xi}_j^0, \quad j = 1, \ldots, N, k = 1, \ldots, n_j,$$

$$t_{j,k}^\epsilon \to t_j^0, \quad j = 1, \ldots, N, k = 1, \ldots, n_j,$$

$$t_{j,s}^\epsilon - t_{j,s-1}^\epsilon = \epsilon \log \frac{1}{\epsilon} - \epsilon \log\left[\frac{\hat{\xi}_j^0}{2D}(s-1)(n_j+1-s)\right] + o(\epsilon), \tag{2.90}$$

$j = 1, \ldots, N$, $s = 2, \ldots, n_j$.

Remark 2.14 Equations (2.89) and (2.90) express the fact that we have two different scalings in the spike locations: the distance between the centers of clusters which is of order $O(1)$ and the distance between spikes within each cluster which is of order $O(\epsilon \log \frac{1}{\epsilon})$.

2.5 Notes on the Literature

The method of Liapunov-Schmidt reduction has been applied in [68, 191, 192] on the semi-classical (i.e. for small parameter h) solution of the nonlinear Schrödinger equation

$$\frac{h^2}{2}\Delta U - (V - E)U + U^p = 0 \tag{2.91}$$

in $\mathbb{R}^N$ where V is a potential function and E is a real constant to construct solutions of (2.91) close to nondegenerate critical points of V for h sufficiently small. Subsequently it has also been used in many other examples of single partial differential equations, such as [8, 9, 68, 80, 81, 191, 192, 248, 256, 257].

The approach of Liapunov-Schmidt reduction to prove the existence of multiple spikes has been applied to the Gierer-Meinhardt system in [273]. For symmetric spikes, Theorem 2.2 recovers the existence result of [226]. Thus the results of [240] have been rigorously established. We note that this approach is very flexible and can be applied to many other reaction-diffusion systems in various situations.

In [27, 43, 44] the authors showed the existence of multiple-spike ground state solutions for the Gierer-Meinhardt system on the real line using Liapunov-Schmidt reduction and dynamical systems methods (geometric singular perturbation theory), respectively. We remark that asymmetric patterns can also be obtained for the Gierer-Meinhardt system on the real line [50]. In [49] it has been shown that a general two-component, singularly perturbed system that exhibits large-amplitude pulse patterns has a leading order normal form which is given by (2.1).

For clustered spikes we have presented results from [272].

http://www.springer.com/978-1-4471-5525-6

Mathematical Aspects of Pattern Formation in Biological Systems
Wei, J.; Winter, M.
2014, XII, 319 p. 20 illus., Hardcover
ISBN: 978-1-4471-5525-6