Chapter 3
Geometric Algebra

That all our knowledge begins with experience, there is indeed no doubt ... but although our knowledge originates with experience, it does not all arise out of experience.
–Immanuel Kant

The real number system $\mathbb{R}$ has a long and august history spanning a host of civilizations over a period of many centuries [17]. It may be considered the rock upon which many other mathematical systems are constructed and, at the same time, serves as a model of desirable properties that any extension of the real numbers should have. The real numbers $\mathbb{R}$ were extended to the *complex numbers* $\mathbb{C} = \mathbb{R}(\mathrm{i})$, where $\mathrm{i}^2 = -1$, principally because of the discovery of the solutions to the quadratic and cubic polynomials in terms of complex numbers during the Renaissance. The powerful *Euler formula* $z = r\exp(\mathrm{i}\theta)$ helps makes clear the geometric significance of the multiplication of complex numbers, as we have seen in Fig. 2.5 in the last chapter. Other extensions of the real and complex numbers have been considered but until recently have found only limited acceptance. For example, the extension of the complex numbers to Hamilton's quaternions was more divisive in its effects upon the mathematical community [14], one reason being the lack of universal commutativity and another the absence of a unique, clear geometric interpretation.

We have seen in the previous chapter that extending the real numbers $\mathbb{R}$ to include a new square root of $+1$ leads to the concept of the hyperbolic number plane $\mathbb{H}$, which in many ways is analogous to the complex number plane $\mathbb{C}$. Understanding the hyperbolic numbers is key to understanding even more general geometric extensions of the real numbers. Perhaps the extension of the real numbers to include a new square root $\mathrm{u} = \sqrt{+1} \notin \mathbb{R}$ only occurred much later because people were happy with the status quo that $\sqrt{1} = \pm 1$ and because such considerations were before the advent of Einstein's *theory of special relativity* and the study of *non-Euclidean* geometries.

Geometric algebra is the extension of the real number system to include new *anticommuting* square roots of ± 1, which represent mutually orthogonal unit

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Hermann Gunther Grassmann (1809-1877) was a high-school teacher. His far reaching Ausdehnungslehre, "Theory of extension" lay the ground work for the development of the exterior or outer product of vectors. William Rowan Hamilton (1805-1865) was an Irish physicist, astronomer and mathematician. His invention of the quaternions as the natural generalization of the complex numbers of the plane to three dimensional space, together with the ideas of Grassmann, set the stage for William Kingdon Clifford's definition of geometric algebra. William Kingdon Clifford (1845-1879) was a professor of mathematics and mechanics at the University College of London. Tragically, he died at the early age of 33 before he could explore his profound ideas.

vectors in successively higher dimensions. The critical new insight is that by assuming that our new square roots of ± 1 are anticommutative, we obtain a more general concept of number that will serve us well in the expression of geometrical ideas [72]. We begin with the extension of the real numbers to include two new anticommuting square roots of $+1$ which are given the interpretation of two *unit vectors* $\mathbf{e}_1$ and $\mathbf{e}_2$ lying along the x- and y-axes, respectively, as shown in Fig. 3.1. More generally, if we introduce n anticommuting square roots of unity, together with their geometric sums and products, we can represent all the directions in an n-dimensional space.

Geometric algebra provides the framework for the rest of the material developed in this book, and it is for this reason that the book is entitled *New Foundations in Mathematics: The Geometric Concept of Number*. A brief history of the development of geometric algebra, also known as Clifford algebra, is given in the box.

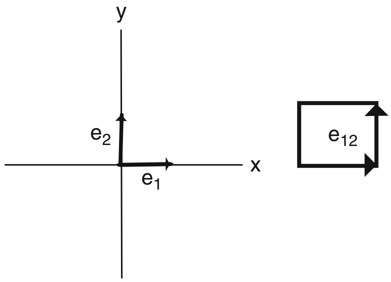


Fig. 3.1 Anticommuting orthogonal unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ along the xy-axes. Also pictured is the unit bivector $\mathbf{e}_{12} = \mathbf{e}_1\mathbf{e}_2$

3.1 Geometric Numbers of the Plane

Let $\mathbf{e}_1 = (1,0)$ and $\mathbf{e}_2 = (0,1)$ be orthonormal unit vectors along the x- and y-axes in $\mathbb{R}^2$ starting at the origin, and let $\mathbf{a} = (a_1,a_2) = a_1\mathbf{e}_1 + a_2\mathbf{e}_2$ and $\mathbf{b} = (b_1,b_2) = b_1\mathbf{e}_1 + b_2\mathbf{e}_2$, for $a_1,a_2,b_1,b_2 \in \mathbb{R}$, be arbitrary vectors in $\mathbb{R}^2$. The familiar *inner product* or "dot product" between the vectors $\mathbf{a}$ and $\mathbf{b}$ is given by

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta = a_1b_1 + a_2b_2, \tag{3.1}$$

where $|\mathbf{a}|$ and $|\mathbf{b}|$ are the *magnitudes* or *lengths* of the vectors $\mathbf{a}$ and $\mathbf{b}$ and θ is the angle between them.

The associative geometric algebra $\mathbb{G}_2 = \mathbb{G}(\mathbb{R}^2)$ of the plane $\mathbb{R}^2$ is generated by the *geometric multiplication* of the vectors in $\mathbb{R}^2$, subjected to the rule that given any vector $\mathbf{x} = (x_1,x_2) \in \mathbb{R}^2$,

$$\mathbf{x}^2 = \mathbf{x}\mathbf{x} = |\mathbf{x}|^2 = \mathbf{x}\cdot\mathbf{x} = x_1^2 + x_2^2. \tag{3.2}$$

This rule means that the square of any vector is equal to its magnitude squared. Geometric addition and multiplication of vectors satisfies all of the rules of an *associative algebra* with the *unity* 1. Indeed, geometric algebra satisfies all of the usual rules of addition and multiplication of real numbers, except that the geometric product of vectors is not universally commutative. The geometric algebra $\mathbb{G}_2$ is the *geometric extension* of the real numbers to include the two *anticommuting unit vectors* $\mathbf{e}_1$ and $\mathbf{e}_2$ along the x- and y-axes. See Fig. 3.1.

The fact that $\mathbf{e}_1$ and $\mathbf{e}_2$ are anticommuting can be considered to be a consequence of the rule (3.2). Since $\mathbf{e}_1$ and $\mathbf{e}_2$ are orthogonal unit vectors, by utilizing the *Pythagorean theorem*, we have

$$(\mathbf{e}_1 + \mathbf{e}_2)^2 = \mathbf{e}_1^2 + \mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1 + \mathbf{e}_2^2 = 1 + \mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1 + 1 = 2,$$

from which it follows that $\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1 = 0$, so $\mathbf{e}_1$ and $\mathbf{e}_2$ are anticommuting. We give the quantity $\mathrm{i} = \mathbf{e}_{12} := \mathbf{e}_1\mathbf{e}_2$ the geometric interpretation of a *unit bivector* or

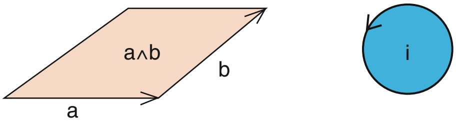


Fig. 3.2 The bivector $\mathbf{a} \wedge \mathbf{b}$ is obtained by sweeping the vector $\mathbf{b}$ out along the vector $\mathbf{a}$. Also shown is the unit bivector $\mathbf{i}$ in the plane of $\mathbf{a}$ and $\mathbf{b}$

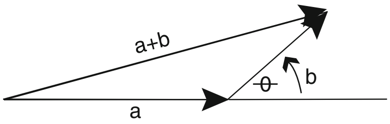


Fig. 3.3 The law of cosines relates the lengths of the sides of the triangle to the cosine of the angle between the sides $\mathbf{a}$ and $\mathbf{b}$

directed plane segment in the xy-plane. Whereas by the rule (3.2) a unit vector has square $+1$, we find that

$$\mathrm{i}^2 = (\mathbf{e}_1\mathbf{e}_2)(\mathbf{e}_1\mathbf{e}_2) = \mathbf{e}_1(\mathbf{e}_2\mathbf{e}_1)\mathbf{e}_2 = -\mathbf{e}_1(\mathbf{e}_1\mathbf{e}_2)\mathbf{e}_2 = -\mathbf{e}_1^2\mathbf{e}_2^2 = -1,$$

so that the unit bivector i has square -1. The unit bivector i is pictured in both Figs. 3.1 and 3.2. Notice that the *shape* of a bivector is unimportant; an oriented disk with area one in the xy-plane provides an equally valid picture of a unit bivector as does the oriented unit square in the xy-plane with the same orientation.

For any vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$, we find by again using (3.2) that

$$(\mathbf{a}+\mathbf{b})^2 = \mathbf{a}^2 + (\mathbf{ab}+\mathbf{ba}) + \mathbf{b}^2, \tag{3.3}$$

or equivalently,

$$\mathbf{a}\cdot\mathbf{b} = \frac{1}{2}(\mathbf{ab}+\mathbf{ba}) = \frac{1}{2}\left(|\mathbf{a}+\mathbf{b}|^2 - |\mathbf{a}|^2 - |\mathbf{b}|^2\right) = |\mathbf{a}||\mathbf{b}|\cos\theta,$$

which is an expression of the *law of cosines*. See Fig. 3.3.

Directly calculating the geometric products $\mathbf{ab}$ and $\mathbf{ba}$ for the vectors $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2$ and $\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2$, and simplifying, gives

$$\begin{aligned}\mathbf{ab} &= (a_1\mathbf{e}_1 + a_2\mathbf{e}_2)(b_1\mathbf{e}_1 + b_2\mathbf{e}_2) = a_1b_1\mathbf{e}_1^2 + a_2b_2\mathbf{e}_2^2 + a_1b_2\mathbf{e}_1\mathbf{e}_2 + a_2b_1\mathbf{e}_2\mathbf{e}_1\\ &= (a_1b_1 + a_2b_2) + (a_1b_2 - a_2b_1)\mathbf{e}_{12}.\end{aligned} \tag{3.4}$$

and

$$\mathbf{ba} = (b_1\mathbf{e}_1 + b_2\mathbf{e}_2)(a_1\mathbf{e}_1 + a_2\mathbf{e}_2) = (a_1b_1 + a_2b_2) - (a_1b_2 - a_2b_1)\mathbf{e}_{12}. \tag{3.5}$$

Using (3.4) and (3.5), we decompose the geometric product $\mathbf{ab}$ into *symmetric* and *antisymmetric* parts, getting

$$\mathbf{ab} = \frac{1}{2}(\mathbf{ab}+\mathbf{ba}) + \frac{1}{2}(\mathbf{ab}-\mathbf{ba}) = \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\wedge\mathbf{b}. \tag{3.6}$$

The symmetric part gives the inner product

$$\mathbf{a}\cdot\mathbf{b} = \frac{1}{2}(\mathbf{ab}+\mathbf{ba}) = \langle\mathbf{ab}\rangle_0 = a_1b_1+a_2b_2,$$

where $\langle\mathbf{ab}\rangle_0$ denotes the 0-*vector part* or *scalar part* of the argument. The antisymmetric part, called the *outer product* or *Grassmann exterior product*, is given by

$$\mathbf{a}\wedge\mathbf{b} = \frac{1}{2}(\mathbf{ab}-\mathbf{ba}) = \langle\mathbf{ab}\rangle_2 = (a_1b_2-a_2b_1)\mathbf{e}_1\mathbf{e}_2 = \mathrm{i}\det\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix},$$

where $\langle\mathbf{ab}\rangle_2$ denotes the 2-*vector part* or *bivector part* of the argument.

The outer product $\mathbf{a}\wedge\mathbf{b}$ defines the *direction* and *orientation* of an oriented or *directed plane segment*, just as a *vector* defines the *direction* and *orientation* of a directed line segment. Note that the bivector $\mathbf{a}\wedge\mathbf{b}$ has the direction of the unit bivector $\mathrm{i} = \mathbf{e}_1\mathbf{e}_2$ and the magnitude of the paralellogram defined by the vectors $\mathbf{a}$ and $\mathbf{b}$. See Fig. 3.2.

We can also express the outer product $\mathbf{a}\wedge\mathbf{b}$ in the form

$$\mathbf{a}\wedge\mathbf{b} = |\mathbf{a}||\mathbf{b}|\mathrm{i}\sin\theta \tag{3.7}$$

which is complimentary to (3.1) for the inner product. In the particular case that

$$0 = \mathbf{a}\cdot\mathbf{b} = \frac{1}{2}(\mathbf{ab}+\mathbf{ba}) \quad\Longleftrightarrow\quad \mathbf{ab} = -\mathbf{ba},$$

we see once again that the geometric product of two nonzero orthogonal vectors is anticommutative. In this case, (3.3) reduces to the familiar Pythagorean theorem for a right triangle with the vectors $\mathbf{a}$ and $\mathbf{b}$ along its sides.

Putting together (3.1) and (3.7) into (3.6), and using (2.3), we find that the geometric product of the vectors $\mathbf{a}$ and $\mathbf{b}$ can be written in the *Euler form*

$$\mathbf{ab} = \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\wedge\mathbf{b} = |\mathbf{a}||\mathbf{b}|(\cos\theta+\mathrm{i}\sin\theta) = |\mathbf{a}||\mathbf{b}|e^{\mathrm{i}\theta}. \tag{3.8}$$

We have made the fundamental discovery that the geometric product of the two vectors $\mathbf{a},\mathbf{b}\in\mathbb{R}^2$ is formally a complex number, most beautifully represented by the Euler formula above. Notice that whereas our derivation was in $\mathbb{R}^2$, the same derivation carries over for any two vectors in $\mathbb{R}^n$, since two linearly independent vectors in $\mathbb{R}^n$ will always determine a two-dimensional subspace of $\mathbb{R}^n$ which we can choose to be $\mathbb{R}^2$.

The *standard basis* of the geometric algebra $\mathbb{G}_2 = \mathbb{G}(\mathbb{R}^2)$ *over the real numbers* $\mathbb{R}$ is

$$\mathbb{G}_2 = \text{span}_{\mathbb{R}}\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_{12}\}, \tag{3.9}$$

where $\mathbf{e}_1^2 = \mathbf{e}_2^2 = 1$ and $\mathbf{e}_{12} := \mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_1$. We have seen that $\mathbf{e}_1$ and $\mathbf{e}_2$ have the interpretation of *orthonormal vectors* along the x- and y-axes of $\mathbb{R}^2$ and that the *imaginary unit* $\mathrm{i} = \mathbf{e}_{12}$ is the *unit bivector* of the plane spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$. The most general geometric number $g \in \mathbb{G}_2$ has the form

$$g = (\alpha_1 + \alpha_2\mathbf{e}_{12}) + (v_1\mathbf{e}_1 + v_2\mathbf{e}_2), \tag{3.10}$$

where $\alpha_i, v_i \in \mathbb{R}$ for $i = 1, 2$, in the standard basis (3.9).

Exercises

1. Let $\mathbf{x}_1 = (2,3)$, $\mathbf{x}_2 = (2,-3)$, and $\mathbf{x}_3 = (4,1)$. Calculate
   (a) $\mathbf{x}_1 \cdot \mathbf{x}_2$.
   (b) $\mathbf{x}_1 \wedge \mathbf{x}_2$. Graph this bivector.
   (c) $\mathbf{x}_2\mathbf{x}_3$. Find the Euler form (3.8) for this product.
   (d) Verify that $\mathbf{x}_1(\mathbf{x}_2 + \mathbf{x}_3) = \mathbf{x}_1\mathbf{x}_2 + \mathbf{x}_1\mathbf{x}_3$.
   (e) Graph $\mathbf{x}_1, \mathbf{x}_2$, and $\mathbf{x}_1 + \mathbf{x}_2$ in $\mathbb{R}^2$.
2. For $\mathbf{a} = (a_1, a_2)$, $\mathbf{b} = (b_1, b_2)$, $\mathbf{c} = (c_1, c_2)$ in $\mathbb{R}^2$, calculate
   (a) $\mathbf{a} \cdot \mathbf{b}$
   (b) $\mathbf{a} \wedge \mathbf{b}$. Graph this bivector.
   (c) Verify that $\mathbf{a}(\mathbf{b} + \mathbf{c}) = \mathbf{ab} + \mathbf{ac}$.
   (d) Verify that $(\mathbf{b} + \mathbf{c})\mathbf{a} = \mathbf{ba} + \mathbf{ca}$.
3. Let $\mathbf{x} = (x, y) = x\mathbf{e}_1 + y\mathbf{e}_2$.
   (a) Find the magnitude $|\mathbf{x}| = \sqrt{\mathbf{x}^2}$.
   (b) Find the unit vector $\hat{\mathbf{x}} := \frac{\mathbf{x}}{|\mathbf{x}|}$, and show that
   $$\mathbf{x}^{-1} = \frac{1}{\mathbf{x}} = \frac{\mathbf{x}}{|\mathbf{x}|^2} = \frac{\hat{\mathbf{x}}}{|\mathbf{x}|}$$
   where $\mathbf{x}^{-1}\mathbf{x} = 1 = \mathbf{x}\mathbf{x}^{-1}$.
   (c) Show that the equation of the **unit circle** in $\mathbb{R}^2$ with center at the point $\mathbf{a} \in \mathbb{R}^2$ is $(\mathbf{x} - \mathbf{a})^2 = 1$.
4. Let $w_1 = 5 + 4\mathbf{e}_1$, $w_2 = 5 - 4\mathbf{e}_2$, and $z_3 = 2 + \mathbf{e}_1\mathbf{e}_2$ be geometric numbers in $\mathbb{G}_2$.
   (a) Show that $w_1 w_2 - z_3 = 23 + 20\mathbf{e}_1 - 20\mathbf{e}_2 - 17\mathbf{e}_{12}$.

(b) Show that $w_1(w_2w_3) = (w_1w_2)w_3 = 66 + 60\mathbf{e}_1 - 20\mathbf{e}_2 - 7\mathbf{e}_{12}$ (geometric multiplication is associative).
(c) Show that $w_1(w_2 + w_3) = w_1w_2 + w_1w_3 = 35 + 28\mathbf{e}_1 - 16\mathbf{e}_2 - 11\mathbf{e}_{12}$ (distributive law).

5. Let $w = x + \mathbf{e}_1 y$ and $w^- = x - \mathbf{e}_1 y$. We define the magnitude $|w| = \sqrt{|ww^-|}$.

(a) Show that $|w| = \sqrt{|x^2 - y^2|}$.
(b) Show that the equation of the unit hyperbola in the hyperbolic number plane $\mathbb{H} = \text{span}_{\mathbb{R}}\{1, \mathbf{e}_1\}$ is $|w|^2 = |ww^-| = 1$ and has four branches.
(c) Hyperbolic Euler formula: Let $x > |y|$. Show that

$$w = x + \mathbf{e}_1 y = |w|\left(\frac{x}{|w|} + \mathbf{e}_1\frac{y}{|w|}\right) = \rho(\cosh\phi + \mathbf{e}_1\sinh\phi) = \rho e^{\mathbf{e}_1\phi}$$

where $\rho = |w|$ is the hyperbolic magnitude of w and ϕ is the hyperbolic angle that w makes with the x-axis. The (ρ, ϕ) are also called the hyperbolic polar coordinates of the point $w = (x, y) = x + \mathbf{e}_1 y$. What happens in the case that $y > |x|$? Compare this with similar results in Chap. 2.
(d) Let $w_1 = \rho_1 \exp(\mathbf{e}_1\phi_1)$ and $w_2 = \rho_2\exp(\mathbf{e}_1\phi_2)$. Show that

$$w_1w_2 = \rho_1\rho_2\exp(\mathbf{e}_1(\phi_1 + \phi_2)).$$

What is the geometric interpretation of this result? Illustrate with a figure. Compare this with similar results in Chap. 2.
(f) Find the *square roots* of the geometric numbers $w = 5 + 4\mathbf{e}_1$ and $z = 2 + \mathbf{e}_{12}$. *Hint:* First express the numbers in Euler form.

6. Calculate

(a) $e^{i\theta}\mathbf{e}_1$ and $e^{i\theta}\mathbf{e}_2$, where $i = \mathbf{e}_1\mathbf{e}_2$, and graph the results on the unit circle in $\mathbb{R}^2$.
(b) Show that $e^{i\theta}\mathbf{e}_1 = \mathbf{e}_1 e^{-i\theta} = e^{\frac{i\theta}{2}}\mathbf{e}_1 e^{\frac{-i\theta}{2}}$.
(c) Show that $(e^{i\theta}\mathbf{e}_1) \wedge (e^{i\theta}\mathbf{e}_2) = \mathbf{e}_1 \wedge \mathbf{e}_2 = i$, and explain the geometric significance of this result.

7. Show that $e^{-i\theta}\mathbf{a}$ rotates the vector $\mathbf{a} = (a_1, a_2)$ counterclockwise in the (x, y)-plane through an angle of θ.
8. Let the *geometric numbers* $A = 1 + 2\mathbf{e}_1 - \mathbf{e}_2 + 3i$ and $B = -2 - \mathbf{e}_1 + 2\mathbf{e}_2 - i$. Calculate the geometric product AB and write it as the sum of its *scalar*, *vector*, and *bivector* parts.
9. (a) Show that $X = \mathbf{a}^{-1}(\mathbf{c} - \mathbf{b}) \in \mathbb{G}_2$ is the solution to the linear equation $\mathbf{a}X + \mathbf{b} = \mathbf{c}$ where $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^2$.
(b) Show that $X = (\mathbf{c} - \mathbf{b})\mathbf{a}^{-1} \in \mathbb{G}_2$ is the solution to the linear equation $X\mathbf{a} + \mathbf{b} = \mathbf{c}$ where $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^2$.
(c) Find the solution to the equation $\mathbf{a}X^2 + \mathbf{b} = \mathbf{c}$.

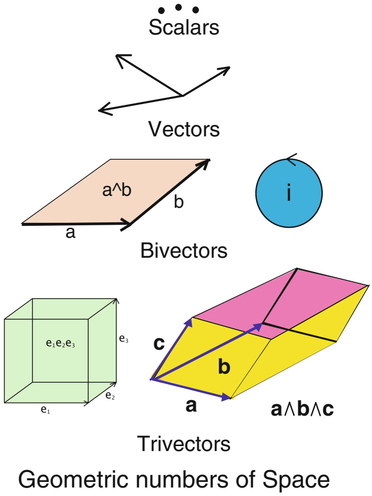


Fig. 3.4 Geometric numbers of space

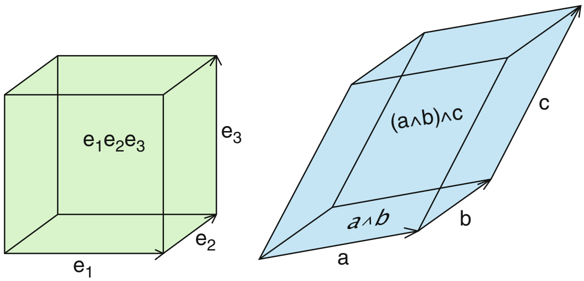


Fig. 3.5 The trivector $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$ is formed by sweeping the bivector $\mathbf{a} \wedge \mathbf{b}$ out along the vector $\mathbf{c}$. Also shown is the unit pseudoscalar $I = \mathbf{e}_{123}$

3.2 The Geometric Algebra $\mathbb{G}_3$ of Space

To arrive at the geometric algebra $\mathbb{G}_3 = \mathbb{G}(\mathbb{R}^3)$ of the 3-dimensional Euclidean space $\mathbb{R}^3$, we simply extend the geometric algebra $\mathbb{G}_2 = \mathbb{G}(\mathbb{R}^2)$ of the vector plane $\mathbb{R}^2$ to include the unit vector $\mathbf{e}_3$ which is orthogonal to the plane of the bivector $\mathbf{e}_{12}$. The *standard basis* of the geometric algebra $\mathbb{G}_3 = \mathbb{G}(\mathbb{R}^3)$ is

$$\mathbb{G}_3 = \operatorname{span}_{\mathbb{R}}\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_{23}, \mathbf{e}_{31}, \mathbf{e}_{12}, I\}, \tag{3.11}$$

where $\mathbf{e}_1^2 = \mathbf{e}_2^2 = \mathbf{e}_3^2 = 1$ and $\mathbf{e}_{ij} := \mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i$ for $i \neq j$ and $1 \le i, j \le 3$, and where the *trivector* $I := \mathbf{e}_{123} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$. The geometric numbers of 3-dimensional space are pictured in Fig. 3.4.

The $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are orthonormal unit vectors along the x, y, z-axes, the $\mathbf{e}_{23}$, $\mathbf{e}_{31}$, and $\mathbf{e}_{12}$ are unit bivectors defining the directions of the yz-, zx-, and xy-planes, respectively, and the unit trivector $I = \mathbf{e}_{123}$ defines the *directed volume element* of space. The trivector I is pictured in Fig. 3.5.

Calculating the square of the unit trivector or *pseudoscalar* of $\mathbb{G}_3$ gives

$$I^2 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = -\mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_3\mathbf{e}_2\mathbf{e}_3 = (\mathbf{e}_1\mathbf{e}_2)^2\mathbf{e}_3^2 = -1,$$

so the trivector I is another geometric square root of minus one. The pseudoscalar I also has the important property that it commutes with all the elements of $\mathbb{G}_3$ and is therefore in the *center* $Z(\mathbb{G}_3)$ of the algebra $\mathbb{G}_3$.

Noting that $\mathbf{e}_3 = -I\mathbf{e}_{12}, \mathbf{e}_{23} = I\mathbf{e}_1$ and $\mathbf{e}_{31} = I\mathbf{e}_2$, the standard basis of $\mathbb{G}_3$ takes the form

$$\mathbb{G}_3 = \mathrm{span}_{\mathbb{R}}\{1, \mathbf{e}_1, \mathbf{e}_2, -I\mathbf{e}_{12}, I\mathbf{e}_1, I\mathbf{e}_2, \mathbf{e}_{12}, I\} = \mathrm{span}_{\mathbb{C}}\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_{12}\} = \mathbb{G}_2(I) \quad (3.12)$$

The last two equalities in the above equation show that the geometric algebra $\mathbb{G}_3$ can be considered to be the geometric algebra $\mathbb{G}_2$ of the plane extended or *complexified* to include the pseudoscalar element $I = \mathbf{e}_{123}$. Any geometric number $g \in \mathbb{G}_3$ can be expressed in the form $g = A + IB$ where $A, B \in \mathbb{G}_2$. Later, it will be shown that there is an algebraic isomorphism between $\mathbb{G}_2$ and the 2×2 matrix algebra over the real numbers and that $\mathbb{G}_3$ is isomorphic to the 2×2 matrix algebra over the complex numbers.

Given the vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$, where $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$, a similar calculation of the geometric products (3.4) and (3.5) shows that

$$\mathbf{ab} = \frac{1}{2}(\mathbf{ab} + \mathbf{ba}) + \frac{1}{2}(\mathbf{ab} - \mathbf{ba}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\wedge\mathbf{b}, \quad (3.13)$$

where the inner product $\mathbf{a}\cdot\mathbf{b} = \langle\mathbf{ab}\rangle_0$ is given by

$$\mathbf{a}\cdot\mathbf{b} = \frac{1}{2}(\mathbf{ab} + \mathbf{ba}) = a_1b_1 + a_2b_2 + a_3b_3 \quad (3.14)$$

and the corresponding outer product $\mathbf{a}\wedge\mathbf{b} = \langle\mathbf{ab}\rangle_2$ is given by

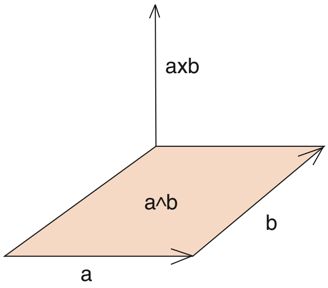
$$\mathbf{a}\wedge\mathbf{b} = \frac{1}{2}(\mathbf{ab} - \mathbf{ba}) = I\det\begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} = I(\mathbf{a}\times\mathbf{b}). \quad (3.15)$$

We have expressed the outer product $\mathbf{a}\wedge\mathbf{b}$ in terms of the familiar *cross product* $\mathbf{a}\times\mathbf{b}$ of the vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$. The vector $\mathbf{a}\times\mathbf{b}$ is the right-handed normal to the plane of the bivector $\mathbf{a}\wedge\mathbf{b}$, and $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}\wedge\mathbf{b}|$. See Fig. 3.6.

Let us now explore the geometric product of the vector $\mathbf{a}$ with the bivector $\mathbf{b}\wedge\mathbf{c}$. We begin by decomposing the geometric product $\mathbf{a}(\mathbf{b}\wedge\mathbf{c})$ into antisymmetric and symmetric parts:

$$\mathbf{a}(\mathbf{b}\wedge\mathbf{c}) = \frac{1}{2}[\mathbf{a}(\mathbf{b}\wedge\mathbf{c}) - (\mathbf{b}\wedge\mathbf{c})\mathbf{a}] + \frac{1}{2}[\mathbf{a}(\mathbf{b}\wedge\mathbf{c}) + (\mathbf{b}\wedge\mathbf{c})\mathbf{a}].$$

Fig. 3.6 The vector $\mathbf{a}$ is swept out along the vector $\mathbf{b}$ to form the bivector $\mathbf{a} \wedge \mathbf{b}$. The vector $\mathbf{a} \times \mathbf{b}$ is the right-handed normal to this plane, whose length $|\mathbf{a} \times \mathbf{b}|$ is equal to the area or magnitude $|\mathbf{a} \wedge \mathbf{b}|$ of the bivector $\mathbf{a} \wedge \mathbf{b}$



For the antisymmetric part, we find that

$$\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c}) := \frac{1}{2}[\mathbf{a}(\mathbf{b} \wedge \mathbf{c}) - (\mathbf{b} \wedge \mathbf{c})\mathbf{a}] = I\frac{1}{2}[\mathbf{a}(\mathbf{b} \times \mathbf{c}) - (\mathbf{b} \times \mathbf{c})\mathbf{a}] = -\mathbf{a} \times (\mathbf{b} \times \mathbf{c}), \tag{3.16}$$

and for the symmetric part, we find that

$$\begin{aligned} \mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) &:= \frac{1}{2}[\mathbf{a}(\mathbf{b} \wedge \mathbf{c}) + (\mathbf{b} \wedge \mathbf{c})\mathbf{a}] = I\frac{1}{2}[\mathbf{a}(\mathbf{b} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c})\mathbf{a}] \\ &= I[\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})]. \end{aligned} \tag{3.17}$$

The quantity $\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c}$ is called the *trivector* obtained by sweeping the bivector $\mathbf{a} \wedge \mathbf{b}$ out along the vector $\mathbf{c}$, as shown in Fig. 3.5.

Whereas we have carried out the above calculations for three vectors in $\mathbb{R}^3$, the calculation remains valid in $\mathbb{R}^n$ for the simple reason that three linearly independent vectors in $\mathbb{R}^n$, $n \geq 3$, define a three-dimensional subspace of $\mathbb{R}^n$. It is important, however, to give a direct argument for the identity

$$\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b})\mathbf{c} - (\mathbf{a} \cdot \mathbf{c})\mathbf{b}, \tag{3.18}$$

which is also valid in $\mathbb{R}^n$.

Decomposing the left side of this equation, using (3.16) and (3.15), we find

$$\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c}) = \frac{1}{4}[\mathbf{abc} - \mathbf{acb} - \mathbf{bca} + \mathbf{cba}].$$

Decomposing the right hand side of the equation, using (3.14), we get

$$\begin{aligned} (\mathbf{a} \cdot \mathbf{b})\mathbf{c} - (\mathbf{a} \cdot \mathbf{c})\mathbf{b} &= \frac{1}{2}[(\mathbf{a} \cdot \mathbf{b})\mathbf{c} + \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) - (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - \mathbf{b}(\mathbf{a} \cdot \mathbf{c})] \\ &= \frac{1}{4}[(\mathbf{ab} + \mathbf{ba})\mathbf{c} + \mathbf{c}(\mathbf{ab} + \mathbf{ba}) - (\mathbf{ac} + \mathbf{ca})\mathbf{b} - \mathbf{b}(\mathbf{ac} + \mathbf{ca})]. \end{aligned}$$

After cancellations, we see that the left and right sides of the equations are identical, so the identity (3.18) is proved.

We now give the definition of the inner product $(\mathbf{a}\wedge\mathbf{b})\cdot(\mathbf{c}\wedge\mathbf{d})$ of the two bivectors $\mathbf{a}\wedge\mathbf{b}$ and $\mathbf{c}\wedge\mathbf{d}$ where $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}\in\mathbb{R}^3$.

Definition 3.2.1. $(\mathbf{a}\wedge\mathbf{b})\cdot(\mathbf{c}\wedge\mathbf{d})=\langle(\mathbf{a}\wedge\mathbf{b})(\mathbf{c}\wedge\mathbf{d})\rangle_0$ so that $(\mathbf{a}\wedge\mathbf{b})\cdot(\mathbf{c}\wedge\mathbf{d})$ is the *scalar part* of the geometric product $(\mathbf{a}\wedge\mathbf{b})(\mathbf{c}\wedge\mathbf{d})$ of the bivectors.

Using the above definition and (3.13), we can find a formula for calculating $(\mathbf{a}\wedge\mathbf{b})\cdot(\mathbf{c}\wedge\mathbf{d})$. We find that

$$\begin{aligned}(\mathbf{a}\wedge\mathbf{b})\cdot(\mathbf{c}\wedge\mathbf{d}) &=<(\mathbf{a}\wedge\mathbf{b})(\mathbf{c}\wedge\mathbf{d})>_0=<(\mathbf{ab}-\mathbf{a}\cdot\mathbf{b})(\mathbf{c}\wedge\mathbf{d})>_0\\ &=<\mathbf{ab}(\mathbf{c}\wedge\mathbf{d})>_0=<\mathbf{a}[\mathbf{b}\cdot(\mathbf{c}\wedge\mathbf{d})]>_0=\mathbf{a}\cdot[\mathbf{b}\cdot(\mathbf{c}\wedge\mathbf{d})].\end{aligned}\tag{3.19}$$

As a special case of this identity, we define the *magnitude* $|\mathbf{a}\wedge\mathbf{b}|$ of the bivector $\mathbf{a}\wedge\mathbf{b}$ by

$$|\mathbf{a}\wedge\mathbf{b}|:=\sqrt{(\mathbf{a}\wedge\mathbf{b})\cdot(\mathbf{b}\wedge\mathbf{a})}=\sqrt{|\mathbf{a}|^2|\mathbf{b}|^2-(\mathbf{a}\cdot\mathbf{b})^2}.\tag{3.20}$$

Exercises

1. Verify the coordinate formulas (3.14) and (3.15) for the inner and outer products in $\mathbb{G}_3$.

   For problems 2–5, do the calculation for the vectors $\mathbf{a}=2\mathbf{e}_1+3\mathbf{e}_2-\mathbf{e}_3$, $\mathbf{b}=-\mathbf{e}_1+2\mathbf{e}_2+\mathbf{e}_3, \mathbf{c}=3\mathbf{e}_1-4\mathbf{e}_2+2\mathbf{e}_3\in\mathbb{R}^3$. The reader is strongly encouraged to check his hand calculations with the Clifford Algebra Calculator (CLICAL) developed by Pertti Lounesto [54]. The software for CLICAL can be downloaded from the site http://users.tkk.fi/ppuska/mirror/Lounesto/CLICAL.htm

2. (a) Calculate $\mathbf{a}\wedge\mathbf{b}$.
   (b) Calculate

   $$(\mathbf{a}\wedge\mathbf{b})^{-1}=\frac{1}{\mathbf{a}\wedge\mathbf{b}}=-\frac{\mathbf{a}\wedge\mathbf{b}}{|\mathbf{a}\wedge\mathbf{b}|^2}.$$

   Why is the minus sign necessary?
3. Calculate $\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c})=-\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ using formula (3.18)
4. Calculate $(\mathbf{a}\wedge\mathbf{b})\cdot(\mathbf{b}\wedge\mathbf{c}):=<(\mathbf{a}\wedge\mathbf{b})(\mathbf{b}\wedge\mathbf{c})>_0$ by using (3.19) and (3.18).
5. (a) Calculate $\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}$ using the formula (3.17).
   (b) Using (3.15), calculate $(\mathbf{a}\wedge\mathbf{b})\boxtimes(\mathbf{b}\wedge\mathbf{c})$, where $A\boxtimes B:=\frac{1}{2}(AB-BA)$. (The symbol $\boxtimes$ is used for the *antisymmetric product* to distinguish it from the symbol $\times$ used for the cross product of vectors in $\mathbb{R}^3$.)

   For problems 6–16, let $\mathbf{a},\mathbf{b},\mathbf{c}$ be arbitrary vectors in $\mathbb{R}^3\subset\mathbb{G}_3$.

6. (a) Show that for any vectors $\mathbf{a},\mathbf{b},\mathbf{c}\in\mathbb{R}^3$,

$$\begin{aligned}\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c} &= \frac{1}{4}(\mathbf{abc}-\mathbf{acb}+\mathbf{bca}-\mathbf{cba})\\ &= \frac{1}{6}(\mathbf{abc}-\mathbf{acb}+\mathbf{bca}-\mathbf{cba}+\mathbf{cab}-\mathbf{bac})\end{aligned}$$

   (b) Show that $(\mathbf{a}\wedge\mathbf{b})(\mathbf{b}\wedge\mathbf{c}) = (\mathbf{a}\wedge\mathbf{b})\cdot(\mathbf{b}\wedge\mathbf{c})+(\mathbf{a}\wedge\mathbf{b})\boxtimes(\mathbf{b}\wedge\mathbf{c})$.
7. Noting that $\mathbf{b}=(\mathbf{ba})\mathbf{a}^{-1}=(\mathbf{b}\cdot\mathbf{a})\mathbf{a}^{-1}+(\mathbf{b}\wedge\mathbf{a})\mathbf{a}^{-1}$, show that $\mathbf{b}_{\|}\mathbf{a}=\mathbf{b}_{\|}\cdot\mathbf{a}$ and $\mathbf{b}_{\perp}\mathbf{a}=\mathbf{b}_{\perp}\wedge\mathbf{a}$ where $\mathbf{b}_{\|}=(\mathbf{b}\cdot\mathbf{a})\mathbf{a}^{-1}$ and $\mathbf{b}_{\perp}=(\mathbf{b}\wedge\mathbf{a})\mathbf{a}^{-1}=\mathbf{b}-\mathbf{b}_{\|}$.
8. Find vectors $\mathbf{c}_{\|}$ and $\mathbf{c}_{\perp}$ such that $\mathbf{c}=\mathbf{c}_{\|}+\mathbf{c}_{\perp}$ and $\mathbf{c}_{\|}(\mathbf{a}\wedge\mathbf{b})=\mathbf{c}\cdot(\mathbf{a}\wedge\mathbf{b})$ and $\mathbf{c}_{\perp}(\mathbf{a}\wedge\mathbf{b})=\mathbf{c}\wedge(\mathbf{a}\wedge\mathbf{b})$. *Hint:* Use the fact that

$$\mathbf{c}=[\mathbf{c}(\mathbf{a}\wedge\mathbf{b})](\mathbf{a}\wedge\mathbf{b})^{-1}=[\mathbf{c}\cdot(\mathbf{a}\wedge\mathbf{b})+\mathbf{c}\wedge(\mathbf{a}\wedge\mathbf{b})](\mathbf{a}\wedge\mathbf{b})^{-1}.$$

9. Using (3.15), show that $\mathbf{a}\times\mathbf{b}=-I(\mathbf{a}\wedge\mathbf{b})$ where $I=\mathbf{e}_{123}$.
10. Show that $\mathbf{a}\cdot[I(\mathbf{b}\wedge\mathbf{c})]=I(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c})$ where $I=\mathbf{e}_{123}$.
11. Show that $\mathbf{a}\wedge[I(\mathbf{b}\wedge\mathbf{c})]=I[\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c})]$ where $I=\mathbf{e}_{123}$.
12. Show that $\mathbf{a}(\mathbf{a}\wedge\mathbf{b})=\mathbf{a}\cdot(\mathbf{a}\wedge\mathbf{b})$
13. Show that $(\mathbf{a}+\mathbf{b}\wedge\mathbf{c})^2=\mathbf{a}^2+2\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}+(\mathbf{b}\wedge\mathbf{c})^2$.
14. Show that $\mathbf{a}(\mathbf{a}\wedge\mathbf{b})=-(\mathbf{a}\wedge\mathbf{b})\mathbf{a}$.
15. Show that $(\mathbf{a}+\mathbf{a}\wedge\mathbf{b})^2=\mathbf{a}^2+(\mathbf{a}\wedge\mathbf{b})^2$.
16. Show that $(\mathbf{a}\wedge\mathbf{b}+\mathbf{b}\wedge\mathbf{c})^2=(\mathbf{a}\wedge\mathbf{b})^2+2(\mathbf{a}\wedge\mathbf{b})\cdot(\mathbf{b}\wedge\mathbf{c})+(\mathbf{b}\wedge\mathbf{c})^2$.

3.3 Orthogonal Transformations

Much of the utility of geometric algebra stems from the simple and direct expression of orthogonal transformations. Specifically, given two vectors $\mathbf{a}$ and $\mathbf{b}$, such that $|\mathbf{a}|=|\mathbf{b}|$, let us find a *reflection* (mirror image) $L(\mathbf{x})$ and a rotation $R(\mathbf{x})$ with the property that $L(\mathbf{a})=R(\mathbf{a})=\mathbf{b}$.

Noting that

$$(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})=\mathbf{a}^2-\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{a}-\mathbf{b}^2=0,$$

and assuming $\mathbf{a}\neq\mathbf{b}$, define the transformation

$$L(\mathbf{x})=-(\mathbf{a}-\mathbf{b})\mathbf{x}(\mathbf{a}-\mathbf{b})^{-1}. \tag{3.21}$$

Writing $\mathbf{a}=\frac{1}{2}(\mathbf{a}+\mathbf{b})+\frac{1}{2}(\mathbf{a}-\mathbf{b})$, it easily follows that

$$L(\mathbf{a})=L\left(\frac{1}{2}(\mathbf{a}+\mathbf{b})+\frac{1}{2}(\mathbf{a}-\mathbf{b})\right)=\frac{1}{2}(\mathbf{a}+\mathbf{b})-\frac{1}{2}(\mathbf{a}-\mathbf{b})=\mathbf{b}$$

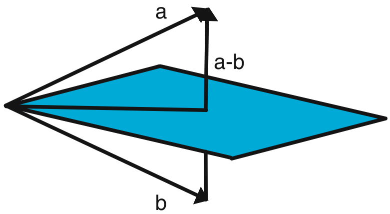


Fig. 3.7 The vector $\mathbf{a}$ is reflected in the plane perpendicular to the vector $\mathbf{a}-\mathbf{b}$ to give the vector $\mathbf{b}$

as required. See Fig. 3.7. The transformation $L(\mathbf{x})$ represents a reflection through the plane normal to the vector $\mathbf{a}-\mathbf{b}$. In the case that $\mathbf{b}=\mathbf{a}$, the required reflection is given by

$$L(\mathbf{x}) = -\mathbf{c}\mathbf{x}\mathbf{c}^{-1},$$

where $\mathbf{c}$ is any nonzero vector orthogonal to $\mathbf{a}$. Another interesting case is when $\mathbf{b} = -\mathbf{a}$, in which case the $L_{\mathbf{a}}(\mathbf{x}) = -\mathbf{a}\mathbf{x}\mathbf{a}^{-1}$ is a reflection with respect to the hyperplane that has $\mathbf{a}$ as its normal vector.

Let us now see how to define a rotation $R(\mathbf{x})$ with the property that $R(\mathbf{a}) = \mathbf{b}$ where $\mathbf{a}$ and $\mathbf{b}$ are any two vectors such that $|\mathbf{a}| = |\mathbf{b}|$. Recalling (3.8), we note that

$$\mathbf{b} = \mathbf{a}(\mathbf{a}^{-1}\mathbf{b}) = (\mathbf{b}\mathbf{a}^{-1})^{\frac{1}{2}}\mathbf{a}(\mathbf{a}^{-1}\mathbf{b})^{\frac{1}{2}} = \mathbf{b}(\mathbf{a}^{-1}\mathbf{a}),$$

where $\mathbf{a}^{-1} = \frac{\mathbf{a}}{|\mathbf{a}|^2}$. The desired rotation is given by

$$R(\mathbf{x}) = (\mathbf{b}\mathbf{a}^{-1})^{\frac{1}{2}}\mathbf{x}(\mathbf{a}^{-1}\mathbf{b})^{\frac{1}{2}}. \tag{3.22}$$

Since $|\mathbf{a}| = |\mathbf{b}|$, it follows that $\mathbf{a}^{-1}\mathbf{b} = \hat{\mathbf{a}}\hat{\mathbf{b}}$ where $\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}$ and $\hat{\mathbf{b}} = \frac{\mathbf{b}}{|\mathbf{b}|}$ are unit vectors.

There is a very simple formula that allows us to calculate $\sqrt{\hat{\mathbf{a}}\hat{\mathbf{b}}}$ algebraically. We have

$$\sqrt{\hat{\mathbf{a}}\hat{\mathbf{b}}} = \pm\hat{\mathbf{a}}\frac{\hat{\mathbf{a}}+\hat{\mathbf{b}}}{|\hat{\mathbf{a}}+\hat{\mathbf{b}}|}. \tag{3.23}$$

To verify (3.23), we simply calculate

$$\left(\hat{\mathbf{a}}\frac{\hat{\mathbf{a}}+\hat{\mathbf{b}}}{|\hat{\mathbf{a}}+\hat{\mathbf{b}}|}\right)^2 = \frac{\hat{\mathbf{a}}(\hat{\mathbf{a}}+\hat{\mathbf{b}})\hat{\mathbf{a}}(\hat{\mathbf{a}}+\hat{\mathbf{b}})\hat{\mathbf{b}}\hat{\mathbf{b}}}{(\hat{\mathbf{a}}+\hat{\mathbf{b}})^2}$$
$$= \frac{\hat{\mathbf{a}}(\hat{\mathbf{a}}+\hat{\mathbf{b}})(\hat{\mathbf{b}}+\hat{\mathbf{a}})\hat{\mathbf{b}}}{(\hat{\mathbf{a}}+\hat{\mathbf{b}})^2} = \hat{\mathbf{a}}\hat{\mathbf{b}}.$$

Equation (3.22) is called the *half angle* or *two-sided* representation of a rotation in the plane of the vectors $\mathbf{a}$ and $\mathbf{b}$. Writing the vector $\mathbf{x} = \mathbf{x}_{\parallel} + \mathbf{x}_{\perp}$ where

$$\mathbf{x}_{\parallel} = [\mathbf{x}\cdot(\mathbf{a}\wedge\mathbf{b})](\mathbf{a}\wedge\mathbf{b})^{-1} = \mathbf{x}\cdot(\mathbf{a}\wedge\mathbf{b})\frac{\mathbf{b}\wedge\mathbf{a}}{|\mathbf{a}\wedge\mathbf{b}|^2}$$

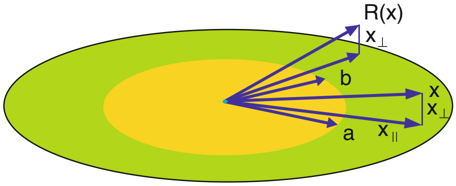


Fig. 3.8 The vector $\mathbf{x} = \mathbf{x}_{\|} + \mathbf{x}_{\perp}$ is rotated in the plane of the bivector $\mathbf{a} \wedge \mathbf{b}$ into $R(\mathbf{x})$, leaving the perpendicular component unchanged. When $|\mathbf{a}| = |\mathbf{b}| = 1$, $R(\mathbf{x}) = \mathrm{e}^{-\mathrm{i}\frac{\theta}{2}}\mathbf{x}\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}$, where $\mathrm{e}^{\mathrm{i}\frac{\theta}{2}} = \frac{\mathbf{a}(\mathbf{a}+\mathbf{b})}{|\mathbf{a}+\mathbf{b}|}$, $\mathrm{e}^{-\mathrm{i}\frac{\theta}{2}} = \frac{(\mathbf{a}+\mathbf{b})\mathbf{a}}{|\mathbf{a}+\mathbf{b}|}$ and θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. If $\mathbf{a}$ and $\mathbf{b}$ lie in the xy-plane, then $\mathrm{i} = \mathbf{e}_{12}$

so that $\mathbf{x}_{\|}$ is in the plane of $\mathbf{a} \wedge \mathbf{b}$, and $\mathbf{x}_{\perp} = \mathbf{x} - \mathbf{x}_{\|}$ is the component of $\mathbf{x}$ perpendicular to that plane, it follows that

$$R(\mathbf{x}) = R(\mathbf{x}_{\|} + \mathbf{x}_{\perp}) = R(\mathbf{x}_{\|}) + \mathbf{x}_{\perp}. \tag{3.24}$$

Thus, the two-sided representation of the rotation leaves the perpendicular part of the vector $\mathbf{x}$ to the plane of the rotation $\mathbf{a} \wedge \mathbf{b}$ invariant. See Fig. 3.8.

The rotation found in (3.22) can equally well be considered to be the composition of the two reflections

$$L_{\mathbf{a}}(x) = -\mathbf{a}\mathbf{x}\mathbf{a}^{-1} \quad \text{and} \quad L_{\mathbf{a}+\mathbf{b}}(x) = -(\mathbf{a}+\mathbf{b})\mathbf{x}(\mathbf{a}+\mathbf{b})^{-1}.$$

Using (3.23), we find that

$$R(\mathbf{x}) = (\mathbf{b}\mathbf{a}^{-1})^{\frac{1}{2}}\mathbf{x}(\mathbf{a}^{-1}\mathbf{b})^{\frac{1}{2}} = L_{\mathbf{a}+\mathbf{b}} \circ L_{\mathbf{a}}(\mathbf{x}). \tag{3.25}$$

Exercises

Given the vectors $\mathbf{a} = (1,2,2)$, $\mathbf{b} = (1,2,-2)$, and $\mathbf{c} = (2,1,2)$ in $\mathbb{R}^3$. The reader is strongly encouraged to check his hand calculations with Pertti Lounesto's CLICAL [54].

1. Calculate
   (a) $\mathbf{a} \cdot \mathbf{b}$ and $\mathbf{a} \wedge \mathbf{b}$
   (b) $\mathbf{ab}$
   (c) $(\mathbf{ab})\mathbf{c}$
   (d) $\mathbf{a}(\mathbf{bc})$
   (e) $\mathbf{a}(\mathbf{b}+\mathbf{c})$
   (f) $\mathbf{ab} + \mathbf{ac}$.

2. (a) Find the magnitude $|\mathbf{a} \wedge \mathbf{b}|$ of the bivector $\mathbf{a} \wedge \mathbf{b}$.
   (b) Graph the bivector $\mathbf{a} \wedge \mathbf{b}$.

(c) Find the Euler form of $\mathbf{ac}$.
(d) Find $\sqrt{\mathbf{ab}}$ and $\sqrt{\mathbf{ac}}$.

3. (a) Find a reflection $L(\mathbf{x})$ and a rotation $R(\mathbf{x})$ which takes $\mathbf{a}$ into $\mathbf{b}$.
(b) Find a reflection $L(\mathbf{x})$ and a rotation $R(\mathbf{x})$ which takes $\mathbf{b}$ into $\mathbf{c}$.
(c) Find a reflection $L(\mathbf{x})$ and a rotation $R(\mathbf{x})$ which takes $\mathbf{a}$ into $\mathbf{c}$.
4. Verify (3.24) for the rotation (3.22).
5. Verify (3.25) for the rotation (3.22).
6. Hamilton's quaternions: In the geometric algebra $\mathbb{G}_3 = \mathbb{G}(\mathbb{R}^3)$, define

$$i = \mathbf{e}_{23}, \quad j = \mathbf{e}_{31}, \quad k = \mathbf{e}_{21}.$$

(a) Show that $i^2 = j^2 = k^2 = ijk = -1$. These are the famous relationships that Hamilton carved into stone on the Bougham Bridge on 16 October 1843 in Dublin.

http://en.wikipedia.org/wiki/Quaternion

(b) Show that

$$ij = -ji = k, \quad jk = -kj = i, \quad ik = -ki = -j.$$

We see that the quaternions $\{1, i, j, k\}$ can be identified as the *even subalgebra* of scalars and bivectors of the geometric algebra $\mathbb{G}_3$. For a further discussion of these issues, see [55, p.68,190].

3.4 Geometric Algebra of $\mathbb{R}^n$

Geometric algebra was introduced by William Kingdon Clifford in 1878 as a generalization and unification of the ideas of Hermann Grassmann and William Hamilton, who came before him [12–14]. Whereas Hamilton's quaternions are known and Grassmann algebras appear in the guise of differential forms and antisymmetric tensors, the utility of Clifford's geometric algebras is just beginning to be widely appreciated [6, 20, 24, 41].

http://en.wikipedia.org/wiki/Differential_form

Let

$$(\mathbf{e})_{(n)} := (\mathbf{e}_i)_{i=1}^{n} = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \dots & \mathbf{e}_n \end{pmatrix} \tag{3.26}$$

be the standard orthonormal basis of the n-dimensional Euclidean space $\mathbb{R}^n$. In anticipation of our work with matrices whose elements are vectors, or even more general geometric numbers, we are deliberately expressing the basis (3.26) as a *row matrix* of the orthonormal basis vectors $\mathbf{e}_i \in \mathbb{R}^n$ for $i = 1, \dots, n$. We will have much more to say about this later.

If the vectors $\mathbf{a},\mathbf{b}\in\mathbb{R}^n$ are represented by *column vectors* of their components,

$$\mathbf{a}=\begin{pmatrix}a^1\\a^2\\\cdot\\\cdot\\a_n\end{pmatrix}=\sum_i a^i\mathbf{e}_i \text{ and } \mathbf{b}=\begin{pmatrix}b^1\\b^2\\\cdot\\\cdot\\b_n\end{pmatrix}=\sum_i b^i\mathbf{e}_i,$$

then their *inner product* is given by

$$\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos\theta=\sum_i a^ib^i$$

where θ is the angle between them.

Just as we write the standard orthonormal basis of $\mathbb{R}^n$ as the row of vectors $(\mathbf{e})_{(n)}$, we write the *standard orthonormal reciprocal basis* as the *column* of vectors:

$$(\mathbf{e})^{(n)}:=\begin{pmatrix}\mathbf{e}^1\\\mathbf{e}^2\\\cdot\\\cdot\\\mathbf{e}^n\end{pmatrix}. \tag{3.27}$$

However, since we are dealing with an orthonormal basis, the reciprocal basis vectors $\mathbf{e}^i$ can be identified with the basis vectors $\mathbf{e}_i$, i.e., $\mathbf{e}^i=\mathbf{e}_i$ for $i=1,\dots,n$. We can also represent a vector $\mathbf{a}\in\mathbb{R}^n$ in the *row vector form*

$$\mathbf{a}=\begin{pmatrix}a_1 & a_2 & \cdots & a_n\end{pmatrix}=\sum_i a_i\mathbf{e}^i.$$

More generally, given any basis $(\mathbf{v})_{(n)}$ of $\mathbb{R}^n$, it is always possible to construct a reciprocal basis $(\mathbf{v})^{(n)}$, as will be discussed in Chap. 7.

The *associative geometric algebra* $\mathbb{G}_n=\mathbb{G}(\mathbb{R}^n)$ of the Euclidean space $\mathbb{R}^n$ is the *geometric extension* of the real number system $\mathbb{R}$ to include n new *anticommuting square roots* of unity, which we identify with the orthonormal basis vectors $\mathbf{e}_i$ of $\mathbb{R}^n$, and their geometric sums and products. Each vector $\mathbf{a}\in\mathbb{R}^n$ has the property that

$$\mathbf{a}^2=\mathbf{a}\mathbf{a}=\mathbf{a}\cdot\mathbf{a}=|\mathbf{a}|^2, \tag{3.28}$$

where $|\mathbf{a}|$ is the Euclidean length or *magnitude* of the vector $\mathbf{a}$.

The fundamental *geometric product* $\mathbf{ab}$ of the vectors $\mathbf{a},\mathbf{b}\in\mathbb{R}^n\subset\mathbb{G}_n$ can be decomposed into the symmetric inner product and the antisymmetric outer product

$$\mathbf{ab}=\frac{1}{2}(\mathbf{ab}+\mathbf{ba})+\frac{1}{2}(\mathbf{ab}-\mathbf{ba})=\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\wedge\mathbf{b}=|\mathbf{a}||\mathbf{b}|e^{i\theta}, \tag{3.29}$$

where

$$\mathbf{a}\cdot\mathbf{b}=\frac{1}{2}(\mathbf{ab}+\mathbf{ba})=|\mathbf{a}||\mathbf{b}|\cos\theta$$

is the inner product and

$$\mathbf{a}\wedge\mathbf{b}=\frac{1}{2}(\mathbf{ab}-\mathbf{ba})=|\mathbf{a}||\mathbf{b}|\mathrm{i}\sin\theta$$

is the outer product of the vectors $\mathbf{a}$ and $\mathbf{b}$. The outer product $\mathbf{a}\wedge\mathbf{b}$ is given the geometric interpretation of a *directed plane segment* or *bivector* and characterizes the direction of the plane of the subspace of $\mathbb{R}^n$ spanned by the vectors $\mathbf{a}$ and $\mathbf{b}$, recall Fig. 3.2. The unit bivector i orients the plane of the vectors $\mathbf{a}$ and $\mathbf{b}$, and, as we have seen in Sect. 3.2, has the property that $\mathrm{i}^2=-1$.

The real 2^n-dimensional geometric algebra $\mathbb{G}_n=\mathbb{G}(\mathbb{R}^n)$ has the *standard basis*

$$\mathbb{G}_n=\mathrm{span}\{\mathbf{e}_{\lambda_{(k)}}\}_{k=0}^{n}, \tag{3.30}$$

where the $\binom{n}{k}$ k-vector basis elements of the form $\mathbf{e}_{\lambda_{(k)}}$ are defined by

$$\mathbf{e}_{\lambda_{(k)}}=e_{\lambda_1,\ldots,\lambda_k}=e_{\lambda_1}\cdots e_{\lambda_k}$$

for each $\lambda_{(k)}=\lambda_1,\ldots,\lambda_k$ where $1\le\lambda_1<\cdots<\lambda_k\le n$. When $k=0$, $\lambda_0=0$ and $\mathbf{e}_0=1$. Thus, as we have already seen in (3.11), the 2^3-dimensional geometric algebra of $\mathbb{R}^3$ has the standard basis

$$\mathbb{G}_3=\mathrm{span}\{\mathbf{e}_{\lambda_{(k)}}\}_{k=0}^{3}=\mathrm{span}\{1,\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_{12},\mathbf{e}_{13},\mathbf{e}_{23},\mathbf{e}_{123}\}$$

over the real numbers $\mathbb{R}$.

In geometric algebra, deep geometrical relationships are expressed directly in terms of the multivectors of the algebra without having to constantly refer to a basis. On the other hand, the language gives geometric meaning to the powerful matrix formalism that has developed over the last 150 years. As a real associative algebra, each geometric algebra is isomorphic to a corresponding algebra or subalgebra of real matrices, and we have advocated elsewhere the need for a uniform approach to both of these structures [65, 86]. Matrices are invaluable for systematizing calculations, but geometric algebra provides deep geometrical insight and powerful algebraic tools for the efficient expression of geometrical properties. Clifford algebras and their relationships to matrix algebra and the classical groups have been thoroughly studied in [55, 64] and elsewhere and will be examined in many of the succeeding chapters in this book.

We denote the *pseudoscalar* of $\mathbb{R}^n$ by the special symbol $I=\mathbf{e}_{1\ldots n}$. The pseudoscalar gives a unique orientation to $\mathbb{R}^n$ and $I^{-1}=\mathbf{e}_{n\ldots 1}$. For an n-vector $\mathbf{A}_n\in\mathbb{G}_n$, the *determinant function* is defined by $\mathbf{A}_n=\det(\mathbf{A}_n)I$ or $\det(\mathbf{A}_n)=\mathbf{A}_nI^{-1}$. The determinant function will be studied in Chap. 5.

In dealing with geometric numbers, three different types of *conjugation operations* are important. Let $A \in \mathbb{G}_n$ be a geometric number. By the *reverse* $A^\dagger$ of A, we mean the geometric number obtained by reversing the order of all the geometric products of vectors in A. For example, if $A = 2\mathbf{e}_1 + 3\mathbf{e}_{12} - 2\mathbf{e}_{123} + \mathbf{e}_{1234}$, then

$$A^\dagger = 2\mathbf{e}_1 + 3\mathbf{e}_{21} - 2\mathbf{e}_{321} + \mathbf{e}_{4321} = 2\mathbf{e}_1 - 3\mathbf{e}_{12} + 2\mathbf{e}_{123} + \mathbf{e}_{1234}. \tag{3.31}$$

By the *grade inversion* B^- of a geometric number $B \in \mathbb{G}_n$, we mean the geometric number obtained by replacing each vector that occurs in B by the negative of that vector. For the geometric number A given above, we find that

$$A^- = -2\mathbf{e}_1 + 3\mathbf{e}_{12} + 2\mathbf{e}_{123} + \mathbf{e}_{1234}. \tag{3.32}$$

The third conjugation, called the *Clifford conjugation* $\widetilde{C} = (C^\dagger)^-$ of $C \in \mathbb{G}_n$, is the composition of the operations of reversal together with grade inversion. For the geometric number A, we find that

$$\widetilde{A} = (A^\dagger)^- = (2\mathbf{e}_1 - 3\mathbf{e}_{12} + 2\mathbf{e}_{123} + \mathbf{e}_{1234})^- = -2\mathbf{e}_1 - 3\mathbf{e}_{12} - 2\mathbf{e}_{123} + \mathbf{e}_{1234}. \tag{3.33}$$

For $A, B, C \in \mathbb{G}_n$, the following properties are easily verified. For the operation of reverse, we have

$$(A+B)^\dagger = A^\dagger + B^\dagger, \quad \text{and} \quad (AB)^\dagger = B^\dagger A^\dagger. \tag{3.34}$$

For the grade involution, we have

$$(A+B)^- = A^- + B^-, \quad \text{and} \quad (AB)^- = A^- B^-. \tag{3.35}$$

Clifford conjugation, like the conjugation of reverse, satisfies

$$\widetilde{(A+B)} = \widetilde{A} + \widetilde{B}, \quad \text{and} \quad \widetilde{AB} = \widetilde{B}\widetilde{A}. \tag{3.36}$$

In general, the geometric product of a vector $\mathbf{a}$ and a k-vector $\mathbf{B}_k$ for $k \geq 1$ can always be decomposed into the sum of an *inner product* and an *outer product*:

$$\mathbf{a}\mathbf{B}_k = \mathbf{a} \cdot \mathbf{B}_k + \mathbf{a} \wedge \mathbf{B}_k. \tag{3.37}$$

The inner product $\mathbf{a} \cdot \mathbf{B}_k$ is the $(k-1)$-vector

$$\mathbf{a} \cdot \mathbf{B}_k := \frac{1}{2}(\mathbf{a}\mathbf{B}_k + (-1)^{k+1}\mathbf{B}_k\mathbf{a}) = \langle \mathbf{a}\mathbf{B}_k \rangle_{k-1}, \tag{3.38}$$

and the outer product $\mathbf{a} \wedge \mathbf{B}_k$ is the $(k+1)$-vector

$$\mathbf{a} \wedge \mathbf{B}_k := \frac{1}{2}(\mathbf{a}\mathbf{B}_k - (-1)^{k+1}\mathbf{B}_k\mathbf{a}) = \langle \mathbf{a}\mathbf{B}_k \rangle_{k+1}. \tag{3.39}$$

More generally, if $\mathbf{A}_r$ and $\mathbf{B}_s$ are r- and s-vectors of $\mathbb{G}_n$, where $r > 0, s > 0$, we define $\mathbf{A}_r \cdot \mathbf{B}_s = \langle \mathbf{A}_r \mathbf{B}_s \rangle_{|r-s|}$ and $\mathbf{A}_r \wedge \mathbf{B}_s = \langle \mathbf{A}_r \mathbf{B}_s \rangle_{r+s}$. In the exceptional cases when $r = 0$ or $s = 0$, we define $\mathbf{A}_r \cdot \mathbf{B}_s = 0$ and $\mathbf{A}_r \wedge \mathbf{B}_s = \mathbf{A}_r \mathbf{B}_s$.

In a similar way that we established the identity (3.18), we now show that for the vectors $\mathbf{a}$ and $\mathbf{b}$ and the k-vector $\mathbf{C}_k$, where $k \geq 1$,

$$\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{C}_k) = (\mathbf{a} \cdot \mathbf{b})\mathbf{C}_k - \mathbf{b} \wedge (\mathbf{a} \cdot \mathbf{C}_k). \tag{3.40}$$

Decomposing the left side of this equation, with the help of (3.38) and (3.39), gives

$$\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{C}_k) = \frac{1}{4}[\mathbf{abC}_k + (-1)^k \mathbf{aC}_k\mathbf{b} + (-1)^k \mathbf{bC}_k\mathbf{a} + \mathbf{C}_k\mathbf{ba}].$$

Decomposing the right hand side of the equation, with the help of (3.38) and (3.39), we get

$$\begin{aligned}(\mathbf{a} \cdot \mathbf{b})\mathbf{C}_k - \mathbf{b} \wedge (\mathbf{a} \cdot \mathbf{C}_k) &= \frac{1}{2}[(\mathbf{a} \cdot \mathbf{b})\mathbf{C}_k + \mathbf{C}_k(\mathbf{a} \cdot \mathbf{b}) - \mathbf{b}(\mathbf{a} \cdot \mathbf{C}_k) + (-1)^k (\mathbf{a} \cdot \mathbf{C}_k)\mathbf{b}] \\ &= \frac{1}{4}[(\mathbf{ab} + \mathbf{ba})\mathbf{C}_k + \mathbf{C}_k(\mathbf{ab} + \mathbf{ba}) - \mathbf{baC}_k + (-1)^k \mathbf{bC}_k\mathbf{a} + (-1)^k \mathbf{aC}_k\mathbf{b} - \mathbf{C}_k\mathbf{ab}].\end{aligned}$$

After cancellations, we see that the left and right sides of the equations are the same, so the identity (3.40) is proved.

In the equations above, three different products have been used, the geometric product $\mathbf{ab}$, the outer or wedge product $\mathbf{a} \wedge \mathbf{b}$, and the inner or dot product $\mathbf{a} \cdot \mathbf{b}$, often more than one such product in the same equation. In order to simplify the writing of such equations and to eliminate the use of a host of parentheses, we establish a convention regarding the order in which these operations are to be carried out. *Outer*, *inner*, and then *geometric* products are to be carried out in that order in evaluating any expression. For example, for the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$,

$$\mathbf{a} \cdot \mathbf{b} \wedge \mathbf{c}\, \mathbf{d} = \mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c})\mathbf{d} = [\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c})]\mathbf{d}. \tag{3.41}$$

One very useful identity, which follows from (3.40), is

$$\mathbf{a} \cdot (\mathbf{A}_r \wedge \mathbf{B}_s) = (\mathbf{a} \cdot \mathbf{A}_r) \wedge \mathbf{B}_s + (-1)^r \mathbf{A}_r \wedge (\mathbf{a} \cdot \mathbf{B}_s) = (-1)^{r+s+1}(\mathbf{A}_r \wedge \mathbf{B}_s) \cdot \mathbf{a} \tag{3.42}$$

and gives the *distributive law* for the inner product of a vector over the outer product of an r- and an s-vector. In Chap. 5, we study the *outer product* $\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_k$ of k vectors $\mathbf{a}_i \in \mathbb{R}^n$. This outer product can be directly expressed as the completely antisymmetric geometric product of those vectors. We have

$$\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_k = \frac{1}{k!} \sum_{\pi \in \Pi} \operatorname{sgn}(\pi) \mathbf{a}_{\pi(1)} \cdots \mathbf{a}_{\pi(k)}, \tag{3.43}$$

where π is a permutation on the indices $(1,2,\ldots,k)$, and $\mathrm{sgn}(\pi)=\pm 1$ according to whether π is an even or odd permutation, respectively.

A *simple k-vector* or k-blade is any geometric number which can be expressed as the outer product of k vectors. For example, the bivector $\mathbf{e}_{12}+\mathbf{e}_{23}$ is a 2-blade because $\mathbf{e}_{12}+\mathbf{e}_{23}=(\mathbf{e}_1-\mathbf{e}_3)\wedge\mathbf{e}_2$. On the other hand, the bivector $\mathbf{e}_{12}+\mathbf{e}_{34}$ is not a simple bivector. The *magnitude* of a k-vector $\mathbf{A}_k$ is defined by

$$|\mathbf{A}_k|=\sqrt{|\mathbf{A}_k\cdot\mathbf{A}_k^{\dagger}|}, \tag{3.44}$$

where $\mathbf{A}_k^{\dagger}=(-1)^{\frac{k(k-1)}{2}}\mathbf{A}_k$. If $\mathbf{A}_k$ is a k-blade for $k\geq 1$, then $\mathbf{A}_k^{-1}=(-1)^{\frac{k(k-1)}{2}}\frac{\mathbf{A}_k}{|\mathbf{A}_k|^2}$.

We have given here only a few of the most basic algebraic identities, others are given in the exercises below. For further discussion of basic identities, we suggest the references [1, 36, 43, 55]. In addition, the following links provide good online references:

http://users.tkk.fi/ppuska/mirror/Lounesto/

http://geocalc.clas.asu.edu/

http://www.mrao.cam.ac.uk/~clifford/pages/introduction.htm

http://www.science.uva.nl/ga/index.html

http://en.wikipedia.org/wiki/Geometric_algebra

Exercises

1. Let $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}\in\mathbb{R}^n$. Show that

$$\mathbf{abc}=(\mathbf{b}\cdot\mathbf{c})\mathbf{a}-(\mathbf{a}\cdot\mathbf{c})\mathbf{b}+(\mathbf{a}\cdot\mathbf{b})\mathbf{c}+\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}.$$

2. Show that

$$\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c}\wedge\mathbf{d})=(\mathbf{a}\cdot\mathbf{b})\mathbf{c}\wedge\mathbf{d}-\mathbf{b}\wedge[\mathbf{a}\cdot(\mathbf{c}\wedge\mathbf{d})].$$

3. Show that

$$\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c}\wedge\mathbf{d})=(\mathbf{a}\cdot\mathbf{b})\mathbf{c}\wedge\mathbf{d}-(\mathbf{a}\cdot\mathbf{c})\mathbf{b}\wedge\mathbf{d}+(\mathbf{a}\cdot\mathbf{d})\mathbf{b}\wedge\mathbf{c}.$$

4. Show that

$$(\mathbf{a}\wedge\mathbf{b})\boxtimes(\mathbf{cd})=[(\mathbf{a}\wedge\mathbf{b})\cdot\mathbf{c}]\mathbf{d}+\mathbf{c}[(\mathbf{a}\wedge\mathbf{b})\cdot\mathbf{d}],$$

where $A\boxtimes B=\frac{1}{2}(AB-BA)$ is the *anti-symmetric part* of the geometric product of A and B.

5. Show that

$$(\mathbf{a}\wedge\mathbf{b})\boxtimes(\mathbf{c}\wedge\mathbf{d})=[(\mathbf{a}\wedge\mathbf{b})\cdot\mathbf{c}]\wedge\mathbf{d}+\mathbf{c}\wedge[(\mathbf{a}\wedge\mathbf{b})\cdot\mathbf{d}].$$

6. Show that

$$(\mathbf{a}\wedge\mathbf{b})(\mathbf{c}\wedge\mathbf{d}) = (\mathbf{a}\wedge\mathbf{b})\cdot(\mathbf{c}\wedge\mathbf{d}) + (\mathbf{a}\wedge\mathbf{b})\boxtimes(\mathbf{c}\wedge\mathbf{d}) + (\mathbf{a}\wedge\mathbf{b})\wedge(\mathbf{c}\wedge\mathbf{d}).$$

7. Prove the identity (3.42) by using (3.40).
8. Show that if $\mathbf{A}_k \in \mathbb{G}_n^k$, then $\mathbf{A}_k^\dagger = (-1)^{\frac{k(k-1)}{2}}\mathbf{A}_k$. Also find a formula for $\mathbf{A}_k^-$ and $\widetilde{\mathbf{A}_k}$.
9. Using (3.44), find the magnitudes of the bivectors $\mathbf{e}_{12}+2\mathbf{e}_{34}$ and $\mathbf{e}_{12}+2\mathbf{e}_{23}$.
10. Let $\mathbf{A}_k$ be a k-blade, and define $\hat{\mathbf{A}}_k = \frac{\mathbf{A}_k}{|\mathbf{A}_k|}$. Show that $\hat{\mathbf{A}}_k^2 = \pm 1$. For what values of k is $\hat{\mathbf{A}}_k^2 = 1$, and for what values of k is $\hat{\mathbf{A}}_k^2 = -1$? We say that $\hat{\mathbf{A}}_k$ is a unit k-vector.
11. Let $\mathbf{F} = \mathbf{F}_k$ be a k-vector where $k \geq 1$, $\mathbf{x}$ a vector, and $\mathbf{B} = \mathbf{B}_r$ an r-vector where $r \geq k$. Prove the identity,

$$(\mathbf{B}\wedge\mathbf{x})\cdot\mathbf{F} = \mathbf{B}\cdot(\mathbf{x}\cdot\mathbf{F}) + (-1)^k(\mathbf{B}\cdot\mathbf{F})\wedge\mathbf{x}.$$

This identity will be needed in Chap. 16 when studying Lie algebras.

12. Let $\mathbf{A}_k, \mathbf{B}_k \in \mathbb{G}_n^k$ be k-vectors and $\mathbf{a}, \mathbf{b} \in \mathbb{G}_n^1$ be 1-vectors.

    (a) Show that
    $$\mathbf{A}_k\cdot(\mathbf{B}_k\wedge\mathbf{b}) = (\mathbf{A}_k\cdot\mathbf{B}_k)\mathbf{b} - (\mathbf{b}\cdot\mathbf{A}_k)\cdot\mathbf{B}_k.$$

    (b) Show that
    $$(\mathbf{a}\wedge\mathbf{A}_k)\cdot(\mathbf{B}_k\wedge\mathbf{b}) = (\mathbf{a}\cdot\mathbf{b})(\mathbf{A}_k\cdot\mathbf{B}_k) - (\mathbf{b}\cdot\mathbf{A}_k)(\mathbf{B}_k\cdot\mathbf{a}).$$

13. (a) Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be bivectors. Show that
    $$\mathbf{A}\boxtimes(\mathbf{B}\boxtimes\mathbf{C}) = (\mathbf{A}\cdot\mathbf{B})\mathbf{C} - (\mathbf{A}\cdot\mathbf{C})\mathbf{B} + (\mathbf{A}\wedge\mathbf{B})\cdot\mathbf{C} - (\mathbf{A}\wedge\mathbf{C})\cdot\mathbf{B}.$$

    (b) Show that $\mathbf{A}\circ(\mathbf{B}\boxtimes\mathbf{C}) = (\mathbf{A}\boxtimes\mathbf{B})\circ\mathbf{C}$, where $\mathbf{A}\circ\mathbf{B} = \frac{1}{2}(\mathbf{AB}+\mathbf{BA})$ is the symmetric part of the geometric product of $\mathbf{A}$ and $\mathbf{B}$.

3.5 Vector Derivative in $\mathbb{R}^n$

A powerful *vector deriviative* $\partial_{\mathbf{x}}$ at a point $\mathbf{x} \in \mathbb{R}^n$ can be easily defined in terms of its basic properties. Given a direction $\mathbf{a} \in \mathbb{R}^n$ at the point $\mathbf{x}$, we first define the $\mathbf{a}$-derivative $\mathbf{a}\cdot\partial_{\mathbf{x}}$ in the direction $\mathbf{a}$. Let $F : \mathscr{D} \longrightarrow \mathbb{G}_n$ be any continuous geometric algebra-valued function defined at all points $\mathbf{x} \in \mathscr{D}$ where $\mathscr{D} \subset \mathbb{R}^n$ is an open domain.[1] Recall that an *open r-ball* around a point $\mathbf{x}_0 \in \mathbb{R}^n$ is defined by

[1]An open domain is an open connected subset of $\mathbb{R}^n$. The topological propererties of $\mathbb{R}^n$ are rigorously defined and discussed in Michael Spivak's book, "Calculus on Manifolds" [92].

$$B_r(\mathbf{x}_0) := \{\mathbf{x} \in \mathbb{R}^n | \ |\mathbf{x}-\mathbf{x}_0| < r\}.$$

We say that $F(\mathbf{x})$ has an **a**-*derivative* $F_{\mathbf{a}}(\mathbf{x}) = \mathbf{a}\cdot\partial_{\mathbf{x}}F(\mathbf{x})$ at the point $\mathbf{x} \in \mathscr{D}$ provided the limit

$$F_{\mathbf{a}}(\mathbf{x}) = \lim_{h\to 0} \frac{F(\mathbf{x}+h\mathbf{a})-F(\mathbf{x})}{h} \tag{3.45}$$

exists, where $h \in \mathbb{R}$. We say that F is C^1 *differentiable* on the open domain $\mathscr{D}$ if $F_{\mathbf{a}}(\mathbf{x})$ exists and is continuous for all directions $\mathbf{a} \in \mathbb{R}^n$ and at all points $\mathbf{x} \in \mathscr{D}$. The definition of the **a**-derivative is equivalent to the *directional derivative* on $\mathbb{R}^n$, but the range of the function F lies in the much larger geometric algebra $\mathbb{G}_n$ of $\mathbb{R}^n$. In Chap. 15, when we study the differential geometry of a vector manifold, we will refine this definition.

To complete the definition of the vector derivative $\partial_{\mathbf{x}}F(\mathbf{x})$ of the geometric algebra-valued function $F(\mathbf{x})$, we require that the vector derivative $\partial_{\mathbf{x}}$ has the algebraic properties of a vector in $\mathbb{R}^n$. This means that for the orthonormal basis $(\mathbf{e})_{(n)}$ of $\mathbb{R}^n$, we have

$$\partial_{\mathbf{x}} = \sum_{i=1}^{n} \mathbf{e}_i\,(\mathbf{e}_i\cdot\partial_{\mathbf{x}}) = \sum_{i=1}^{n} \mathbf{e}_i\,\frac{\partial}{\partial x^i}, \tag{3.46}$$

where $\mathbf{e}_i\cdot\partial_{\mathbf{x}} = \frac{\partial}{\partial x^i}$ is the ordinary partial derivatives in the directions of the coordinate axis at the point $\mathbf{x} = \begin{pmatrix} x^1 & x^2 & \cdots & x^n \end{pmatrix} \in \mathbb{R}^n$. The geometric function $F(\mathbf{x})$ is said to have *order* C^p at the point $\mathbf{x} \in \mathbb{R}^n$ for a positive integer $p \in \mathbb{N}$, if all p^{th} order partial derivatives of $F(\mathbf{x})$ exist and are continuous at the point $\mathbf{x}$. If $F(\mathbf{x})$ has continuous partial derivatives of all orders, we say that $F(\mathbf{x})$ is a C^∞ function [92, p.26].

As our first example, consider the identity function $f(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. We calculate

$$f_{\mathbf{a}}(\mathbf{x}) = \mathbf{a}\cdot\partial_{\mathbf{x}}\mathbf{x} = \lim_{h\to 0}\frac{f(\mathbf{x}+h\mathbf{a})-f(\mathbf{x})}{h} = \lim_{h\to 0}\frac{\mathbf{x}+h\mathbf{a}-\mathbf{x}}{h} = \mathbf{a} \tag{3.47}$$

for all $\mathbf{x} \in \mathbb{R}^n$ and all $\mathbf{a} \in \mathbb{G}^1$. It follows that the identity function is differentiable on $\mathbb{R}^n$. From (3.46) and (3.47), we calculate the vector derivative of the identity function

$$\partial_{\mathbf{x}}\mathbf{x} = \sum_{i=1}^{n} \mathbf{e}_i(\mathbf{e}_i\cdot\partial_{\mathbf{x}}\mathbf{x}) = \sum_{i=1}^{n} \mathbf{e}_i\mathbf{e}_i = n, \tag{3.48}$$

which gives the number of *degrees of freedom* at the point $\mathbf{x} \in \mathbb{R}^n$. It follows immediately from (3.48) that

$$\partial_{\mathbf{x}}\cdot\mathbf{x} = n, \quad \text{and} \quad \partial_{\mathbf{x}}\wedge\mathbf{x} = 0. \tag{3.49}$$

From the basic identities (3.47) and (3.49), together with Leibniz product rule for directional or partial deriviatives, other identities easily follow. For example, since $\partial_{\mathbf{x}} \wedge \mathbf{x} = 0$, it follows that

$$0 = \mathbf{a} \cdot (\partial_{\mathbf{x}} \wedge \mathbf{x}) = \mathbf{a} \cdot \partial_{\mathbf{x}} \mathbf{x} - \partial_{\mathbf{x}} \mathbf{x} \cdot \mathbf{a},$$

so that

$$\partial_{\mathbf{x}} \mathbf{x} \cdot \mathbf{a} = \mathbf{a} \cdot \partial_{\mathbf{x}} \mathbf{x} = \mathbf{a}. \tag{3.50}$$

We also have

$$\mathbf{a} \cdot \partial_{\mathbf{x}} |\mathbf{x}|^2 = \mathbf{a} \cdot \partial_{\mathbf{x}} (\mathbf{x} \cdot \mathbf{x}) = 2\mathbf{a} \cdot \mathbf{x}, \tag{3.51}$$

$$\partial_{\mathbf{x}} \mathbf{x}^2 = \dot{\partial}_{\mathbf{x}}\, \dot{\mathbf{x}} \cdot \mathbf{x} + \dot{\partial}_{\mathbf{x}}\, \mathbf{x} \cdot \dot{\mathbf{x}} = 2\mathbf{x}, \tag{3.52}$$

and consequently,

$$\partial_{\mathbf{x}} |\mathbf{x}| = \partial_{\mathbf{x}} (\mathbf{x}^2)^{1/2} = \frac{1}{2} (\mathbf{x}^2)^{-1/2} 2\mathbf{x} = \hat{\mathbf{x}}. \tag{3.53}$$

The dots over the arguments denote which argument is being differentiated.

There is one further property of the vector derivative $\partial_{\mathbf{x}}$ in $\mathbb{R}^n$ that needs to be discussed, and that is the *integrability condition*

$$\partial_{\mathbf{x}} \wedge \partial_{\mathbf{x}} = \sum_{i,j} \mathbf{e}_{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} = \sum_{i<j} \mathbf{e}_{ij} \left(\frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} - \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^i} \right) = 0, \tag{3.54}$$

since partial derivatives commute for differentiable functions. This property for the vector derivative in $\mathbb{R}^n$ depends heavily upon the fact that in a flat space a constant orthonormal basis, in this case the standard orthonormal basis $(\mathbf{e})_{(n)}$, can be chosen at all points $\mathbf{x} \in \mathbb{R}^n$, thus making the derivatives $\frac{\partial \mathbf{e}_j}{\partial x^i} = 0$ for all $1 \le i, j \le n$. When we discuss the vector derivative on a curved space, say, a cylinder or a sphere, this property is no longer true as we will later see.

The above differentiation formulas will be used in the study of the structure of a linear operator on $\mathbb{R}^n$. In Chap. 13, and in later chapters, we generalize the concept of both the $\mathbf{a}$-derivative and the vector derivative in $\mathbb{R}^n$ to the concept of the corresponding derivatives on a k-dimensional surface in $\mathbb{R}^n$ or in an even more general *pseudo-Euclidean space* $\mathbb{R}^{p,q}$. Because the $\mathbf{a}$-derivative and the vector derivative are defined ultimately in terms of partial derivatives, the usual rules of differentiation remain valid. However, care must be taken when applying these rules since we no longer have universal commutativity in the geometric algebra $\mathbb{G}_n$. Other differentiation formulas, which follow easily from the basic formulas above, are given as exercises below. All of these formulas remain valid when we generalize to the vector derivative for a curved surface, except formulas using the property (3.54) which only apply in flat spaces.

Exercises

Calculate or verify the following vector derivatives:

1. $\partial_{\mathbf{x}}|\mathbf{x}|^k = k|\mathbf{x}|^{k-2}\mathbf{x}$,
2. $\mathbf{a}\cdot\partial_{\mathbf{x}}|\mathbf{x}|^k = k|\mathbf{x}|^{k-2}\mathbf{a}\cdot\mathbf{x}$,
3. $\partial_{\mathbf{x}}\frac{\mathbf{x}}{|\mathbf{x}|^k} = \frac{n-k}{|\mathbf{x}|^k}$,
4. $\mathbf{a}\cdot\partial_{\mathbf{x}}\frac{\mathbf{x}}{|\mathbf{x}|^k} = \frac{\mathbf{a}-k(\mathbf{a}\cdot\hat{\mathbf{x}})\hat{\mathbf{x}}}{|\mathbf{x}|^k}$,
5. $\partial_{\mathbf{x}}\log|\mathbf{x}| = \frac{\mathbf{x}}{|\mathbf{x}|^2} = \mathbf{x}^{-1}$.
6. $\mathbf{a}\cdot\partial_{\mathbf{x}}\log|\mathbf{x}| = \frac{\mathbf{a}\cdot\mathbf{x}}{|\mathbf{x}|^2}$.
7. $\mathbf{a}\cdot\partial_{\mathbf{x}}\hat{\mathbf{x}} = \frac{(\mathbf{a}\wedge\hat{\mathbf{x}})\hat{\mathbf{x}}}{|\mathbf{x}|}$.
8. $\mathbf{a}\cdot\partial_{\mathbf{x}}\sin\mathbf{x} = \mathbf{a}\cdot\partial_{\mathbf{x}}\hat{\mathbf{x}}\sin|\mathbf{x}| = \frac{(\mathbf{a}\wedge\hat{\mathbf{x}})\hat{\mathbf{x}}}{|\mathbf{x}|}\sin|\mathbf{x}| + (\mathbf{a}\cdot\hat{\mathbf{x}})\hat{\mathbf{x}}\cos|\mathbf{x}|$.
9. $\mathbf{a}\cdot\partial_{\mathbf{x}}\sinh\mathbf{x} = \mathbf{a}\cdot\partial_{\mathbf{x}}\hat{\mathbf{x}}\sinh|\mathbf{x}| = \frac{(\mathbf{a}\wedge\hat{\mathbf{x}})\hat{\mathbf{x}}}{|\mathbf{x}|}\sinh|\mathbf{x}| + (\mathbf{a}\cdot\hat{\mathbf{x}})\hat{\mathbf{x}}\cosh|\mathbf{x}|$.
10. $\partial_{\mathbf{x}}\sin\mathbf{x} = \frac{n-1}{|\mathbf{x}|}\sin|\mathbf{x}| + \cos|\mathbf{x}|$.
11. $\partial_{\mathbf{x}}\sinh\mathbf{x} = \frac{n-1}{|\mathbf{x}|}\sinh|\mathbf{x}| + \cosh|\mathbf{x}|$.
12. $\mathbf{a}\cdot\partial_{\mathbf{x}}\exp\mathbf{x} = (\mathbf{a}\cdot\hat{\mathbf{x}})\hat{\mathbf{x}}e^{\mathbf{x}} + \frac{(\mathbf{a}\wedge\hat{\mathbf{x}})\hat{\mathbf{x}}}{|\mathbf{x}|}\sinh|\mathbf{x}|$.
13. $\partial_{\mathbf{x}}\exp\mathbf{x} = e^{\mathbf{x}} + \frac{n-1}{|\mathbf{x}|}\sinh|\mathbf{x}| = (1+\frac{n-1}{|\mathbf{x}|}e^{-\mathbf{x}}\sinh|\mathbf{x}|)e^{\mathbf{x}}$.
14. $\mathbf{a}\cdot\partial_{\mathbf{x}}(\mathbf{x}\wedge\mathbf{b})^2 = 2(\mathbf{a}\wedge\mathbf{b})\cdot(\mathbf{x}\wedge\mathbf{b})$.
15. $\partial_{\mathbf{x}}(\mathbf{x}\wedge\mathbf{b})^2$.
16. $\mathbf{a}\cdot\partial_{\mathbf{x}}|\mathbf{x}\wedge\mathbf{b}| = -\frac{(\mathbf{a}\wedge\mathbf{b})\cdot(\mathbf{x}\wedge\mathbf{b})}{|\mathbf{x}\wedge\mathbf{b}|}$.
17. $\partial_{\mathbf{x}}|\mathbf{x}\wedge\mathbf{b}|$.
18. $\mathbf{a}\cdot\partial_{\mathbf{x}}\frac{\mathbf{x}\wedge\mathbf{b}}{|\mathbf{x}\wedge\mathbf{b}|} = \frac{\mathbf{a}\wedge\mathbf{b}+(\mathbf{a}\wedge\mathbf{b})\cdot(\widehat{\mathbf{x}\wedge\mathbf{b}})\widehat{\mathbf{x}\wedge\mathbf{b}}}{|\mathbf{x}\wedge\mathbf{b}|}$.
19. $\partial_{\mathbf{x}}\frac{\mathbf{x}\wedge\mathbf{b}}{|\mathbf{x}\wedge\mathbf{b}|}$.
20. $\mathbf{a}\cdot\partial_{\mathbf{x}}\exp(\mathbf{x}\wedge\mathbf{b})$.
21. $\partial_{\mathbf{x}}\exp(\mathbf{x}\wedge\mathbf{b})$.
22. Using the property (3.54), show that the vector derivative in $\mathbb{R}^n$ satisfies the property
$$(\mathbf{a}\wedge\mathbf{b})\cdot(\partial_{\mathbf{x}}\wedge\partial_{\mathbf{x}}) = [\mathbf{b}\cdot\partial_{\mathbf{x}},\mathbf{a}\cdot\partial_{\mathbf{x}}] - [\mathbf{b},\mathbf{a}]\cdot\partial_{\mathbf{x}} = 0,$$
where the brackets are defined by
$$[\mathbf{b}\cdot\partial_{\mathbf{x}},\mathbf{a}\cdot\partial_{\mathbf{x}}] = \mathbf{b}\cdot\partial_{\mathbf{x}}\,\mathbf{a}\cdot\partial_{\mathbf{x}} - \mathbf{a}\cdot\partial_{\mathbf{x}}\,\mathbf{b}\cdot\partial_{\mathbf{x}}, \tag{3.55}$$
and
$$[\mathbf{b},\mathbf{a}] = \mathbf{b}\cdot\partial_{\mathbf{x}}\mathbf{a} - \mathbf{a}\cdot\partial_{\mathbf{x}}\mathbf{b}, \tag{3.56}$$
and where $\mathbf{a}=\mathbf{a}(\mathbf{x})$ and $\mathbf{b}=\mathbf{b}(\mathbf{x})$ are any differentiable vector-valued functions at the point $\mathbf{x}\in\mathbb{R}^n$.

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