Preface

I am sure that something must be found. There must exist a notion of generalized functions which are to functions what the real numbers are to the rationals (G. Peano, 1912)

Not that much effort is needed, for it is such a smooth and simple theory (F. Trèves, 1975)

In undergraduate physics a lecturer will be tempted to say on certain occasions: “Let \( \delta(x) \) be a function on the line that equals 0 away from 0 and is infinite at 0 in such a way that its total integral is 1. The most important property of \( \delta(x) \) is exemplified by the identity

\[
\int_{-\infty}^{\infty} \phi(x) \delta(x) \, dx = \phi(0),
\]

where \( \phi \) is any continuous function of \( x \).” Such a function \( \delta(x) \) is an object that one frequently would like to use, but of course there is no such function, because a function that is 0 everywhere except at one point has integral 0. All the same, it is important to realize what our lecturer is trying to accomplish: to describe an object in terms of the way it behaves when integrated against a function. It is for such purposes that the theory of distributions, or “generalized functions,” was created. It can be formulated in all dimensions, its mathematical scope is vast, and it has revolutionized modern analysis.

One way to elaborate on the distributional point of view\(^1\) is to note that a pointwise definition of functions is not very relevant to many situations arising in engineering or physics. This is due to the fact that physical observations often do not represent sharp computations at a single point in space-time but rather averages of fluctuations in small but finite regions in space-time. This is an essential point in signal theory, where there are limitations to the determination of pulse lengths, and

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\(^1\) Here we follow the masterly exposition of Varadarajan [22, p. 185].
in quantum theory, where the electromagnetic fields of elementary particles cannot be measured unless one uses a macroscopic test body. From the mathematical point of view one can say that a measurement of a physical quantity by means of a test body yields an average of the values of that quantity in a very small region, the latter being represented by a smooth function that is zero outside a small domain. One replaces the test bodies by these functions, which are naturally called test functions. The value thus measured is a function on the space of test functions, and the interpretation of the measurement as an average makes it clear that this function must be linear. Thus, if $T$ is the space of test functions (unspecified at this point), physical quantities assign real or complex values to functions in $T$. In keeping with our idea that measurements are averages, we recognize that sometimes things are not so bad and that actual point measurements are possible. Thus ordinary functions are also allowed to be viewed as functionals on $T$. If $f$ is such an ordinary function, it represents the following functional on $T$:

$$\phi \mapsto \int f(x)\phi(x) \, dx \in \mathbb{C}.$$ 

However, since we admit measurements that are too singular to be represented by ordinary functions, we refer to the general functionals on $T$ as generalized functions or distributions. We have been vague about what the space is in which we are operating and also what functions are chosen as test functions. This actually is a great strength of these ideas, because the methods evidently apply without any restriction on the nature or the dimensions of the space. In this book, however, we restrict ourselves to the most important case, that of open subsets of the Euclidean spaces $\mathbb{R}^n$.

Distributions are to functions what the real numbers $\mathbb{R}$ are to the rational numbers $\mathbb{Q}$. In $\mathbb{R}$, the cube root of any number also belongs to $\mathbb{R}$, as does the logarithm of the absolute value of a nonzero number; by contrast, $\sqrt{2}$ and $\log 2$ do not belong to $\mathbb{Q}$. Moreover, $\mathbb{R}$ is the smallest extension of $\mathbb{Q}$ having such properties, while every real number can be approximated by rationals with arbitrary precision. Similarly, distributions are always infinitely differentiable, which is not true of all functions. Here, too, distributions are the smallest possible extension of the test functions satisfying this property, while every distribution can be approximated in the appropriate sense by test functions with arbitrary precision. Continuing the analogy, we mention that differential equations may have distributional solutions in situations where there are no classical solutions, that is, given by differentiable functions. In numerous problems it is of great advantage that solutions exist, even at the penalty of introducing new objects such as distributions, because the solutions can be subject to further study. The theory of distributions provides many tools for the investigation of these so-called weak solutions; for example, these tools enable one to determine when and where distributions are actually functions. One of the early triumphs of distribution theory was the result that every partial differential equation with constant coefficients has a fundamental solution in the sense of distributions: classically, nothing comparable is available.
Fourier theory is another branch of analysis in which a suitable subclass of all distributions helps to clarify many issues. This theory is a far-reaching generalization of writing a vector \( x = (x_1, \ldots, x_n) \) in \( \mathbb{R}^n \) as

\[
x = \sum_{k=1}^{n} x_k e_k,
\]

that is, as a superposition of a finite sum of multiples of the basis vectors \( e_k \). Analogously, in Fourier analysis one attempts to write functions or even distributions as superpositions of basic functions. In this case, finitely many functions do not suffice, but the collection of all bounded exponential functions turns out to be a good choice: bounded, because unbounded exponentials grow too fast at infinity, and exponential, because such functions are simultaneous eigenvectors of all partial derivatives.

The sense in which the infinite superposition represents the original object then becomes an important issue: is the convergence pointwise or uniform, or in a smeared sense? Fourier analysis in the distributional setting enables one to handle problems that classically were out of reach, as well as many new ones. So one obtains, working modulo \( 2\pi \),

\[
\delta(x) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{ikx}.
\]

This formula goes back to Euler, except that he found the sum to be equal to 0 when \( x \) is away from 0.

Hörmander’s monumental treatise [11] on linear partial differential equations and Harish-Chandra’s pioneering work [10] on harmonic analysis on semisimple Lie groups over the fields of real, complex, or \( p \)-adic numbers are but two of the rich fruits borne by Schwartz’s text [20], which gave birth to the theory of distributions.

This book aims to be a thorough, yet concise and application-oriented, introduction to the theory of distributions that can be covered in one semester. These constraints forced us to make choices: we try to be rigorous but do not construct a complete theory that prepares the reader for all aspects and applications of distributions. It supplies a certain degree of rigor for a kind of calculation that people long ago did completely heuristically, and it establishes what is legitimate and what is not. The amount of functional analysis that is needed in our treatment is reduced to a bare minimum: only the principle of uniform boundedness is used, while the Hahn–Banach theorems are applied to give alternative proofs, with one exception, of results obtained by different methods. On the other hand, in our exposition of the theory and, in particular, in the problems, we stress applications and interactions with other parts of mathematics.

As a result of this approach our text is complementary to the books [13] and [14] by A.W. Knapp, also published in the Cornerstones series. Building on firm foundations in functional analysis and measure theory, Knapp develops the theory rigorously and in greater depth and wider context than we do, by treating pseudodifferential operators on manifolds, for instance. In many ways our text is introductory;
on the other hand, it presents students of (theoretical) physics or electrical engineering with an idea of what distributions are all about from the mathematical point of view, while giving applied or pure mathematicians a taste of the power of distributions as a natural method in analysis. Our aim is to make the reader familiar with the essentials of the theory in an efficient and fairly rigorous way, while emphasizing the applications.

Solutions of important ordinary and partial differential equations, such as the equation for an electrical LRC network, those of Cauchy–Riemann, Laplace, and Helmholtz and the heat and wave equations, are studied in great detail. Tools for the investigation of the regularity of the solution, that is, its smoothness, are developed. Topics in signal reconstruction have also been treated, such as the mathematical theory underlying CT (= computed tomography) scanners as well as results on band-limited functions. The fundamentals of the theory of complex-analytic functions in one variable are efficiently derived in the context of distributions. In order to make the book self-contained, various results on special functions that are used in our treatment are deduced as consequences of the theory itself, wherever possible.

A large number of problems is included; they are found at the end of each chapter. Some of these illustrate the theory itself, while others explore its relevance to other parts of mathematics. They vary from straightforward applications of the theory to theorems or projects examining a topic in some depth. In particular, important aspects of multidimensional real analysis are studied from the point of view of distributions. Complete solutions to 146 of the 281 problems are provided; problems for which solutions are available are marked by the symbol *. A great number of the remaining exercises are supplied with copious hints, and many of the more difficult problems have been tested in take-home examinations.

In more technical terms, the first eleven chapters cover the basics of general distributions. Specifically, Chap. 10 presents a systematic calculus of pullback and pushforward for the transformation of distributions under a change of variables, whereas Chap. 13 considers complex-analytic one-parameter families of distributions with the aim of obtaining fundamental solutions of certain partial differential operators. Chap. 14 then goes on to treat the Fourier transform of the subclass of tempered distributions in the general, aperiodic case, which is of fundamental importance for the subsequent Chaps. 15–19. Chap. 15 discusses the notion of a distribution kernel of a continuous linear mapping. This notion enables an elegant verification of many properties of such mappings. More generally, it enables aspects of the theory of distributions to be surveyed from a fresh and unifying point of view, as is exemplified by many of the problems in the chapter. The Fourier inversion formula is used in a novel proof of the Kernel Theorem. The Fourier transform is applied in Chap. 16 to study the periodic case and in Chap. 17 to construct additional fundamental solutions. Chap. 18 deals with the Fourier transforms of compactly supported distributions, and Chap. 19 considers rudiments of the theory of Sobolev spaces.

Mathematically sophisticated readers, having perused the first ten chapters, might prefer to proceed immediately to Chaps. 14 and 15.
Important characteristics of the present treatment of the theory of distributions are the following. The theory as presented provides a highly coherent context with a strong potential for unification of seemingly distant parts of analysis. A systematic use of the operations of pullback and pushforward enables the development of a very clean and concise notation. A survey of distribution theory in the framework of distribution kernels allows a description that is algebraic rather than analytic in nature, and makes it possible to study distributions with a minimal use of test functions. In particular, within this framework some more advanced aspects of distribution theory can be developed in a highly efficient manner and transparent proofs can be given. The treatment emphasizes the role of symmetry in obtaining short arguments. In addition, distributions invariant under the actions of various groups of transformations are investigated.

Our preferred theory of integration is that of Riemann, because it will be more familiar to most readers than that of Lebesgue. In some instances, however, our arguments might be slightly shortened by the use of Lebesgue’s theory. In the very limited number of cases in which Lebesgue integration is essential, we mention this explicitly. The reader who is not familiar with measure theory may safely skip these passages.

On the other hand, in the theory of distributions Radon measures arise naturally as linear forms defined on compactly supported continuous functions, and therefore the Daniell approach to the theory of integration, which emphasizes linear forms acting on functions instead of functions acting on sets, is very natural in this setting. In the Appendix, Chap. 20, we survey the theory of Lebesgue integration with respect to a measure from this point of view. Although the approach seems very appropriate in our context, we are aware of the fact that it is of limited value to the mathematical probabilist, who primarily requires a theory of integration on function spaces, which are not usually locally compact.

We strongly feel that a mathematical style of writing is appropriate for our purposes, so the book contains a certain amount of theorem–proof text. The reader of a text at this level of mathematical sophistication rightly expects to find all the information needed to follow the argument as well as clear expositions of difficult points, and the theorem–proof format is a time-honored vehicle for conveying these. Furthermore, in theorems one summarizes useful information for future application. Important results (for instance, the Fourier inversion formula) often get several proofs; in this manner different aspects or unexpected relations are brought to the fore.

The present text has evolved from a set of notes for courses taught at Utrecht University over the last twenty years, mainly to bachelor-degree students in their third year of theoretical physics and/or mathematics. In those courses, familiarity with measure theory, functional analysis, or even some of the more theoretical aspects of real analysis, such as compactness, could not be assumed. Since this book addresses the same type of audience, the present text was therefore designed to be essentially self-contained: the reader is assumed to have merely a working knowledge of linear algebra and of multidimensional real analysis (see [7], for instance), while only a
few of the problems also require some acquaintance with the residue calculus from complex analysis in one variable. In some cases, the notion of a group will be encountered, mainly in the form of a (one-parameter) group of transformations acting on $\mathbb{R}^n$.

Each time the course was taught, the notes were corrected and refined, with the help of the students; we are grateful to them for their remarks. In particular, J.J. Kuit made a considerable number of original contributions and we benefitted from fruitful discussions with him. M.A. de Reus suggested many improvements. Also, we express our gratitude to our colleagues E.P. van den Ban, for making available the notes for his course in 1987 on distributions and Fourier transform and for very constructive criticism of a preliminary draft, and R.W. Bruggeman, for the improvements and additional problems that he contributed over the past few years. In addition, T.H. Koornwinder read substantial parts of the manuscript with great care when preparing a course on distributions, and contributed significantly, by many valuable queries and comments, to the accuracy of the final version.

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The responsibility for any imprecisions remains entirely ours; we would be grateful to be told of them, at j.a.c.kolk@uu.nl.

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