CHAPTER 4 Paper-Folding, Polyhedra-Building, and Number Theory

4.1 INTRODUCTION: FORGING THE LINK BETWEEN GEOMETRIC PRACTICE AND MATHEMATICAL THEORY

In this chapter we carry the paper-folding procedures and the mathematics of paper-folding further than we did in [2]. However, in order to make this account as self-contained as possible, we will recall, in Section 2, the systematic folding procedures from Chapter 4 of [2] that enabled us to approximate, to any degree of accuracy desired, any regular convex N-gon.[1] We will see, from the examples, that the process also enables us to fold certain regular star $\{\frac{b}{a}\}$-gons,[2] some of which are shown in Figure 1. For brevity we will refer to the approximations we obtain for both the regular convex N-gons and the regular star $\{\frac{b}{a}\}$-gons (when $a \geq 2$) as ***quasi-regular polygons***. In most cases the context will make it unnecessary to state whether or not they are genuine convex polygons. Sometimes we

[1] It is not uncommon for people to adopt special conventionally-permitted paper-folding procedures that produce exact constructions for certain families of regular polygons (see, for example, [13]).

[2] We will give a more precise definition of these star polygons in Section 2. Note that, when we speak of a $\{\frac{b}{a}\}$-gon, we assume that a, b are coprime; but, if a, b emerge from a *calculation* (see (3), (4) of Section 2), they may not, at that stage, be coprime.

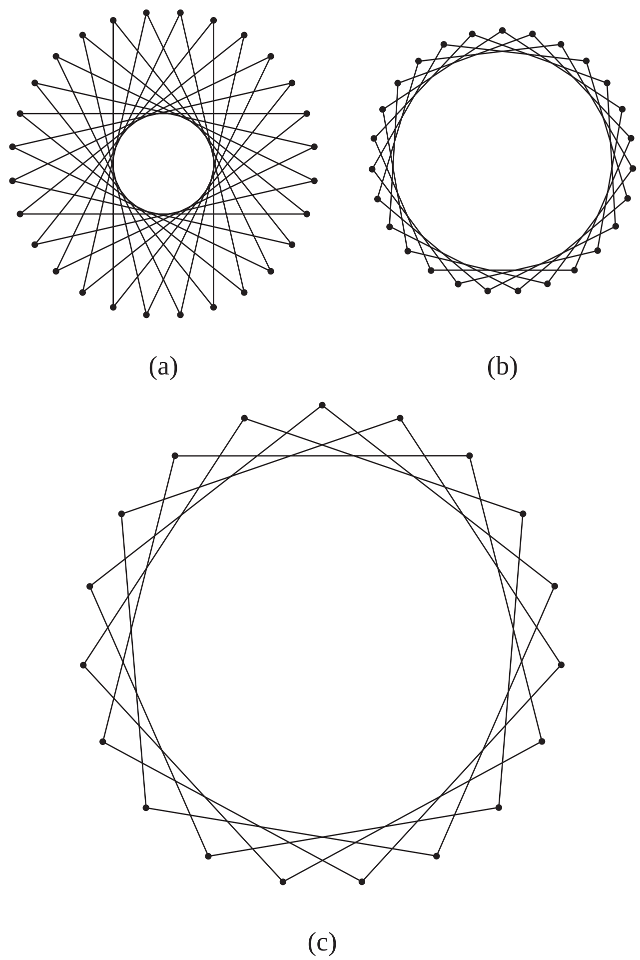

FIGURE 1 **Some star $\{\frac{b}{a}\}$-gons. (a) $a = 28, b = 11$. (b) $a = 27, b = 5$. (c) $a = 19, b = 4$.**

refer to a *star b-gon* to mean a star $\{\frac{b}{a}\}$-gon for some a prime to b and satisfying $a < \frac{b}{2}$.

In the process of recalling the folding procedures we will also describe again how to construct a ***symbol*** that enables one to read off the folding instructions for constructing any quasi-regular N-gon, where N is odd and ≥ 3. Surprisingly, the construction of this symbol led us, in [2], to the Quasi-Order Theorem in base 2 in number theory (the “2” came naturally from the paper-folding, since the folding procedure always involved *bisecting* angles). We will restate that remarkable theorem in Section 2 (the proof begins on page 130 of [2]) and illustrate it with some examples.

In Section 3 we describe precisely how to make secondary fold lines, to augment the primary fold lines described in Section 2, so that we can construct *any* quasi-regular polygon.

Section 4 gives instructions for how to use the folded strips of paper that produce 3-, 4-, 5-gons to build certain polyhedra. Many of these models are very striking to behold — but they have other unusual characteristics. Some of them come apart into straight strips, thus making them easy to store, others collapse in unexpected ways, and, as we show in Chapter 8, some are also connected with beautiful mathematical ideas; and we believe that this connection is no accident. Thus Section 4 is not itself mathematical in nature, but it can stimulate and enliven some fine mathematics.

In Section 5 we generalize the Quasi-Order Theorem to a general base t. In doing this we then are forced to give up the interpretation of folding paper, but the analogy is clear, and the result is truly remarkable. The proof of the general theorem is scarcely more difficult than that of the Quasi-Order Theorem in base 2, but the statement of the generalization is not at all obvious. If you try generalizing the Quasi-Order Theorem of Section 2 before reading Section 5, we think you will see what we mean.

• • • BREAK 1

Why is it no restriction on the notion of a star $\{\frac{b}{a}\}$-gon to insist that $a < \frac{b}{2}$? [Hint: What would be the difference between the star $\{\frac{5}{2}\}$-gon and the star $\{\frac{5}{3}\}$-gon?]

4.2 WHAT CAN BE DONE WITHOUT EUCLIDEAN TOOLS

We begin by recalling how the question of whether or not, for a given N, it is possible to construct a regular N-gon using Euclidean tools (straight edge and compass) has fascinated people since the time of the ancient Greeks. In fact, Gauss (1777–1855) completely settled the question by proving that a Euclidean construction of a regular N-gon is possible *if and only if* the number of sides N is of the form $N = 2^c \prod \rho_i$, where the numbers ρ_i are distinct Fermat primes — that is, primes of the form $F_n = 2^{2^n} + 1$. Now, since F_n is only known to be prime for

$$F_0 = 3, \quad F_1 = 5, \quad F_2 = 17, \quad F_3 = 257, \quad F_4 = 65537,$$

it is clear that a Euclidean construction of a regular N-gon is known to exist for very few values of N; and even for these N we do not know, at the time of writing, an explicit construction in all cases.

Despite this restrictive result we still would like, somehow, to construct *all* regular polygons. Our approach (as in [2]) is to modify the question so that, instead of asking for an exact construction,[3] we ask:

For which $N \geq 3$ ***is it possible, systematically and explicitly, to construct quasi-regular (convex)*** N***-gons by paper-folding?***

Surprisingly, as we will show, the answer to this question is: *all* $N \geq 3$. Furthermore, in showing precisely how this is done, we receive a bonus, that is, we will also be able to construct all possible quasi-regular $\{\frac{b}{a}\}$-gons.

Let us now begin by recalling the precise and fundamental folding procedure, involving a straight strip of paper with parallel edges. We suggest that you will find it useful to have a long strip of paper handy. Adding-machine tape or ordinary unreinforced gummed tape work well.

Assume that we have a straight strip of paper that has certain vertices marked on its top and bottom edges, at equally spaced intervals, and that also has *creases* or *folds* along straight lines emanating from the vertices at the top edge of the strip. Further assume that the creases at those vertices labeled A_{nk}, $n = 0, 1, 2, \cdots$ (see Figure 2), which are on the top edge, form identical angles of $\frac{a\pi}{b}$ with the top edge, with an identical angle of $\frac{a\pi}{b}$ between the crease along the lines $A_{nk}A_{nk+2}$ and the crease along $A_{nk}A_{nk+1}$ (as shown in Figure 2(a)). If we fold this strip on $A_{nk}A_{nk+2}$, as shown in Figure 2(b), and then twist the tape so that it folds on $A_{nk}A_{nk+1}$, as shown in Figure 2(c), the direction of the *top edge* of the tape will be rotated through an angle of $2(\frac{a\pi}{b})$. We call this process of **f**olding **a**nd **t**wisting the ***FAT***-algorithm (see any of [4, 5, 6, 7, 8, 11]).

Now consider the A_{nk} along the top of the tape, with k fixed and n varying. If the FAT-algorithm is performed on a sequence of angles, each of measure $\frac{a\pi}{b}$, at the vertices given by $n = 0, 1, 2, \cdots, b-1$, then the top of the tape will have turned through an angle of $2a\pi$. Thus the vertex A_{bk} will come into coincidence with A_0; and the top edge of the tape will have visited every ath vertex of a bounding regular convex b-gon, thus creating a quasi-regular $\{\frac{b}{a}\}$-gon. As an example, see Figure 6(c), where $a = 2$ and $b = 7$. (In order to fit with our usage of "N-gon" we make a slight adaptation of the Coxeter notation for star polygons (see [1]), so that

[3]Of course, in many cases, such as when $N = 2^c$ (with $c \geq 2$), we can give exact constructions.

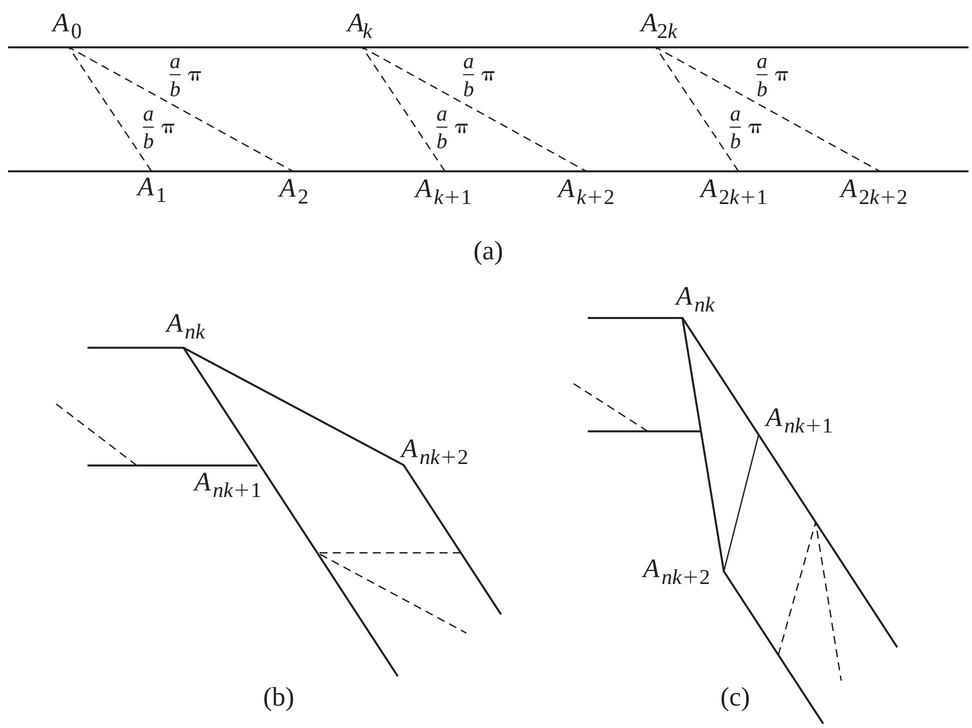

FIGURE 2

when we refer to a ***quasi-regular*** $\{\frac{b}{a}\}$***-gon*** we mean a connected sequence of edges that visits every ath vertex of a quasi-regular b-gon. Thus our N-gon is the special star $\{\frac{N}{1}\}$-gon. When labeling a convex polygon this way we may well use a lower case letter instead of N.)

Figure 3 illustrates how a suitably creased strip of paper may be folded by the FAT-algorithm to produce a quasi-regular p-gon, (or $\{\frac{p}{1}\}$-gon). In Figure 3 we have written V_k instead of A_{nk}, since it is more natural in this particular context.

Let us now illustrate how the FAT-algorithm may be used to fold a regular convex 8-gon. Figure 4(a) shows a straight strip of paper on which the dotted lines indicate certain special exact crease lines. In fact, these crease lines occur at equally spaced intervals along the top of the tape, so that the angles occurring at the top of each vertical line are (from left

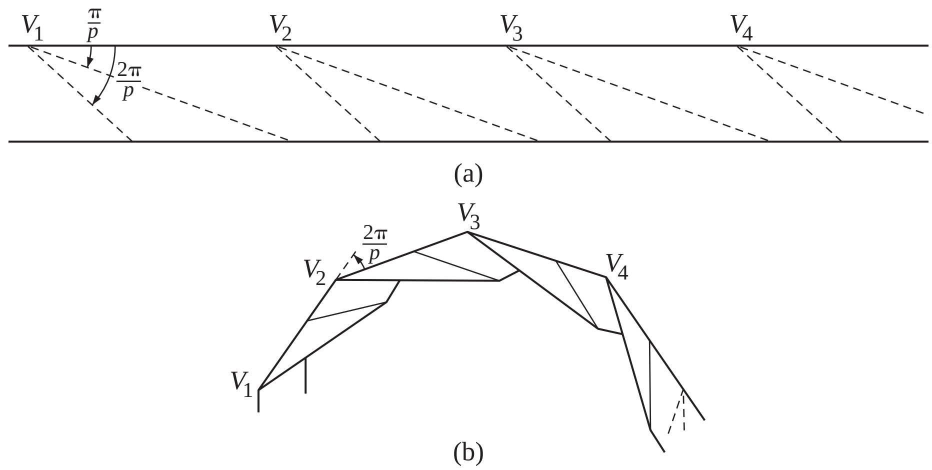

FIGURE 3

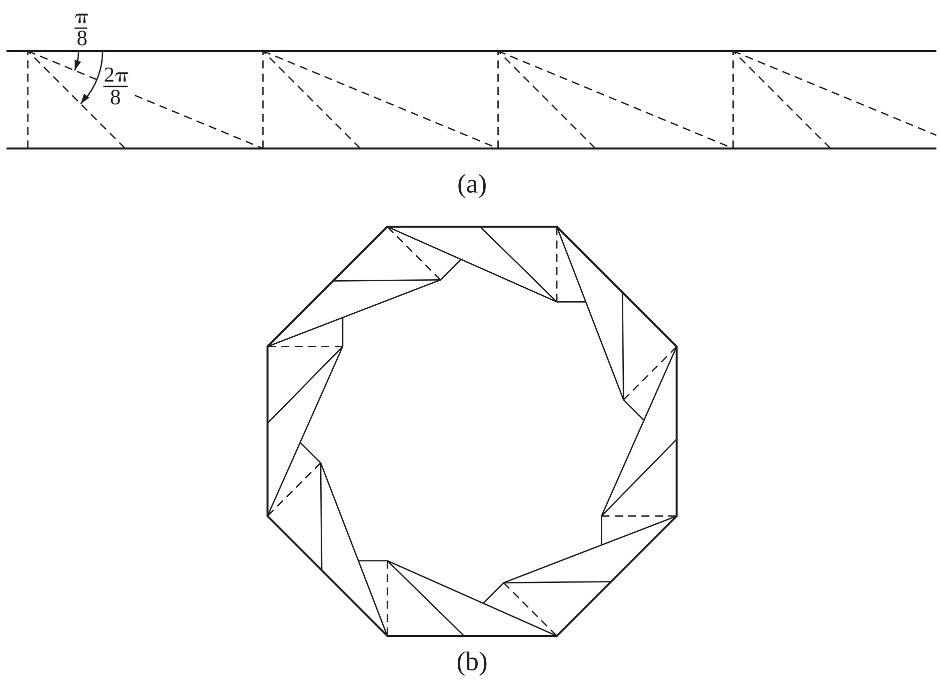

FIGURE 4

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to right) $\frac{\pi}{2}$, $\frac{\pi}{4}$, $\frac{\pi}{8}$, $\frac{\pi}{8}$. Figuring out how to fold a strip of tape to obtain this arrangement of crease lines is very unlikely to cause the reader any difficulty, but complete instructions are given in [8]. Our immediate interest is focused on the observation that this tape has, at equally spaced intervals along the top edge, adjacent angles each measuring $\frac{\pi}{8}$, and we can therefore execute the FAT-algorithm at 8 consecutive vertices along the top of the tape to produce an exact regular convex 8-gon, as shown in Figure 4(b). (Of course, in constructing the model, one would cut the tape on the first vertical line and glue a section at the end to the beginning so that the model would form a closed polygon.)

Notice that the tape shown in Figure 4(a) also has suitable crease lines that make it possible to use the FAT-algorithm to fold a regular convex 4-gon. We leave this as an exercise for the reader and turn to a more challenging construction, the regular convex 7-gon.

Now, since the 7-gon is the first regular polygon that we encounter for which there does not exist a Euclidean construction, we are faced with a real difficulty in creating a crease line making an angle of $\frac{\pi}{7}$ with the top edge of the tape. We proceed by adopting a general policy we call our ***optimistic strategy***. Assume that we *can* crease an angle of $\frac{2\pi}{7}$ (certainly we can come close) as shown in Figure 5(a). Given that we have the angle $\frac{2\pi}{7}$, it is then a trivial matter to fold the top edge of the strip DOWN to bisect this angle, producing two adjacent angles of $\frac{\pi}{7}$ at the top edge as shown in Figure 5(b). (We say that $\frac{\pi}{7}$ is the ***putative*** angle on this tape.) Then, since we are content with this arrangement, we go to the bottom of the tape, where we observe that the angle to the right of the last crease line is $\frac{6\pi}{7}$ — and we decide, as paper folders, that we will always avoid leaving even multiples of π in the numerator of any angle next to the edge of the tape, so we bisect this angle of $\frac{6\pi}{7}$, by bringing the bottom edge of the tape UP to coincide with the last crease line and creating the new crease line sloping up shown in Figure 5(c). We settle for this (because we are content with an odd multiple of π in the numerator) and go to the top of the tape, where we observe that the angle to the right of the last crease line is $\frac{4\pi}{7}$ — and, since we have decided against leaving an even multiple of π in any angle next to an edge of the tape, we are forced to bisect this angle twice, each time bringing the top edge of the tape DOWN to coincide with the last crease line, obtaining the arrangement of crease lines shown in Figure 5(d). But now we notice that something miraculous has occurred! If we had really started with an angle of exactly $\frac{2\pi}{7}$, and if we now continue introducing crease lines by repeatedly folding the tape DOWN TWICE at the top and UP ONCE at the bottom, we get precisely what

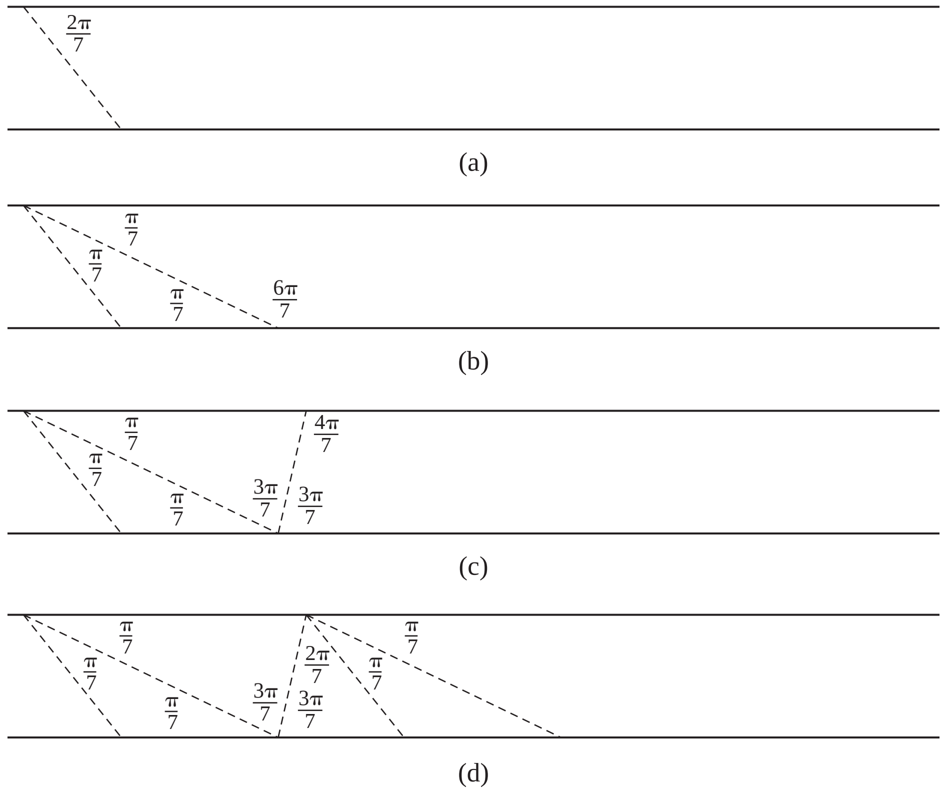


FIGURE 5

we want, namely, pairs of adjacent angles, measuring $\frac{\pi}{7}$, at equally spaced intervals along the top edge of the tape. Let us call this folding procedure the ***D^2U^1-folding procedure*** (or, more simply — and especially when we are concerned merely with the related number theory — the ***(2, 1)-folding procedure***) and call the strip of creased paper it produces ***D^2U^1-tape*** (or, again more simply, ***(2, 1)-tape***). The crease lines on this tape are called the ***primary crease lines***.

• • • BREAK 2

(1) We suggest that before reading further you get a piece of paper and fold an acute angle which you call an approximation to $\frac{2\pi}{7}$. Then fold about 40 triangles using the D^2U^1-folding procedure

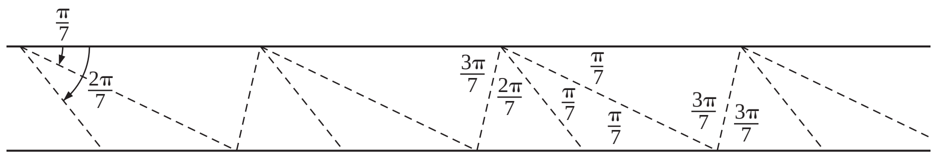
as shown in Figures 5 and 6(a) and described above, throw away the first 10 triangles, and see if you can tell that the first angle you get between the top edge of the tape and the adjacent crease line is *not* $\frac{\pi}{7}$. Then try to construct the FAT 7-gon shown in Figure 6(b). You may then *believe* that the D^2U^1-folding procedure produces tape on which the smallest angle does approach $\frac{\pi}{7}$, in fact rather rapidly.

(2) Try executing the FAT-algorithm at every other vertex along the top of this tape to produce a quasi-regular $\{\frac{7}{2}\}$-gon. (Hint: Look at Figure 6(c).)

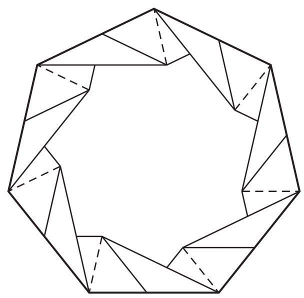
How do we *prove* that this evident convergence actually takes place? A very direct approach is to admit that the first angle folded down from the top of the tape in Figure 5(a) might not have been precisely $\frac{2\pi}{7}$. Then the bisection forming the next crease would make the two acute angles nearest the top edge in Figure 5(b) only approximately $\frac{\pi}{7}$; let us call them $\frac{\pi}{7} + \epsilon$ (where the error ϵ may be either positive or negative). Consequently, the angle to the right of this crease, at the bottom of the tape, would measure $\frac{6\pi}{7} - \epsilon$. When this angle is bisected, by folding up, the resulting acute angles nearest the bottom of the tape, labeled $\frac{3\pi}{7}$ in Figure 5(c), would in fact measure $\frac{3\pi}{7} - \frac{\epsilon}{2}$, forcing the angle to the right of this crease line at the top of the tape to have measure $\frac{4\pi}{7} + \frac{\epsilon}{2}$. When this last angle is bisected twice by folding the tape down, the two acute angles nearest the top edge of the tape will measure $\frac{\pi}{7} + \frac{\epsilon}{2^3}$. This makes it clear that every time we repeat a D^2U^1-folding on the tape the error is reduced by a factor of 2^3.

We see that our ***optimistic strategy*** has paid off — by blandly *assuming* we have an angle of $\frac{\pi}{7}$ at the top of the tape to begin with, and folding accordingly, we *get what we want* — successive angles at the top of the tape that, as we fold, rapidly get closer and closer to $\frac{\pi}{7}$, whatever angle we had, in fact, started with!

In practice, the approximations we obtain by folding paper are quite as accurate as the *real world* constructions with a straight edge and compass — for the latter are only perfect in the mind. In both cases the real world result is a function of human skill, but our procedure, unlike the Euclidean procedure, is very forgiving in that it tends to reduce the effects of human error — and, for many people (even the not so young), it is far easier to bisect an angle by folding paper than it is with a straight edge and compass.



(a)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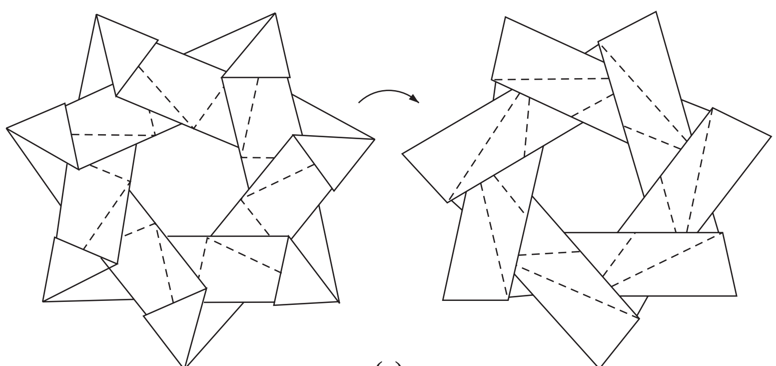
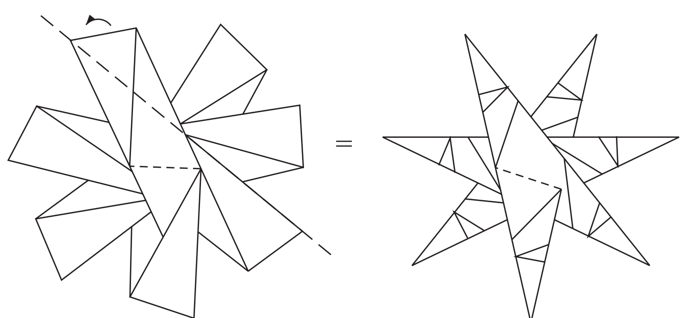

(c)



(d)

FIGURE 6

Figures 6(c), 6(d) show the regular $\{\frac{7}{2}\}$- and $\{\frac{7}{3}\}$-gons that are produced from the D^2U^1-tape by executing the FAT-algorithm on the crease lines that make angles of $\frac{2\pi}{7}$ and $\frac{3\pi}{7}$, respectively, with an edge of the tape (if the angle needed is at the bottom of the tape, as with $\frac{3\pi}{7}$, simply turn the tape over so that the required angle appears on the top). In Figures 6(c), 6(d) the FAT-algorithm was executed on every other suitable vertex along the edge of the tape so that, in (c), the resulting figure, or its flipped version, could be woven together in a more symmetric way and, in (d), the excess could be folded neatly around the points.

It is now natural to ask:

1. Can we use the same general approach used for folding a convex 7-gon to fold a convex N-gon with N odd, at least for certain specified values of N? If so, can we always prove that the actual angles on the tape really converge to the putative angle we originally sought?

2. Do we always get a quasi-regular $\{\frac{b}{a}\}$-gon with any general folding procedure, perhaps with other periods, such as those represented by

$$D^3U^3, D^4U^2, \quad \text{or} \quad D^3U^1D^1U^3D^1U^1?$$

How does the folding procedure determine $\frac{b}{a}$?

(The ***period*** is determined by the repeat of the *exponents*, so these examples have periods 1, 2, and 3, respectively.)

The answer to (1) is *yes*, and we will soon show you an algorithm for determining the folding procedure that produces tape from which you can construct *any* given quasi-regular $\{\frac{b}{a}\}$-gon, if a, b are odd with $a < \frac{b}{2}$. The complete answer to (2) appears in [2], but here we will simply note that an iterative folding procedure of this type will always produce one and only one quasi-regular $\{\frac{b}{a}\}$-gon (see page 135 of [2]).

Let us now look at the general 1-*period* folding procedure D^nU^n. A typical portion of the tape would appear as illustrated in Figure 7(a).

It turns out that the smallest angle u_k at the top, or bottom, of this tape approaches $\frac{\pi}{2^n+1}$; that is,

$$u_k \longrightarrow \frac{\pi}{2^n+1} \quad \text{as} \quad k \longrightarrow \infty \tag{1}$$

A proof of (1) similar to the one provided above for the tape whose smallest angle approached $\frac{\pi}{7}$ may be given. In fact, we can see that, if the original fold down (supposedly making an angle of $\frac{2\pi}{2^n+1}$ with the top of the tape) were such that it produced an angle that differed from the

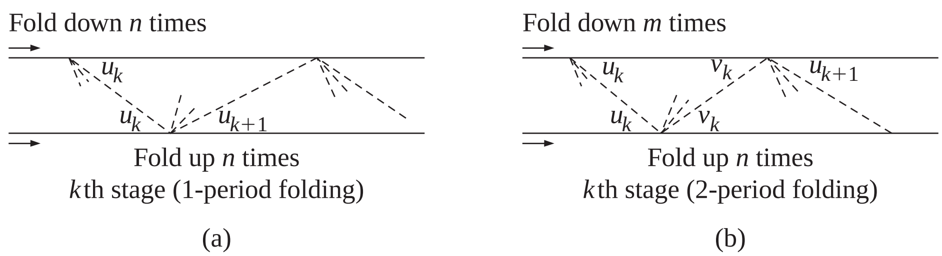


FIGURE 7

putative angle $\frac{\pi}{2^n+1}$ by an error ϵ_0, then the error ϵ_k at the kth stage of the D^nU^n-folding procedure would be given by

$$|\epsilon_k| = \frac{|\epsilon_0|}{2^{nk}} \tag{2}$$

Hence we see that the D^nU^n-folding procedure produces tape from which we may construct quasi-regular $(2^n + 1)$-gons — and, of course, these include those N-gons for which N is a Fermat number, prime or not. We would like to believe that the ancient Greeks and Gauss would have appreciated the fact that, when $n = 1$, 2, 4, 8, and 16, the D^nU^n-folding procedure produces tape from which we can obtain, by means of the FAT-algorithm, a quasi-regular 3-, 5-, 17-, 257-, and 65537-gon, respectively. What's more, if $n = 3$, we approximate the regular 9-gon, whose non-constructibility by Euclidean tools is very closely related to the non-trisectibility of an arbitrary angle.

The case $N = 2^n + 1$ is atypical, since we may construct $(2^n + 1)$-gons from our folded tape by special methods (not involving the FAT-algorithm), in which, however, the top edge does not describe the polygon, as it does in the FAT-algorithm. Figure 8 shows how the D^2U^2-tape shown in part (a) may be folded along just the short lines of the creased tape to form the *outline* of a quasi-regular pentagon shown in (b), and along just the long lines of the creased tape to form the *outline* of the slightly larger quasi-regular pentagon shown in (c); and, finally, we show in (d) the quasi-regular pentagon formed by an edge of the tape when the FAT-algorithm is executed.

• • • BREAK 3

Fold a length of D^nU^n-tape, for various values of n, and *experiment* with the folded tape to see how many differently-sized regular

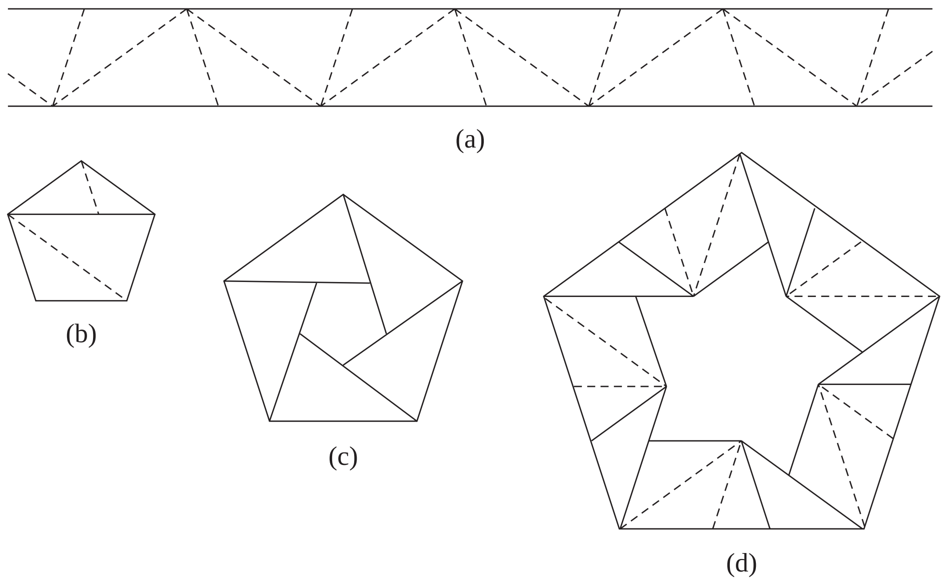


FIGURE 8

$(2^n + 1)$-gons you can create from each tape. You will need the D^1U^1- and the D^2U^2-tape in the section on building models, so be sure to include $n = 1$ and 2 in your experiments. Figure 9 shows three possibilities for constructing a quasi-regular 9-gon from D^3U^3-tape, without using the FAT algorithm. (The shaded portion of the figure indicates that the reverse side of the tape is visible.)

We next demonstrate how we construct quasi-regular polygons with $2^c N$ sides, N odd, if we already know how to construct quasi-regular N-gons. If, for example, we wished to construct a quasi-regular 10-gon, then we take the D^2U^2-tape (which, as you may recall, produced FAT 5-gons) and introduce a ***secondary crease line*** by bisecting each of the angles of $\frac{\pi}{5}$ next to the top (or bottom) edge of the tape. The FAT-algorithm may be used on the resulting tape to produce the quasi-regular convex FAT 10-gon, as illustrated in Figure 10. It should now be clear how to construct a quasi-regular 20-gon, 40-gon, 80-gon, $\cdots$.

This argument shows that we only need construct quasi-regular N-gons for N odd in order to be able to construct quasi-regular N-gons for any N.

Now we turn to the general **2-*period*** folding procedure, D^mU^n, which we may abbreviate to (m, n). (Recall that the tape that produced the quasi-

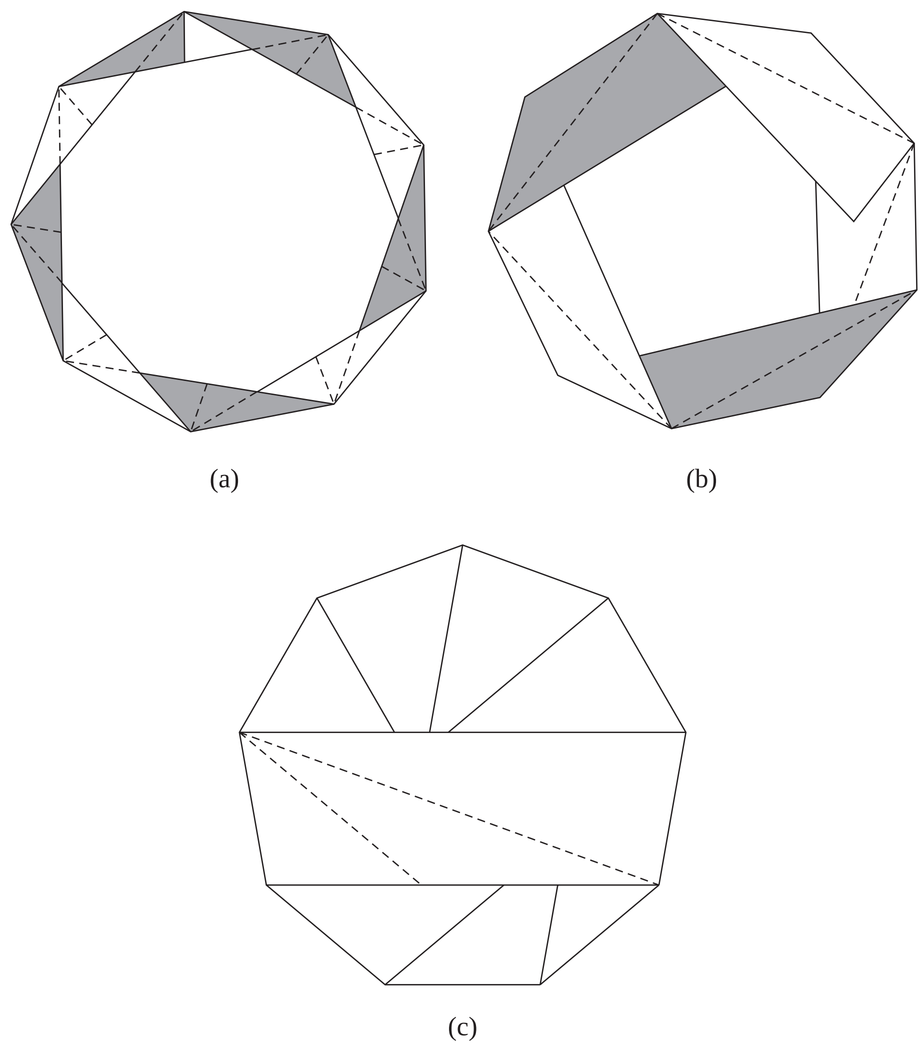
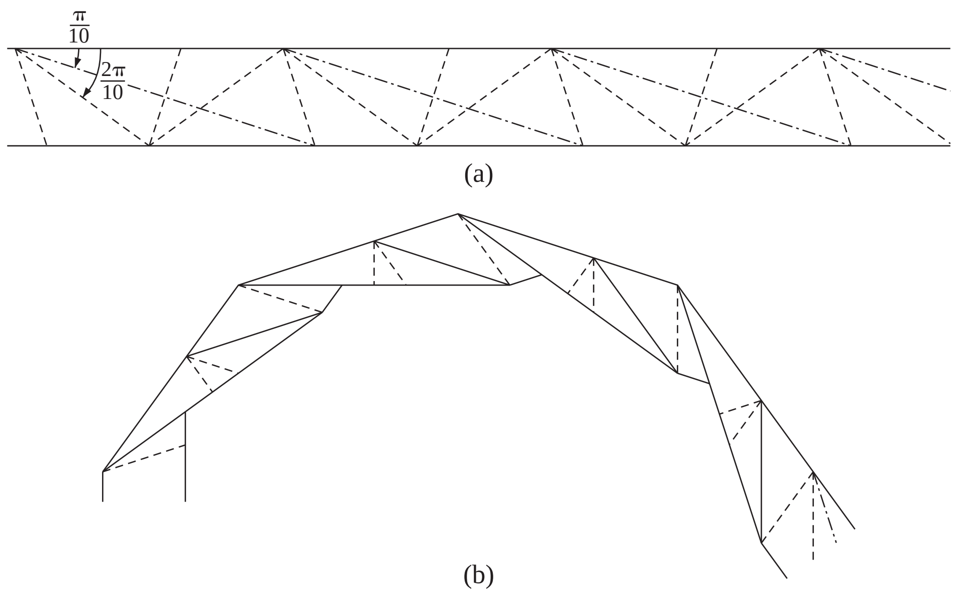


FIGURE 9 (a) A *long*-line 9-gon. (b) A *medium*-line 9-gon. (c) A *short*-line 9-gon.



(a)

(b)

FIGURE 10

regular 7-gon was a 2-period tape employing the $(2, 1)$ procedure.) A typical portion of the 2-period tape, in the general case, may be illustrated as shown in Figure 7(b). If the folding procedure had been started with an arbitrary angle u_0 at the top of the tape, and continued producing angles $u_1, u_2, \cdots$ at the top and $v_0, v_1, \cdots$ at the bottom, we would have, at the kth stage,

$$u_k + 2^n v_k = \pi,$$
$$v_k + 2^m u_{k+1} = \pi,$$

and it is shown in [2] that then

$$u_k \longrightarrow \frac{2^n - 1}{2^{m+n} - 1}\pi \quad \text{as} \quad k \to \infty \tag{3}$$

so that $\frac{2^n-1}{2^{m+n}-1}\pi$ is the putative angle. Thus the FAT-algorithm will produce, from this tape, a star $\{\frac{b}{a}\}$-gon, where the fraction $\frac{b}{a}$ may turn out not to be reduced (for example, when $n = 2$, $m = 4$), with $b = 2^{m+n} - 1$, $a = 2^n - 1$. By symmetry we infer that

$$v_k \longrightarrow \frac{2^m - 1}{2^{m+n} - 1}\pi \quad \text{as} \quad k \to \infty \tag{4}$$

Furthermore, if we assume an initial error of ϵ_0, then it can be shown (see [2]) that the error at the kth stage (when the folding $D^m U^n$ has been done exactly k times) will be given by[4]

$$\epsilon_k = \frac{\epsilon_0}{2^{(m+n)k}} \tag{5}$$

Hence, we see that in the case of our D^2U^1-folding (Figures 5, 6(a)) any initial error ϵ_0 is, as we already saw from our other argument, reduced by a factor 8 between consecutive stages. It should now be clear why we advised throwing away the first part of the tape — but, likewise, it should also be clear that it is never necessary to throw away very much of the tape. In practice, convergence is very rapid indeed, and if one made it a rule of thumb to always throw away the first 20 crease lines on the tape for any iterative folding procedure, it would turn out to be a very conservative rule.

We see that, however wonderful these results may be, they haven't completely solved our problem. For example, as we will have you show in the next break, we would be unable to fold a quasi-regular 11-gon with either the 1- or 2-period folding procedure. So the question remains: ***how do we know which sequence of folds to make in order to produce a particular quasi-regular polygon with the FAT-algorithm?***

• • • BREAK 4

(1) Show that $\frac{2^{m+n}-1}{2^n-1}$ will be an integer if and only if $n|m$. (Hint: Write the top and bottom of $\frac{2^{m+n}-1}{2^n-1}$ in base 2 and carry out the division.. Try some examples and pay particular attention to the form the *remainder* takes.)

(2) Show that, even if $n|m$, the number $\frac{2^{m+n}-1}{2^n-1}$ is never equal to 11. (Hint: In the same division, pay special attention to the *quotient*.)

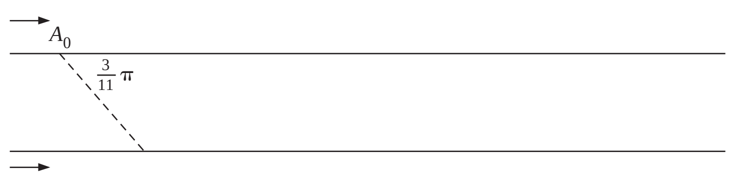
We now show, with a particular but not special case, how to determine the folding instructions for producing tape from which we can construct a quasi-regular $\{\frac{b}{a}\}$-gon, with a, b odd and $a < \frac{b}{2}$.

[4]The discrepancy between formulae (2) and (5) is due to the fact the with D^nU^n folding procedure is really a period 1 procedure.

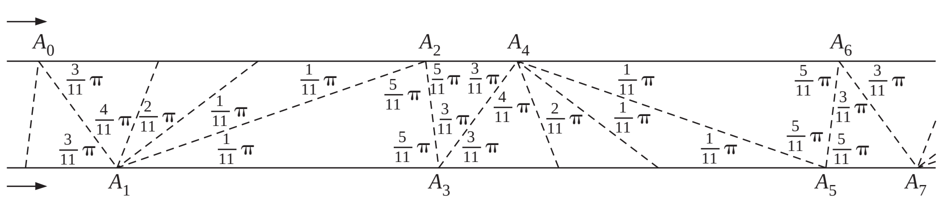
Thus, suppose we want to construct a quasi-regular $\{\frac{11}{3}\}$-gon. Then, of course, $b = 11$, $a = 3$, and we proceed precisely as we did when we wished to construct the regular convex 7-gon; that is, we adopt our ***optimistic strategy*** which, as you recall, means that we *assume* that we've got what we want, and, as we will show, we then actually *get* an arbitrarily good approximation to what we want! This time we assume that we can fold the desired putative angle of $\frac{3\pi}{11}$ at A_0 (see Figure 11(a)), and we adhere to the same principles that we used in constructing the quasi-regular 7-gon, namely, we adopt the following rules.

1. Each new crease line goes in the forward (left to right) direction along the strip of paper.
2. Each new crease line always *bisects* the angle between the last crease line and the edge of the tape from which it emanates.
3. The bisection of angles at any vertex continues until a crease line produces an angle of the form $\frac{a'\pi}{b}$ where a' is an *odd* number; then the folding stops at that vertex and commences at the intersection point of the last crease line with the other edge of the tape.

Once again the ***optimistic strategy*** works; and following this procedure results in tape whose angles converge to those shown in Figure 11(b). We could denote this folding procedure by $D^1U^3D^1U^1D^3U^1$, interpreted in the obvious way on the tape — that is, the first exponent "1" refers to the one bisection (producing a line in a downward direction) at the vertices A_{6n} (for $n = 0, 1, 2, \ldots$) on the top of the tape; similarly, the "3" refers to the 3



(a)

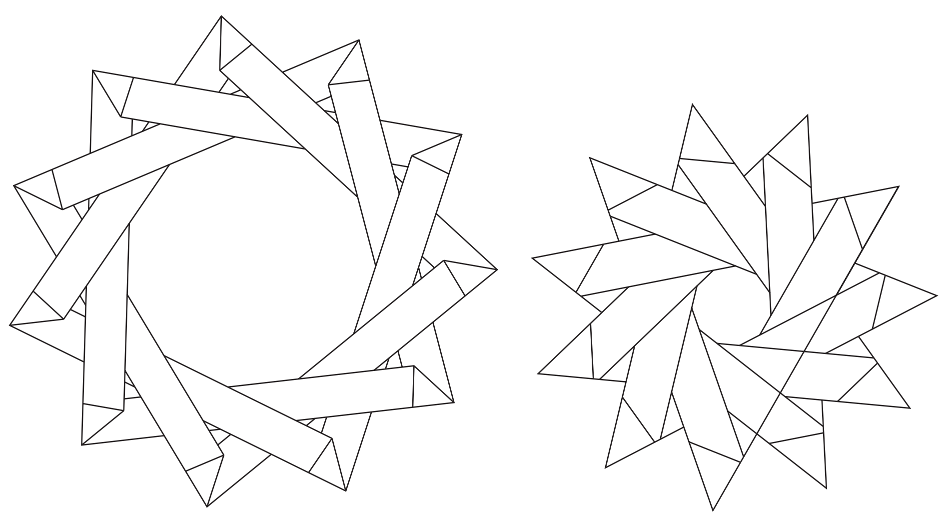


(b)

FIGURE 11 (Note that the indexing of the vertices is not the same as that in Figure 3.)

bisections (producing creases in an upward direction) made at the bottom of the tape through the vertices A_{6n+1}; etc. However, since the folding procedure is *duplicated* halfway through, we can abbreviate the notation and write simply {1, 3, 1}, with the understanding that we alternately fold from the top and bottom of the tape as described, with the *number* of bisections at each vertex running, in order, through the values $1, 3, 1, \cdots$. We call this a ***primary folding procedure of period* 3** or a **3-*period folding***.

To prove the convergence we can use an error-correction type of proof like that given earlier in this section for the 7-gon. We leave the details to the reader, and explore here what we can do with this $(1, 3, 1)$-tape. First, note that, starting with the putative angle $\frac{3\pi}{11}$ at the top of the tape, we produce a putative angle of $\frac{\pi}{11}$ at the bottom of the tape, then a putative angle of $\frac{5\pi}{11}$ at the top of the tape, then a putative angle of $\frac{3\pi}{11}$ at the *bottom* of the tape, and so on. A careful inspection of this tape shows that we could use the FAT algorithm on it to fold quasi-regular $\left\{\frac{11}{a}\right\}$-gons, when $a = 1, 2, 3, 4, 5$. To put the result in a form that suggests the generalization, we may say that if there are crease lines enabling us to fold a star $\{\frac{11}{a}\}$-gon, there will be crease lines enabling us to fold star $\{\frac{11}{2^k a}\}$-gons, where $k \geq 0$ takes any value such that $2^{k+1}a < 11$. These features, described for $b = 11$, would be found with any odd number b. However, this tape has a special



(a) (b)

FIGURE 12

symmetry as a consequence of its *odd* period; namely, if it is "flipped" about the horizontal line halfway between its parallel edges, the result is a *translate* of the original tape. As a practical matter this special symmetry of the tape means that we can use either the top edge or the bottom edge of the tape to construct our polygons. On tapes with an *even* period the top edge and the bottom edge of the tape are not translates of each other (under the horizontal flip), which simply means that care must be taken in choosing the edge of the tape used to construct a specific polygon. Figures 12(a, b) show the completed $\{\frac{11}{3}\}$-, $\{\frac{11}{4}\}$-gons, respectively.

Now, to set the scene for the number theory of Section 5, and to enable us to systematically determine the folding procedure for any given a and b, let us look at the patterns in the *arithmetic* of the computations when $a = 3$ and $b = 11$. Referring to Figure 10(b) we observe that

the smallest angle to the right of A_n, where	is of the form $\frac{a}{11}\pi$, where	and the number of bisections at the *next* vertex[5]
$n = 0$	$a = 3$	$= 3$
1	1	1
2	5	1
3	3	3
4	1	1
5	5	1

We could write this in shorthand form as follows:

$$(b =)11 \left| \begin{array}{ccc} (a =)3 & 1 & 5 \\ 3 & 1 & 1 \end{array} \right| \tag{6}$$

Observe that, had we started with the putative angle of $\frac{\pi}{11}$, then the *symbol* (6) would have taken the form

$$(b =)11 \left| \begin{array}{ccc} (a =)1 & 5 & 3 \\ 1 & 1 & 3 \end{array} \right| \tag{6$'$}$$

In fact, it should be clear that we can *start anywhere* (with $a = 1$, 3, or 5), and the resulting symbol, analogous to (6), will be obtained by cyclic

[5]Notice that, referring to Figure 10(b), to obtain an angle of $\frac{3\pi}{11}$ at $A_0, A_6, A_{12}, \cdots$, the folding instructions would more precisely be $U^3D^1U^1D^3U^1D^1\cdots$. But we don't have to worry about this distinction.

permutation of the matrix component of the symbol, placing our choice of a in the first position along the top row.

In general, suppose we wish to fold a $\{\frac{b}{a}\}$-gon, with b, a odd and $a < \frac{b}{2}$. Then we may construct a symbol[6] as follows. Let us write

$$b \left| \begin{array}{cccc} a_1 & a_2 & \cdots & a_r \\ k_1 & k_2 & \cdots & k_r \end{array} \right| \tag{7}$$

where b, a_i $(a_1 = a)$ are odd, $a_i < \frac{b}{2}$, and

$$b - a_i = 2^{k_i} a_{i+1}, \quad i = 1, 2, \cdots, r, \quad a_{r+1} = a_1 \tag{8}$$

We proved in Chapter 4 of [2], that, given any two odd numbers a and b, with $a < \frac{b}{2}$, there is always a completely determined unique symbol (7) with $a_1 = a$. At this stage, we do not assume that $\gcd(b, a) = 1$, but we have assumed that the list $a_1, a_2, \cdots, a_r$ is without repeats. Indeed, if $\gcd(b, a) = 1$, we say that the symbol (7) is *reduced*, and, if there are no repeats among the a_i's, we say that the symbol (7) is *contracted*. (It is, of course, theoretically possible to consider symbols (7) in which repetitions among the a_i are allowed.) We regard (7) as encoding the general folding procedure to which we have referred.

Example 1 If we wish to fold a 31-gon we may start with $b = 31, a = 1$ and construct the symbol

$$(b =)31 \left| \begin{array}{cc} (a =)1 & 15 \\ 1 & 4 \end{array} \right|$$

which tells us that folding D^1U^4 will produce tape (usually called $(1, 4)$-tape) that can be used to construct a FAT 31-gon. In fact, this tape can also be used to construct FAT

$$\left\{\frac{31}{2}\right\}\text{-}, \quad \left\{\frac{31}{4}\right\}\text{-}, \quad \text{and} \quad \left\{\frac{31}{8}\right\}\text{-gons.}$$

However, if we wish to fold a $\{\frac{31}{3}\}$-gon, we start with $b = 31, a = 3$ and construct the symbol

$$(b =)31 \left| \begin{array}{cc} (a =)3 & 7 \\ 2 & 3 \end{array} \right|$$

[6] More exactly, a 2-symbol. Later on, we introduce a more general t-symbol, $t \geq 2$.

which tells us to fold D^2U^3 — or, more simply, to use the $(2, 3)$-folding procedure — to produce $(2, 3)$-tape from which we can fold the FAT $\{\frac{31}{3}\}$-gon. Again, we get more than we initially sought, since we can also use the $(2, 3)$-tape to construct FAT

$$\left\{\frac{31}{6}\right\}\text{-},\quad \left\{\frac{31}{12}\right\}\text{-},\quad \left\{\frac{31}{7}\right\}\text{-},\quad \text{and}\quad \left\{\frac{31}{14}\right\}\text{-gons.}$$

However, we don't have a folding procedure that produces the $\{\frac{31}{5}\}$-gon. Thus we construct another symbol, this time with $b = 31$, $a = 5$. We get

$$\begin{array}{r|cccc|} (b=)31 & (a=)5 & 13 & 9 & 11 \\ & 1 & 1 & 1 & 2 \end{array}$$

which tells us to fold $D^1U^1D^1U^2$ — or, more simply, to use the 4-period $(1, 1, 1, 2)$-folding procedure — to produce $(1, 1, 1, 2)$-tape from which we can fold the FAT $\{\frac{31}{5}\}$-gon. Once again, we get more than we asked for; we can also use the $(1, 1, 1, 2)$-tape to construct FAT

$$\left\{\frac{31}{10}\right\}\text{-},\quad \left\{\frac{31}{13}\right\}\text{-},\quad \left\{\frac{31}{9}\right\}\text{-},\quad \text{and}\quad \left\{\frac{31}{11}\right\}\text{-gons.}$$

We can combine all the possible symbols for $b = 31$ into one *complete* symbol, adopting the notation

$$\begin{array}{r|cc|cc|cccc|} 31 & 1 & 15 & 3 & 7 & 5 & 13 & 9 & 11 \\ & 1 & 4 & 2 & 3 & 1 & 1 & 1 & 2 \end{array} \tag{9}$$

Notice in (9) that the total amount of folding would be the same to produce any quasi-regular (convex or star) 31-gon. Since it is very difficult to bisect an angle 4 times, you may prefer to use the second or third parts of this symbol to produce the tape. Even if you really want a convex 31-gon it may be easier, in practice, to produce the star polygon first and then use the vertices of that polygon to determine the convex polygon.

• • • BREAK 5

(1) Show that the tape folded in accordance with the folding instruction, or symbol,

$$\begin{array}{c|cccc|} b & a_1 & a_2 & \cdots & a_r \\ & k_1 & k_2 & \cdots & k_r \end{array}$$

contains fold lines allowing us, by means of the FAT-algorithm, to fold any quasi-regular $\{\frac{b}{2^n a_i}\}$-gon, where $2^{n+1}a_i < b$.

Show that the complete symbol for $b = 31$ gives us, in this sense, folding instructions for folding all possible quasi-regular star 31-gons. Give an argument for why this must happen for any odd b. (Note that we are only considering star $\{\frac{b}{a}\}$-gons where a is prime to b.)

(2) Check your understanding of the complete symbol by doing the calculations to fill in the blanks below for $b = 91$:

$$\begin{array}{c|cccccc|cccc|cccccccc|} 91 & 1 & 45 & 23 & 17 & 37 & 27 & 3 & 11 & 5 & 43 & 9 & 41 & 25 & 33 & 29 & 31 & 15 & 19 \\ & 1 & 1 & 2 & 1 & 1 & 6 & 3 & ? & ? & 4 & 1 & ? & ? & ? & ? & ? & ? & ? \end{array}$$

(Notice that no multiples of 7 or 13 appear in the top row. Why do you suppose this is so?)

(3) Calculate other complete symbols for odd b and look for patterns in them.

(4) Calculate the symbol with $b = 33, a = 9$. Compare with item (6). What part of this symbol tells you that the rational number $\frac{33}{9}$ is equal to $\frac{11}{3}$? (You should now see why it is pointless to allow unreduced symbols.)

(5) Guess what the values of k_i would be for the symbol with $b = 91$, $a = 13$. Construct the symbol to see if your guess was correct.

More formally, we can say that given positive odd integers b, a with $a < \frac{b}{2}$, there is always a unique contracted symbol (the proof is in [2])

$$b \left| \begin{array}{cccc} a_1 & a_2 & \cdots & a_r \\ k_1 & k_2 & \cdots & k_r \end{array} \right|, \qquad a_1 = a, \quad a_i \neq a_j \quad \text{if} \quad i \neq j, \tag{10}$$

where each a_i is odd, $a_i < \frac{b}{2}$, and

$$b - a_i = 2^{k_i} a_{i+1}, \quad i = 1, 2, \cdots, r, \quad a_{r+1} = a_1 \tag{11}$$

The proof involves fixing b and letting S be the set of positive odd numbers $a < \frac{b}{2}$. Given $a \in S$, then a' is defined by the rule

$$b - a = 2^k a', \quad k \textit{ maximal}; \tag{12}$$

that is, we take as many factors of 2 as we can out of $b - a$. Then (12) describes a function $\Psi : S \longrightarrow S$ such that $\Psi(a) = a'$. In [2] we show that Ψ is a ***permutation*** of the finite set S.

The permutation Ψ has the important property

$$\gcd(b, a) = \gcd(b, a'). \tag{13}$$

For it is clear from (12) that if $d \mid b$ and $d \mid a'$, then $d \mid b$ and $d \mid a$. Conversely, if $d \mid b$ and $d \mid a$, then d is *odd* and $d \mid 2^k a'$, so $d \mid b$ and $d \mid a'$. Thus if a_1 in (10) is prime to b, so are $a_2, a_3, \cdots, a_r$, and we may, if we wish, confine our choices of a_i to those odd numbers such that $\gcd(a_i, b) = 1$; that is, we may confine ourselves to ***reduced*** symbols, now to be defined as those symbols in which *each* a_i is prime to b. Moreover, given any odd numbers b, a with $a < \frac{b}{2}$, we may construct the symbol (10) and then reduce the symbol by dividing b and each a_i by $\gcd(b, a)$; the bottom row will be unaffected.

4.3 CONSTRUCTING ALL QUASI-REGULAR POLYGONS

We have described procedures for folding any quasi-regular convex polygon and for folding any quasi-regular star $\{\frac{b}{a}\}$-gon if a, b are both odd with, of course, $a < \frac{b}{2}$. To complete our program we must show how to fold any quasi-regular star $\{\frac{b}{a}\}$-gon, where $a < \frac{b}{2}$ and (i) b is odd, a is even or (ii) b is even, a is odd (for we may, of course, assume a, b coprime).

The case where b is odd and a is even is quickly dealt with. Let $a = 2^k a'$ with a' odd, and suppose we have tape creased to fold a $\{\frac{b}{a'}\}$-gon. We claim that this tape already has a crease line making an angle $\frac{a\pi}{b}$ with the forward direction of the top of the tape, with another crease line making an angle $\frac{a\pi}{b}$ with it. For, if ℓ is minimal such that $2^\ell a' > \frac{b}{2}$, then $\ell \geq 1$ and the tape has a crease line making an angle of $\frac{2^\ell \pi a'}{b}$ with the forward direction of the top of the tape, appropriately bisected by crease lines ℓ times. (See Figure 13 for a typical example, with $\ell = 2$.) Now $2^\ell a' > \frac{b}{2}$, so $\ell > k$; thus the stated crease lines appear on the tape, and the FAT algorithm may be applied

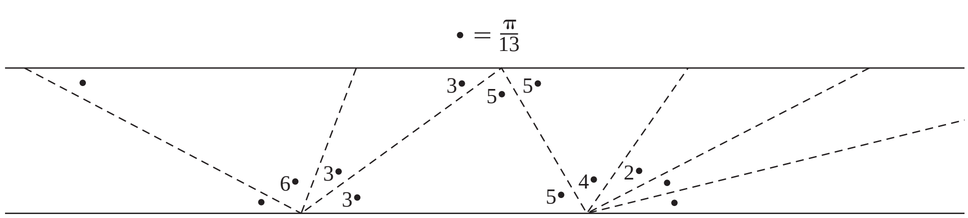


FIGURE 13 $a = 6, a' = 3, k = 1, b = 13$

using these two crease lines to fold a $\{\frac{b}{a}\}$-gon. No further crease lines are necessary.

The situation is very different, however, when b is even and a is odd — of course, we have already seen this on page 101 of Chapter 4 of [2] when $a = 1$. We now assume that $b = 2^k b'$, with b' odd. We write a in base 2 as

$$a = \epsilon_0 \epsilon_1 \cdots \epsilon_n, \quad \epsilon_0 = \epsilon_n = 1, \quad \epsilon_i = 0 \text{ or } 1, \quad 1 \le i \le n-1. \tag{14}$$

We now describe the *initial configuration* of our tape. It will consist of two crease lines, the first making an angle of $\alpha_0 = \frac{\pi}{2^{k-n}b'}$ with the top of the tape, the second making an angle of $\sigma_0 = \frac{\pi}{2^{k-n}b'}$ with the first. To explain how we achieve this initial configuration, we must consider 3 cases.

Case 1: $b' = 1$, so that $b = 2^k$. Then $2^n < a < 2^{k-1}$, so $n \le k-2$. We can then certainly achieve our initial configuration exactly.

Case 2: $b' > 1$, $k > n$. Start with the b'-tape and create the initial configuration by introducing $(k-n)$ secondary crease lines by successively bisecting the top angle on the tape.

Case 3: $b' > 1$, $k \le n$. Now the b'-tape will itself already have the initial configuration on it — the argument is exactly like that above (when b is odd and a is even).

Thus we can suppose the initial configuration achieved. We are now going to define inductively the ***i*th proximand** α_i and the ***i*th support** σ_i, $0 \le i \le n$, and explain how they are achieved by folding the tape. We claim that on achieving the nth proximand we will have a crease line making an angle of $\frac{\pi a}{b}$ with the forward direction of the tape. Obviously, we can then duplicate this angle under this crease line and then apply the FAT algorithm to complete the construction of the $\{\frac{b}{a}\}$-gon.

Thus we suppose α_i, σ_i already defined and achieved, $0 \le i < n$, with a crease line making an angle of α_i with the top of the tape and another crease line making an angle of σ_i with it; see Figure 14. Bisect σ_i by a new crease line into two (equal) angles λ_i, ρ_i, with λ_i to the left of ρ_i. If $\epsilon_{i+1} = 0$, define $\alpha_{i+1} = \alpha_i$, $\sigma_{i+1} = \rho_i$; if $\epsilon_{i+1} = 1$, define $\alpha_{i+1} = \alpha_i \cup \rho_i$, $\sigma_{i+1} = \lambda_i$.

We now prove our claim that $\alpha_n = \frac{\pi a}{b}$. In fact, we prove inductively that

$$\alpha_i = \frac{\pi a_i}{2^k b'}, \quad \sigma_i = \frac{\pi}{2^{k-n+i} b'}, \tag{15}$$

where

$$a_i = \epsilon_0 \epsilon_1 \cdots \epsilon_i 0 \cdots 0, \quad \text{with } (n-i) \text{ zeros}. \tag{16}$$

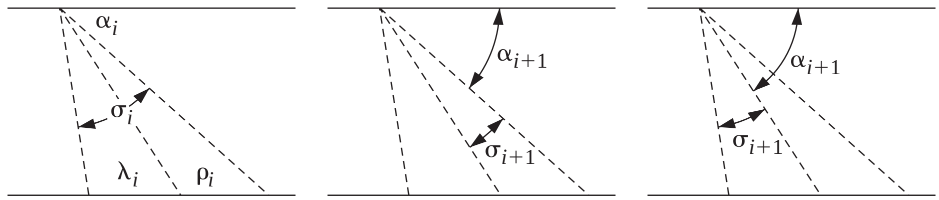


FIGURE 14

Then the equations (15) hold if $i = 0$. Given (15) for a fixed i, assume first that $\epsilon_{i+1} = 0$. Then $\alpha_{i+1} = \alpha_i = \frac{\pi a_{i+1}}{2^k b'}$, since $a_{i+1} = a_i$, and $\sigma_{i+1} = \frac{1}{2}\sigma_i = \frac{\pi}{2^{k-n+i+1}b'}$.

Assume instead that $\epsilon_{i+1} = 1$. Then

$$\alpha_{i+1} = \alpha_i + \frac{1}{2}\sigma_i = \frac{\pi a_i}{2^k b'} + \frac{\pi}{2^{k-n+i+1}b'} = \frac{\pi}{2^k b'}\left(a_i + 2^{n-i-1}\right)$$

$$= \frac{\pi}{2^k b'}\left(\epsilon_0\epsilon_1 \cdots \epsilon_i 10 \cdots 0\right) = \frac{\pi a_{i+1}}{2^k b'}.$$

Again $\sigma_{i+1} = \frac{1}{2}\sigma_i = \frac{\pi}{2^{k-n+i+1}b'}$. This establishes the inductive step and hence (15). Thus $\alpha_n = \frac{\pi a_n}{2^k b'} = \frac{\pi a}{b}$, and our construction has been vindicated.

Of course, what we have given above are algorithms for producing any quasi- regular star polygon by folding paper. Thus it is not to be expected that our recipes will give you the *simplest* procedures in all cases. Indeed, as you will see, they do not.

• • • BREAK 6

(1) Give an algorithm that works if $b = 2^k b'$, $a < \frac{b'}{2}$, b' odd, which is simpler than that given in the absence of the condition $a < \frac{b'}{2}$.

(2) Compare the algorithm in the text with the one you discovered in tackling problem (1) above, when $a = 1$.

4.4 HOW TO BUILD SOME POLYHEDRA (HANDS-ON ACTIVITIES)

In this section we give you explicit instructions for using the D^1U^1-tape to construct regular pentagonal (and triangular) dipyramids, tetrahedra, octahedra, and icosahedra. We also show you how to make exact folds on the tape in order to be able to construct two different kinds of cube. Finally, we will show you how to use the D^2U^2-tape to construct two kinds of regular dodecahedron. All of these models may be taken apart and stored flat.

The models described in this section are referred to in Chapter 8, where some of the mathematics connected with their symmetries is discussed.

First we will tell you what you need. After you have assembled the materials you should choose which model you wish to make and then carefully read, and execute, the instructions for preparing the pattern pieces. Then we suggest that, after a rest, you read the assembly instructions.

What you will need

- A 75 ft (or more) roll of 2-in gummed mailing tape (or a wider and longer roll if you want larger models). The glue on the tape should be the type that needs to be moistened to become sticky. ***Caution:*** Don't try to use tape that is sticky to the touch when it is dry — you would find it very frustrating.
- Scissors
- Sponge (or washcloth)
- Shallow bowl
- Water
- Hand towel (or rag)
- Some books
- Colored paper of your choosing. Construction paper works well, but it may need to be cut into strips and glued together to get long enough pieces. Gift wrapping or butcher paper that comes in rolls are particularly easy to use for these models. The Sunday funnies also work.
- Bobby pins

General instructions for preparing pattern pieces

For each of the models described in this section you will need to glue the pattern piece (or pieces) onto colored paper. Of course, if a model involves more than one piece, the finished model will be more interesting if you use a different color for each piece. In each case, to accomplish the gluing, first prepare the pieces of paper onto which you plan to glue the *prepared* pattern piece. Make certain that each piece is long enough for the pattern piece and that it all lies on a flat surface.

Place a sponge (or washcloth) in a bowl. Add water to the bowl so that the top of the sponge is very moist indeed.[7] Moisten one end of the pattern piece by pressing it onto the sponge; then, holding that end (yes, it's messy!), pull the rest of the strip across the sponge. Make certain the entire strip gets wet and then place it on the colored paper. Use a hand towel (or rag) to wipe up the excess moisture while smoothing the tape into contact with the colored paper.

Put some books on top of the pieces so that they will dry flat. When the tape is dry, cut out the pattern pieces, ***trimming off a small amount of the gummed tape*** (about $\frac{1}{16}$ of an inch or 1.5 mm will do) from the edge as you do so (this serves to make the model look neater and, more importantly, allows for the increased thickness produced by gluing the strip to another piece of paper). Refold the piece *firmly* on all the fold lines that are going to be fold lines on the finished model. This should be done so that the raised ridges, called *mountain folds*, are *on the colored side* of the pattern piece. You will now be ready to construct your model.

We now describe the specific details for each model:

Pentagonal Dipyramid constructed from one strip

Begin by folding the gummed mailing tape to produce D^1U^1-tape (see Break 3). Continue folding until you have 50, or more, triangles. Throw away the first 10 triangles, and then cut off a strip containing 31 triangles.[8] This is the pattern piece you need for this model. Prepare it as described above and then place your strip so that the left-hand end appears as shown in Figure 15(a) *with the colored side visible*. Mark the first and eighth triangles *exactly* as shown (note the orientation of the various letters within their respective triangles).

Begin by placing the first triangle *over* the eighth triangle so that the corner labeled Ⓐ is over the corner labeled A, Ⓑ is over the corner labeled B, and Ⓒ is over the corner labeled C. Hold these two triangles together, in that position, and observe that you have the beginning of a double pyramid for which there will be five triangles above and five triangles below the horizontal plane of symmetry, as shown in Figure 15(b). Now you can hold the model up and let the long strip of triangles fall around this frame. If the strip is folded well, the remaining triangles will simply fall into place. When you get to the last triangle, there will be a crossing of

[7] Perhaps we should have told you to wear some very old, or at least washable, clothes while doing this.

[8] You can now continue folding triangles on this piece of tape to produce triangles for building other models.

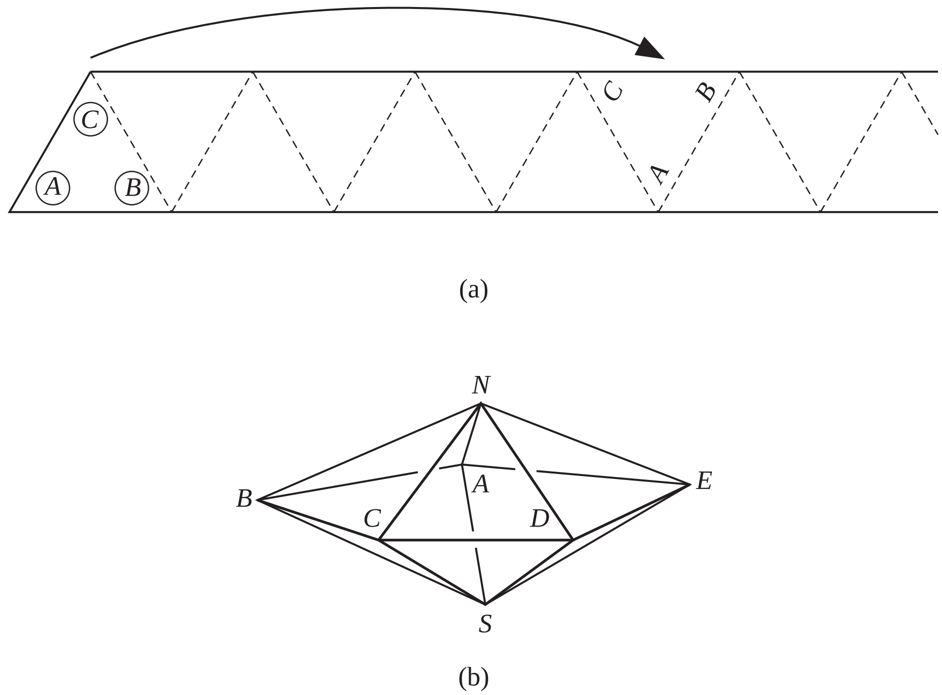


FIGURE 15

the strip that the last triangle can tuck into, and *your pentagonal dipyramid is complete!* It should look like Figure 15(b)

If you have trouble because the strip doesn't seem to fall into place, there are two most likely explanations. The first (and more likely) is that you haven't folded the crease lines firmly enough. This situation is easily remedied by refolding each crease line with more conviction. The second common difficulty occurs when the tape seems too short to reach around the model and tuck in. This problem can be remedied by trimming off a tiny amount more from each edge of the tape.

Triangular dipyramid constructed from one strip

You may wish to figure out how to make the analogous constuction of a triangular dipyramid from a strip of 19 equilateral triangles. Of course, you prepare the pattern strip exactly the same way, and then knowing that the

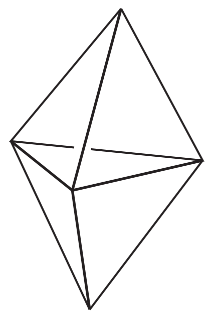


FIGURE 16

finished model should appear as shown in Figure 16 and that you should begin by forming the *top* three faces with one end of the strip should get you off to a good start.

Experiment and see if you can construct either of the above dipyramids with fewer than the number of triangles specified here. This can, in fact, be done, but the real question is: Will they be balanced, in the sense that every face is covered by the same number of triangles? This is not a difficult question to answer — and it is therefore left to the reader.

The rest of the models for which we will describe constructions are each braided from two or more straight strips of paper, and each of them is a *regular convex* polyhedron; that is, each is a model of a polyhedron known as a *Platonic Solid*.[9] We give one construction here for the tetrahedron, octahedron, and icosahedron (and suggest another, more complicated, but arguably more symmetric, construction in Chapter 8). We give two constructions here for the cube and the dodecahedron. Here are the details.

Tetrahedron constructed from 2 strips

Prepare two strips of 5 triangles each, as shown in Figure 17(b). Then, on a flat surface, with the colored surfaces down so that they are not visible, lay one strip *over* the other strip exactly as shown in Figure 17(c). Think of triangle *ABC* as the *base* of the tetrahedron being formed; for the moment, triangle *ABC* remains on the table. Then fold the bottom strip into a tetrahedron by lifting up the two triangles labeled *X* and overlapping

[9]One, non-technical, way of characterizing a Platonic solid is to say that it is a convex polyhedron having the property that it appears the same when it is viewed looking straight on at any vertex, or looking straight on at any edge, or looking straight on at any face. You can easily see that although the pentagonal and triagonal dipyramids are convex, they do not satisfy the rest of the conditions.

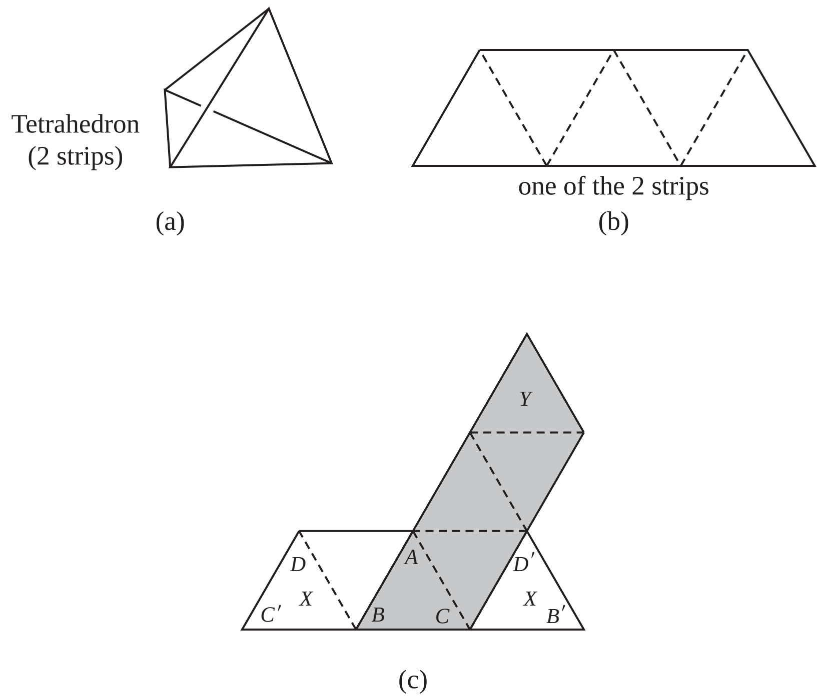


FIGURE 17

them, so that C' meets C, B' meets B, and D' meets D. Don't worry about what is happening to the top strip, as long as it stays in contact with the bottom strip where the two triangles originally overlapped. Now you will have a tetrahedron, with three triangles sticking out from one edge. Complete the model by carefully picking up the whole configuration, holding the overlapping triangles X in position, wrapping the protruding strip around two faces of the tetrahedron and tucking the Y triangle into the open slot along the edge BC. Your model should have 4 triangular faces and look like Figure 17(a), with 2 triangles from each strip visible on its surface.

A suggestion for how to build a "more symmetric" tetrahedron with three strips appears in Chapter 8.

Octahedron constructed from 4 strips

Prepare four strips of 7 triangles each, as shown in Figure 18(b).

To construct the octahedron, begin with a pair of overlapping strips held together with a paper clip with the colored side visible, as indicated in Figure 19(a). Fold these two strips into a double pyramid by placing

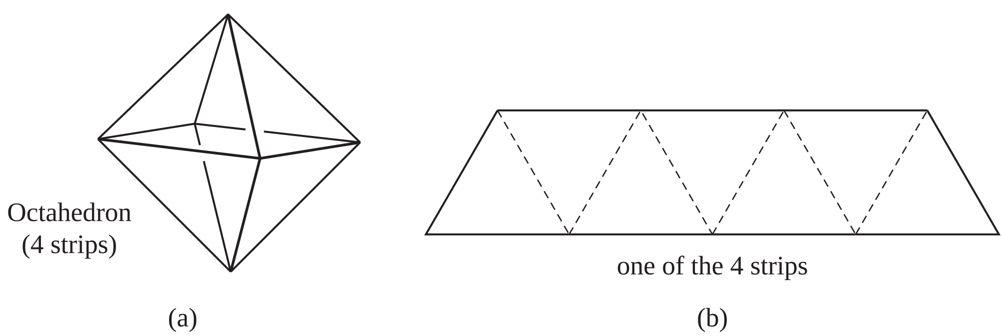


FIGURE 18

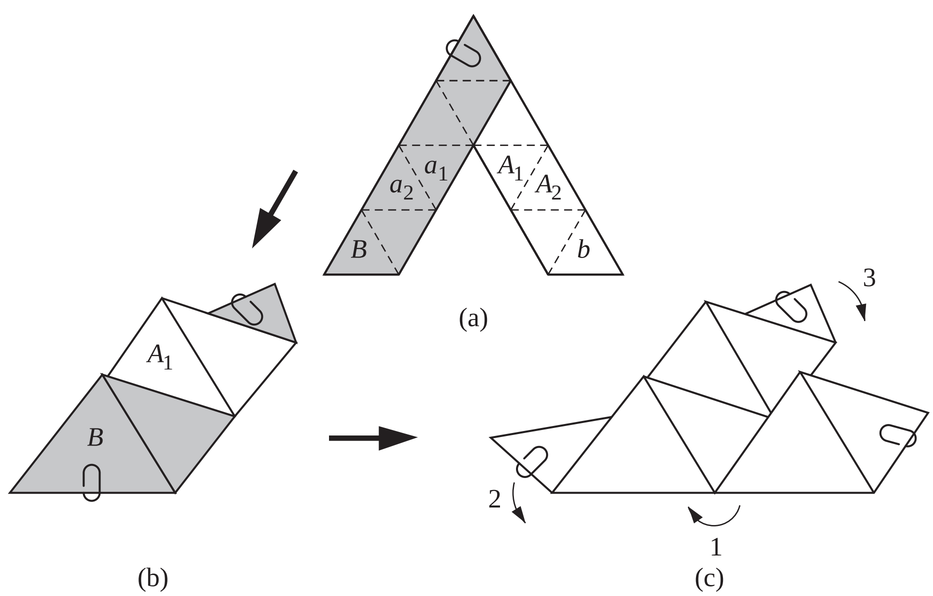
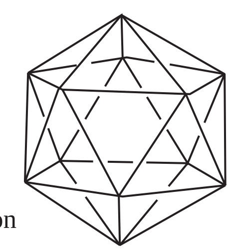


FIGURE 19

triangle a_1 under triangle A_1, and triangle b under triangle B. The overlapping triangles b and B are secured with another paper clip, so that the configuration looks like Figure 19(b). Repeat this process with the second pair of strips, and place the second pair of braided strips over the first pair, as shown in Figure 19(c). When doing this, make certain the flaps with the paper clips are oriented exactly as shown. Complete the octahedron by following the steps indicated by the arrows in Figure 19(c). You will note that after step 1 you have formed an octahedron; performing step 2 simply places the flap with the paper clip on it against a face of the octahedron; in step 3 you should tuck the flap *inside the model*.

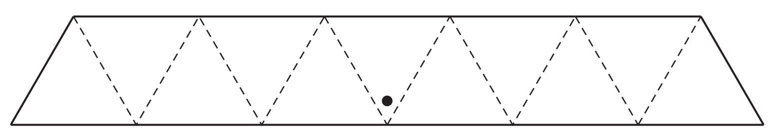
When you become adept at this process you will be able to slip the paper clips off as you perform these last three steps — but they won't show, so this is only an aesthetic consideration. Your finished model should have 8 triangular faces and look like Figure 18(a), with 2 triangles from each strip visible on its surface.

A suggestion for how to build a "more symmetric" octahedron, by putting slits in the triangles of each strip, is given in Chapter 8.



Icosahedron
(5 strips)

(a)



one of the 5 strips

(b)

FIGURE 20

Icosahedron constructed from 5 strips[10]

Prepare five strips of 11 triangles each, as shown in Figure 20(b). Lay the pieces down so that the colors are not visible and mark a heavy dot on the center triangle of each strip, as shown in Figure 20(b).

Then (without moving the pattern pieces) label one of the 5 strips with a "1" on each of its 11 triangles (make sure you are writing on the surface that will be on the *inside* of the finished model). Then label the next strip

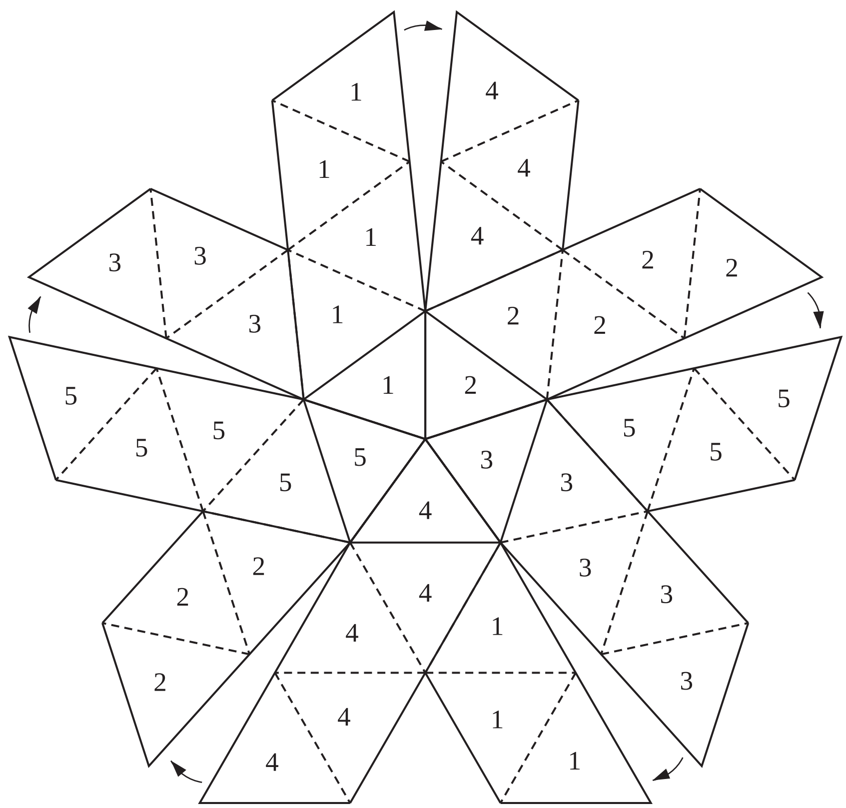


FIGURE 21

[10]Of all the models described in this section, this is, by far, the most difficult to build. So tackle this one only when you have plenty of time and patience available.

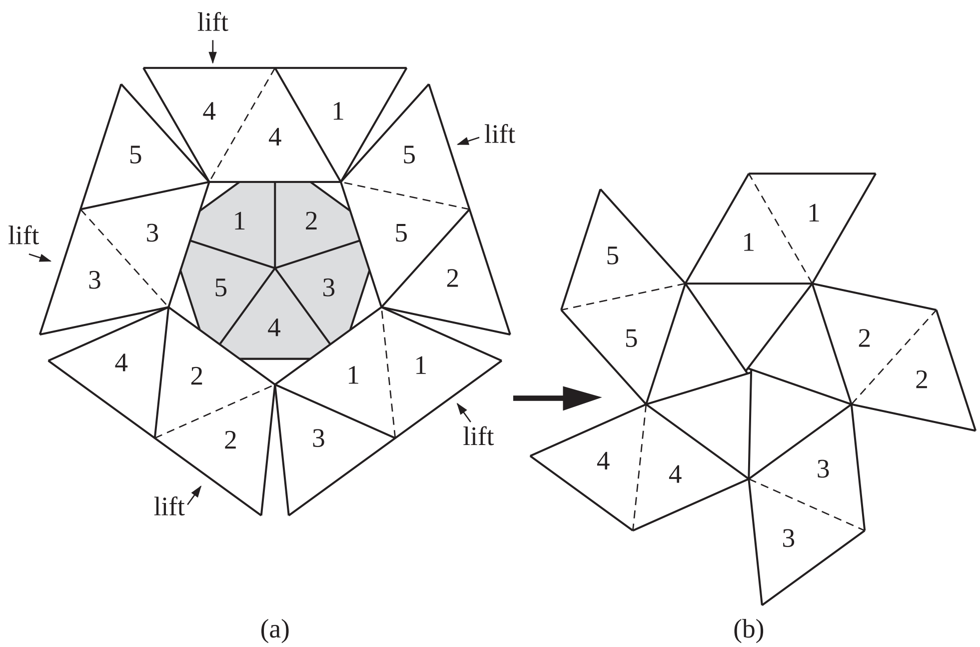


FIGURE 22

with a “2” on each of its triangles, the next with a “3”, the next with a “4”, and, finally, the last with a “5”.

Now lay the five strips out so that they overlap each other *precisely* as shown in Figure 21, making sure that the center five triangles form a shallow cup that points *away* from you. If you do this correctly, all five dots will be hidden. You may wish to use some transparent tape to hold the strips in this position. If you do need the tape, it works best to put a small strip along the middle of each of the five lines coming form the center of the configuration (this tape won’t show when the model is finished).

Now study the situation carefully before making your next move. You must bring the ten ends up so that the part of the strip at the tail of the arrow goes under the part of the strip at the head of the arrow (this means “under” as you look down on the diagram; it is really on the outside of the model you are creating, because we are looking at the inside of the model). Half the strips wrap in a clockwise direction, and the other end of each of those strips wraps in a counterclockwise direction. What finally happens is that each strip overlaps itself at the top of the model. But, in

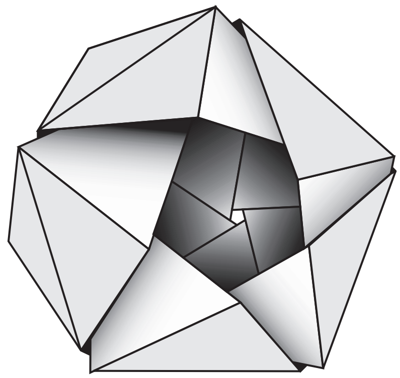


FIGURE 23

the intermediate stage, the model will look like Figure 22(a). At this point it may be useful to slide a rubber band, from the bottom, up around the emerging polyhedron to just below where the flaps are sticking out from the open pentagon (be careful not to use a rubber band that fits too tightly). Then lift the flaps as indicated by the arrows and bring them toward the center so that they tuck in, as shown in Figure 22(b).

Now simply lift flap 1 and smooth it into position. Do the same with flaps 2, 3, and 4. Complete the model by tucking flap 5 into the obvious slot. The vertex of the icosahedron nearest you will look like Figure 23.

Your finished model should have 20 triangular faces and look like Figure 20(a), with 4 triangles from each strip visible on its surface.

Congratulations! You have completed the most difficult model in this section.

A suggestion for how to build a "more symmetric" icosahedron with 6 strips appears in Chapter 8.

Cube constructed from 3 strips

Prepare three strips of 5 squares each, as shown in Figure 24(b). Figure 24(c) shows one possible set of exact fold lines that produces the desired 5 squares. Be sure only to fold on the *short* lines of the tape after you cut out the pattern piece.

The construction may be accomplished by first taking one strip and clipping the end squares together with a paper clip. Then take a second strip and wrap it around the outside of the "cube" so that one square covers the clipped squares from the first strip and the end squares cover one of

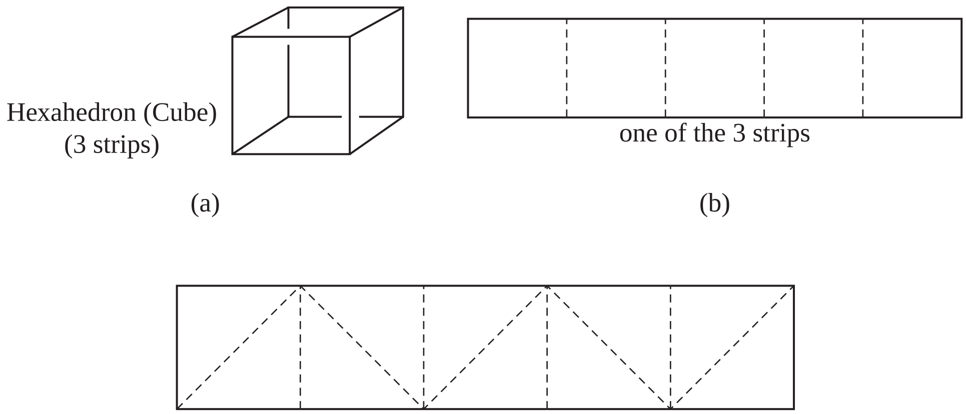


FIGURE 24

the square holes. Secure this second strip with a paper clip. Make certain that the overlapping squares of the second strip do not cover any squares from the first strip and that the first paper clip is covered. The pieces should now appear as shown in Figure 25(a). Now slide the third strip under the top square so that two squares of the third strip stick out on both the right and left sides of the cube. Tuck the end squares of this third strip inside the model through the slits along the bottom of the cube, as indicated in Figure 25(b). When the completed cube of Figure 25(b) is turned upside down, it may be opened by pulling up on the strip that covers the top face (this square will be attached inside the model by a paper clip, so you may have to pull firmly) and folding back the flaps that were the last to be tucked

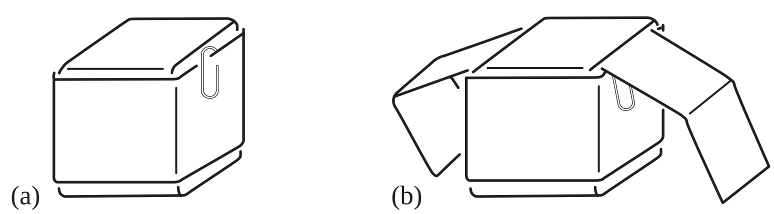


FIGURE 25

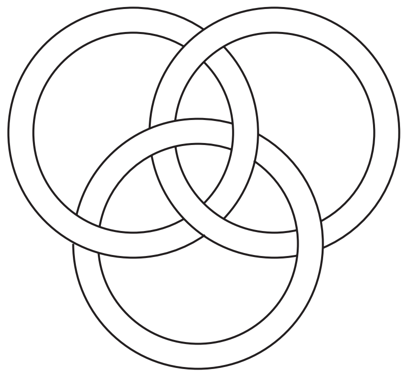


FIGURE 26

inside the model. You can remove the paper clips and put the model back together. Friction should hold the strips in place.

The cube, constructed as described above, will have 6 square faces, with opposite faces of the same color. Furthermore, if you remove any one of the strips, the other two will fall free — this is just the situation with the well-known Borromean rings, shown in Figure 26. It is also possible to construct a cube so that there are three pairs of adjacent faces having the same color. We leave this as an exercise for the interested reader.

Cube constructed from 4 strips

Prepare four strips of 7 right isosceles triangles as shown in Figure 27(a). Consult Figure 24(c) to see how to make exact folds on the tape in order to produce the pattern pieces. This time, however, you will need a longer piece of the tape, and you will only fold on the *long* lines after you cut out each pattern piece.

Begin the construction by laying the 4 strips on a table with the colored side *down*, exactly as shown in Figure 27(b). The first time you do this it may be helpful to secure the center (where the 4 strips cross each other) with some transparent tape. It is sometimes useful to put a can of unopened soda pop on this square to hold the arrangement in place and allow you to work on the vertical faces. Remove the can before you complete the top face! Now think of the center square in Figure 27(b) as the base of the cube you are constructing and note that the strip near the tail of each arrow should go *under* the strip at the head of the arrow (thus the strip near the tail will be on the outside of the model when it crosses the vertical edge of the

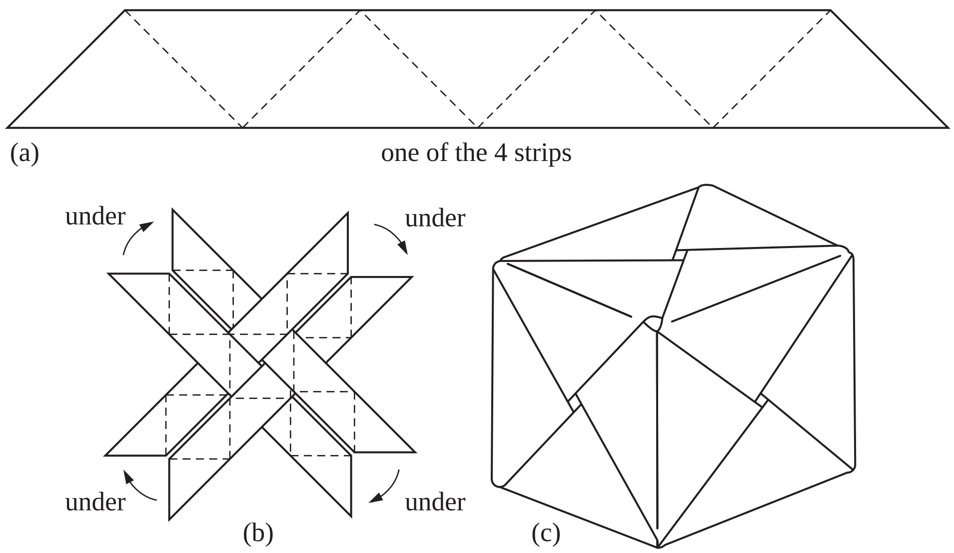


FIGURE 27

cube). The procedure for completing the cube is now almost self-evident, especially if you remember that every strip must go alternately over and under the other strips on the model. It may help to secure the centers of the vertical faces with transparent tape as you complete them, but as you become experienced at this construction, you will soon abandon such aids. The final triangles will tuck in to produce the cube of Figure 27(c).

This cube, which we naturally call the ***diagonal cube***, has some remarkable combinatorial properties. For example, how many ways can you arrange 4 colors in a circle? Look at the faces of this cube. How many ways can you take 3 of 4 colors and arrange them in a circle? Look at the vertices of this cube. How many ways can you take 4 colors 2 at a time? Look at the edges of the cube that are opposite each other, with respect to the center of the cube.

There are other remarkable facts connected with the diagonal cube that are discussed in Chapter 8.

Dodecahedron constructed from 6 strips

Prepare 6 strips from the D^2U^2- tape so that you have 3 pairs of strips like the pair shown in Figure 28(b).

Notice that in preparing the pattern pieces for the dodecahedron shown in Figure 28(a) we only need to use the short lines on the D^2U^2-tape, but

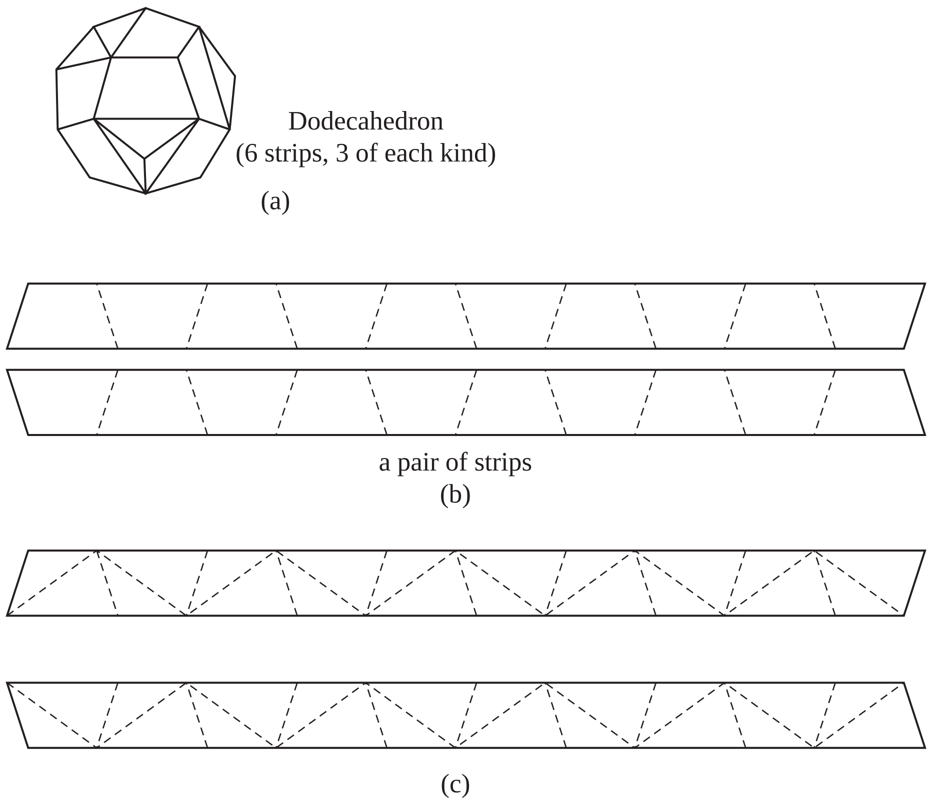


FIGURE 28

the gummed tape will look like that shown in Figure 28(c). Thus, after you glue the gummed tape to the colored paper you should only refold on the short lines after you cut out the pieces.

To construct this model take one pair of the strips and cross them with the colors visible as shown in Figure 29.

Secure the overlapping edge with a bobby pin (a paper clip will *NOT* work) or stick some transparent tape along each of the edges that are within the two center pentagons. Then make a bracelet out of each of the strips in such a way that

1. four sections of each strip overlap, and
2. the strip that is *under* on one side of the bracelet is *over* on the other side of the bracelet. (This will be true for both strips.)

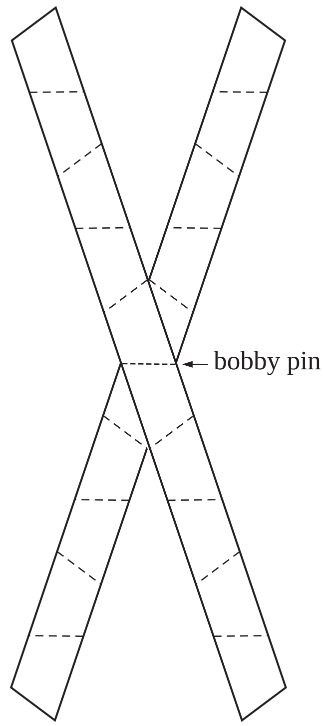


FIGURE 29

Use another bobby pin to hold all four thicknesses of tape together on the edge that is opposite the one already secured by a bobby pin.

Repeat the above steps with another pair of strips. You now have two identical bracelet-like arrangements. Slip one inside the other one as illustrated in Figure 30, so that it looks like a dodecahedron with triangular holes in four faces.

Take the last two strips and cross them precisely as you did earlier (reversing the crossing would destroy some of the symmetry on the finished model); then secure them with a bobby pin. Carefully put two of the loose ends (either the top two or the bottom two) through the top hole and pull them out the other side so that the bobby pin lands on *CD*. Then put the other two ends through the bottom hole and pull them out the other side (see Figure 31(a)). Now you can tuck in the loose flaps, but make certain to reverse the order on the strips — that is, whichever one was *under* at *CD* should be on *top* when you do the final tucking (and, of course, the top

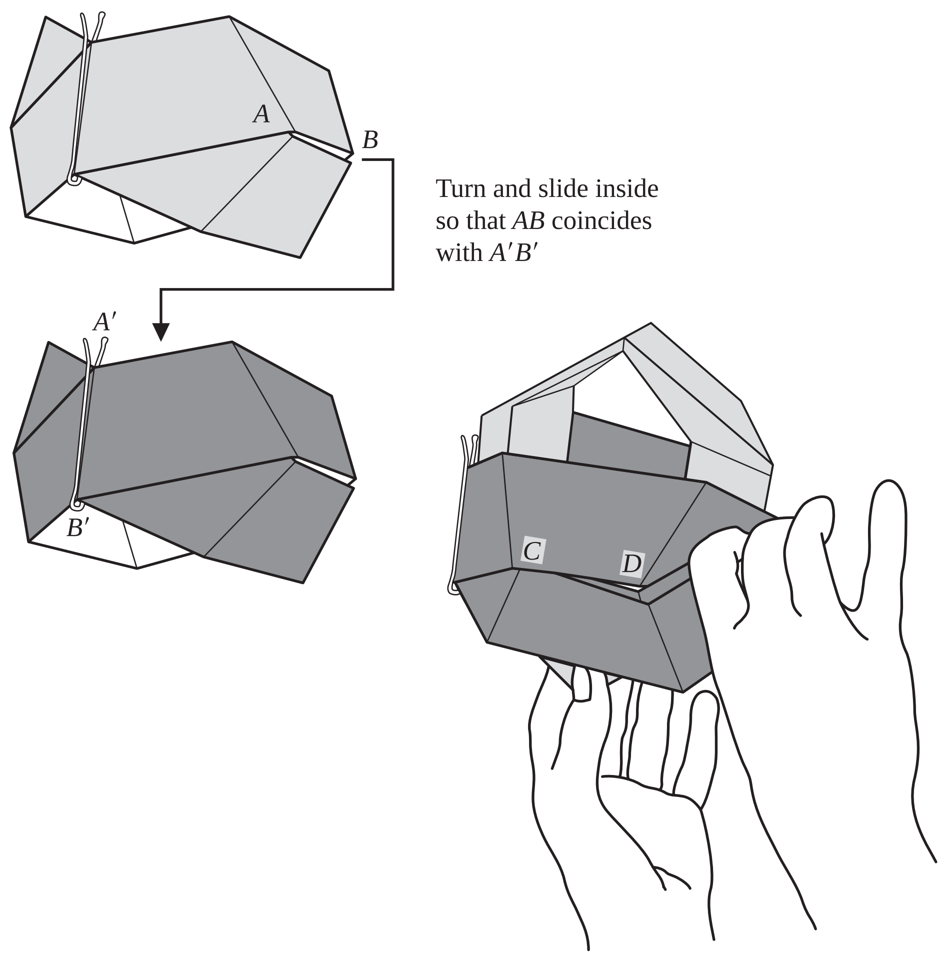
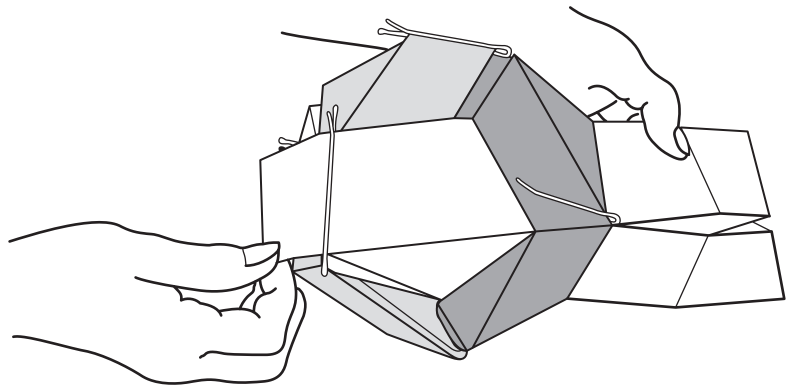


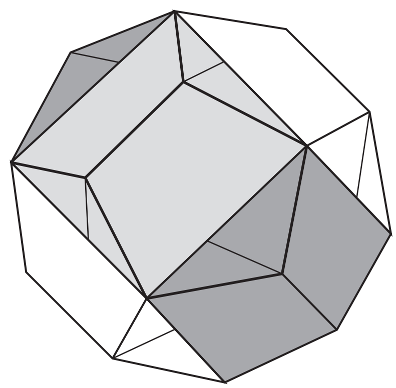
FIGURE 30

strip at *CD* will be the bottom strip when you do the tucking). Your model should look like Figure 28(a), with each face having two colors on it.

After you have mastered this construction you may wish to try to construct the model with tricolored faces, shown in Figure 31(b). This construction and the one just described are both very similar to the construction for the cube with 3 strips. The difference is that in the case of the dodecahedron, the three "bracelets" that are braided together are each composed of two strips. This illustrates, rather vividly, exactly how to inscribe the cube



(a)



(b)

FIGURE 31

symmetrically inside the dodecahedron. To put it another way, it shows how the dodecahedron may be constructed form the cube by placing a "hip roof" on each of the 6 faces of the cube. You should be able to see exactly what the hip roof looks like by examining the dodecahedron with two colors on each face.

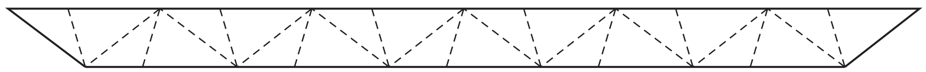


FIGURE 32

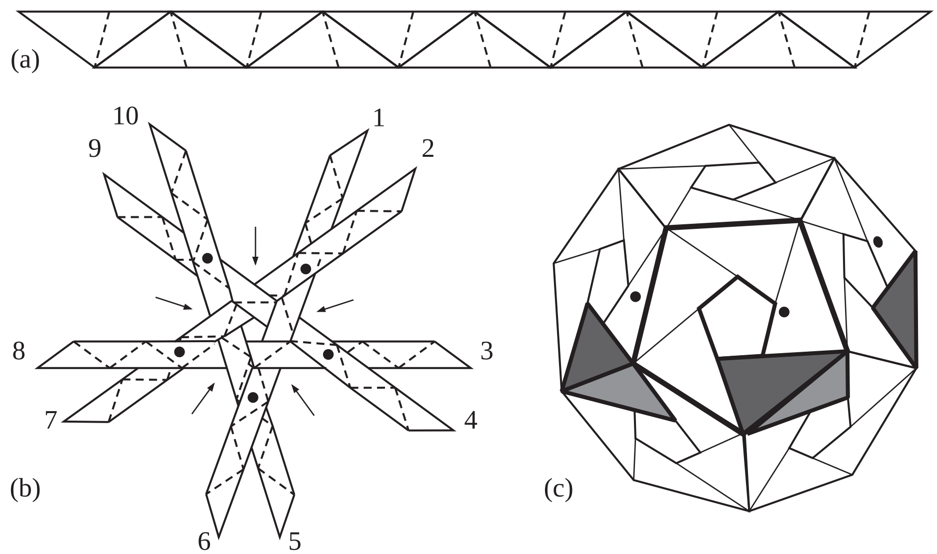


FIGURE 33

Golden Dodecahedron constructed from 6 strips

Prepare 6 strips from the D^2U^2- tape so that each strip has 22 triangles, exactly as shown in Figure 32.

When you have cut out the pattern pieces remember that, this time, you should refold each piece only on the *long* fold lines, so that the mountain fold is on the colored side of the strip. Leave the short lines *uncreased*, so that each of your 6 strips looks like Figure 33(a).

To complete the construction, begin by taking five of the strips and arranging them, with the colors visible, as shown in Figure 33(b). Secure this arrangement with paper clips at the points marked with arrows. View the center of the configuration as the North Pole. Lift this arrangement and slide the even-numbered ends clockwise over the odd-numbered ends to form the five edges coming south from the arctic pentagon. Secure the strips with paper clips at the points indicated by dots (notice where these dots are located on the finished model in Figure 33(c)). Now weave in the sixth (equatorial) strip, shown shaded in Figure 33(c), and continue braiding and clipping, where necessary, until the ends of the first five strips are tucked in securely around the South Pole. During this last phase of the construction, *keep calm and take your time!* Just make certain that every strip goes alternately over and under every other strip all the way around the model. When the model is complete, all the paper clips may be removed, and the model will remain stable.

We think this model is remarkably elegant, and it has lovely symmetry. We have named it the Golden Dodecahedron because the ratio of the length of the long line to the length of the short line on the D^2U^2-tape is, in fact, the golden ratio. See Chapter 8 for more discussion about this model and its symmetry.

★ 4.5 THE GENERAL QUASI-ORDER THEOREM

Now, back to mathematics! In Section 2 we obtained a universal algorithm for folding a $\{\frac{b}{a}\}$-gon, where a, b are coprime odd integers with $a < \frac{b}{2}$. But, from the number-theoretic point of view, it turns out that we have much more. For, from our definition of the *symbol* (10) associated with a given set of folding instructions for our tape, we were able (in [2]) to state and prove

Theorem 2 ***(The Quasi-Order Theorem in base 2)*** *If a, b are coprime integers with $a < \frac{b}{2}$, and if*

$$b \left| \begin{matrix} a_1 & a_2 & \cdots & a_r \\ k_1 & k_2 & \cdots & k_r \end{matrix} \right|, \qquad a_1 = a, \quad a_i \neq a_j \quad \text{if} \quad i \neq j, \tag{17}$$

with $b - a_i = 2^{k_i} a_{i+1}$, $i = 1, 2, \cdots, r$, $(a_{r+1} = a_1)$ is a ***reduced*** *and* ***contracted*** *symbol, and if $k = \sum_{i=1}^{r} k_i$, then k is the quasi-order of* 2 mod b *and, indeed,*

$$2^k \equiv (-1)^r \bmod b \tag{18}$$

Here the ***quasi-order*** of $t \bmod b$, where t, b are coprime, is the smallest positive integer k such that[11]

$$t^k \equiv \pm 1 \bmod b \tag{19}$$

Before we pursue the generalization of this theorem to a general base t we will give you an example and let you experiment with some particularly interesting numbers in a break.

Example 1 (continued) Notice that we could find $k = 5$ from any part of the complete symbol for 31 (see item (9)). Also note that r is an even number in each of the three parts of the symbol. Thus the Quasi-Order Theorem tells us that $k = 5$ is the smallest positive integer such that $2^k \equiv \pm 1 \bmod 31$ and, moreover, that

$$2^k \equiv 1 \bmod 31$$

It is certainly easy to confirm, in this case, that this is true.

• • • BREAK 7

(1) Use your result from Break 5 (2) to find out what the Quasi-Order Theorem tells you when $b = 91$.

(2) A number of the form $2^p - 1$, where p is prime, is called a ***Mersenne number*** because Abbé Mersenne studied these numbers in his search for primes. Construct the symbol (17) for $b = 23$ (with $a = 1$) and see what this tells you about one of the Mersenne numbers. How do you think Mersenne would have liked this result?

(3) Recall that Fermat expected that all numbers of the form $F_n = 2^{2^n} + 1$ would be prime. Construct the symbol (17) for $b = 641$ (with $a = 1$) and see what this tells you about F_5. Notice that this result is achieved without ever calculating with numbers greater than 641.

You may have inferred from the above examples that the symbol (17) gives us, in each case, only one factor of $2^k \pm 1$. However, we described in Section 4.6 of [2] how the symbol could be used to obtain the *complementary* factor. (It is just A_1 as in item (38), p. 135 of [2], since $a = 1$.)

Now we return to our main question: *how do we generalize the Quasi-Order Theorem?* It is interesting, and not altogether surprising, that our main difficulty in generalizing this theorem to a general base t lies not in *proving* the generalization but in *stating* it. For generalization is an art, not

[11] It is not very difficult to prove that such a number k always exists.

an algorithmic procedure, so judicious choices must be made in formulating the generalization. Let us therefore first recall how the symbol (17) was constructed.

We start with an odd positive integer b and an odd positive integer $a < \frac{b}{2}$; at this stage we do *not* insist that a is prime to b. We then execute the ***Ψ-algorithm*** as follows. We choose k to be maximal such that $2^k | b - a$, so that $k \geq 1$, and set $\Psi(a) = a'$, where $b = a + 2^k a'$. We then show that Ψ is a permutation of the set S_b of odd positive integers $a < \frac{b}{2}$ which preserves $\gcd(a, b)$; that is, $\gcd(a', b) = \gcd(a, b)$ if $\Psi(a) = a'$. As a tool in proving that Ψ is a permutation, we introduce the ***reverse algorithm Φ***. Thus, given $c \in S_b$, we proceed as follows: we choose ℓ minimal such that $2^\ell c > \frac{b}{2}$, so that $\ell \geq 1$, and we set $c' = b - 2^\ell c$. Then $\Phi(c) = c'$. We may then prove that Φ is the *inverse* of Ψ; that is, $\Phi(a') = a$ if and only if $\Psi(a) = a'$. Do you see why c' belongs to S_b?

Now, given b, a we construct (17) by setting $\Psi(a_i) = a_{i+1}, i = 1, \ldots, r$, *where* $a_{r+1} = a_1 = a$; more precisely,

$$b = a_i + 2^{k_i} a_{i+1}, \quad i = 1, \cdots, r \tag{20}$$

Note that, since Ψ is a permutation of S_b, we must eventually find r such that $a_{r+1} = a_1$; at that point we stop.

It is thus the Ψ- and Φ-algorithms that must be generalized. We describe how this is done so that you may see that we are indeed generalizing to a general base $t \geq 2$ from the special case $t = 2$, although the choice of generalization is not always obvious. It turns out to be easier to generalize the Φ-algorithm than the Ψ-algorithm, so we tackle that first.

We start with a positive integer b prime to t and we choose a positive integer $c < \frac{b}{2}$ such that $t \nmid c$. (Notice that the original restriction that c be odd generalizes differently from the condition that b be odd; and that the "generalization" of the condition $c < \frac{b}{2}$ is precisely $c < \frac{b}{2}$.) We write $c \in S_b$, where now S_b stands for the set of integers $< \frac{b}{2}$ that are not divisible by t, and define ℓ to be minimal for the property

$$t^\ell c > \frac{b}{2} \tag{21}$$

Notice that $\ell \geq 1$. We now define qb to be the *integer multiple of* b *nearest to* $t^\ell c$. Notice that

(i) when $t = 2$ we always have $q = 1$ and $2^\ell c < b$;
(ii) there is a *unique* such multiple. For $t^\ell c$ cannot be a multiple of $\frac{b}{2}$, because this would imply that $t^\ell c = \frac{sb}{2}$, for some s, so $sb = 2t^\ell c$, $b | 2t^\ell c$. But b, t are coprime, so $b | 2c$, contradicting $c < \frac{b}{2}$.

We now introduce a quantity ϵ such that $\epsilon = 0$ or $\epsilon = 1$. If $t^\ell c > qb$, then $\epsilon = 0$; and if $t^\ell c < qb$, then $\epsilon = 1$. In any case, we define $c' > 0$ by

$$t^\ell c - qb = (-1)^\epsilon c', \tag{22}$$

and set

$$\Phi(c) = c'. \tag{23}$$

Notice that, as stated earlier, with $t = 2$, we always have $q = 1$, $2^\ell c < b$, so $\epsilon = 1$. We now claim

Theorem 3 *The algorithm Φ, defined above, is a permutation of S_b such that* $\gcd(c, b) = \gcd(c', b)$, *where* $\Phi(c) = c'$.

Proof First we must show that $c' \in S_b$. It is obvious from the definition of Φ that $c' < \frac{b}{2}$. Now $t^{\ell-1}c < \frac{b}{2}$, so $t^\ell c < \frac{tb}{2}$. Thus (see Figure 34)

$$\text{if } t \text{ is odd,} \quad 2q + 1 \le t; \qquad \text{if } t \text{ is even,} \quad 2q \le t. \tag{24}$$

In either case, (24) shows that $t \nmid q$. But if $t | c'$ then, from (22), $t | qb$, so that t, b being coprime, $t | q$. Hence, as required, $t \nmid c'$, so $c' \in S_b$, and Φ maps S_b to itself.

• • • BREAK 8

Draw the figure corresponding to Figure 34 for t even.

It remains to show that Φ is a permutation of S_b. Thus we seek another function $\Psi : S_b \to S_b$, inverse to Φ, that is, such that

$$\Phi\Psi = \mathrm{Id}, \quad \Psi\Phi = \mathrm{Id}, \tag{25}$$

where Id represents the appropriate identity function.

However, it is of great practical importance to point out that each of the relations in (25) implies the other (after all, mathematicians strongly dislike doing unnecessary work!). For if, say, $\Phi\Psi = \mathrm{Id}$, then

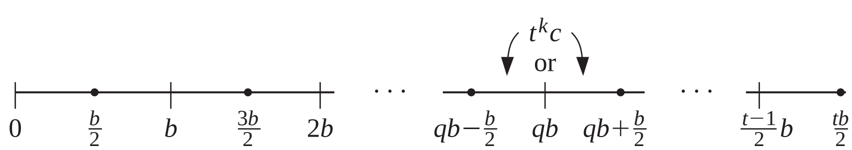


FIGURE 34 The Φ-algorithm with t odd.

Φ maps S_b *onto* itself. But S_b is a *finite* set, so Φ must be a one-one correspondence, with inverse Ψ.

We now proceed to define Ψ. Here we must distinguish between the two cases (i) t odd, (ii) t even.

(i) ***t* odd.** Let $a \in S_b$. We claim that, among the $(t-1)$ positive integers

$$\left\{qb - a, \quad qb + a, \quad 1 \leq q \leq \frac{t-1}{2}\right\}$$

there is *exactly one* divisible by t. For, trivially, the t integers $-\frac{t-1}{2} \leq r \leq \frac{t-1}{2}$ run through all residue classes mod$t$. Therefore, since b is prime to t, so do the t integers rb with r in the given range; and so too, therefore, do the t integers $rb + a$ with r in the given range. However, $r = 0$ does not yield the residue class 0, since $t \nmid a$. Thus, to obtain $rb + a \equiv 0 \bmod t$, we must take r strictly positive or negative. If $r > 0$, set $r = q$. Then, for one value of q in $1 \leq q \leq \frac{t-1}{2}$, we have $qb + a \equiv 0 \bmod t$. If $r < 0$, set $r = -q$. Then, for one value of q in $1 \leq q \leq \frac{t-1}{2}$, we have $qb - a \equiv 0 \bmod t$. Moreover, if there is a value of q yielding $qb + a \equiv 0 \bmod t$, there is no value of q yielding $qb - a \equiv 0 \bmod t$, and conversely. This establishes our claim.

(ii) ***t* even.** Let $a \in S_b$. We claim that, among the $(t-1)$ positive integers

$$\left\{qb + a, \quad 1 \leq q \leq \frac{t}{2} - 1; \qquad qb - a, \quad 1 \leq q \leq \frac{t}{2}\right\}$$

there is *exactly one* divisible by t. Now we start with the range $-\frac{t}{2} \leq r \leq \frac{t}{2} - 1$ and proceed just as in the case when t is odd. We confidently leave the details to the reader.

We now complete the definition of Ψ, but will be content to describe Ψ explicitly only when t is *odd*. We expect the reader to supply the modification needed when t is even.[12] We choose q as explained above, and define a' by the rule

$$qb \pm a = t^k a', \qquad k \quad \text{maximal} \quad (\text{so that} \quad k \geq 1) \tag{26}$$

[12] This is an important exercise, since we would wish to be sure that we are generalizing the case $t = 2$.

We claim that $a' \in S_b$. Now obviously, $t \nmid a'$, by the maximality of k. Also

$$ta' \leq t^k a' = qb \pm a \leq \frac{t-1}{2} b \pm a < \frac{tb}{2} \tag{27}$$

so that $a' < \frac{b}{2}$. We set $\Psi(a) = a'$.

We now prove that $\Phi\Psi = \text{Id}$; recall that this will establish that Φ, Ψ are mutually inverse permutations of S_b. Assume t odd, and let $\Psi(a) = a'$. We want to show that $\Phi(a') = a$. Now, from (27), we easily see that $t^{k-1}a' < \frac{b}{2}$, and from (26) we easily see that

$$t^k a' - qb = \pm a$$

This establishes that $\Phi(a') = a$, so $\Phi\Psi = \text{Id}$ as claimed. The proof of Theorem 3 will be complete (but, of course, we have proved much more) if we show that $\gcd(c, b) = \gcd(c', b)$, or, equivalently, that $\gcd(a, b) = \gcd(a', b)$, where $\Psi(a) = a'$. We choose to prove the former.

Now we have $t^k c - qb = (-1)^\epsilon c'$. Thus, if $d|c, d|b$, then $d|c', d|b$. Conversely, if $d|c', d|b$, then $d|t^k c, d|b$. But, since t, b are coprime, it follows that if $d|b$, then d is prime to t, so $d|c$. We have thus established that $\gcd(c, b) = \gcd(c', b)$. □

In describing the Φ-algorithm, we introduced the quantity ϵ, which takes the value 0 or 1 according to whether $t^k c - qb$ is positive or negative (see (22)). For consistency we must introduce it again in (26), where we must choose between $qb + a$ or $qb - a$, when seeking the allowed number divisible by t. Thus we refine (26) to

$$qb + (-1)^\epsilon a = t^k a' \tag{28}$$

We are now in a position to define a ***t*-symbol**, which we will still just call a symbol if there is no doubt what base t is being used. However, we will add an extra row to the symbol, by comparison with the case $t = 2$, to incorporate the quantity ϵ (recall that if $t = 2$, we always have $\epsilon = 1$). On the other hand, we will not need to include the quantity q from (28) in the symbol if our purpose is just to state and prove the general Quasi-Order Theorem.

We suppose that b, a are coprime, with b prime to t, $t \nmid a$, and $a_1 = a < \frac{b}{2}$. We proceed to obtain a contracted, reduced symbol

$$b \left| \begin{array}{cccc} a_1 & a_2 & \cdots & a_r \\ k_1 & k_2 & \cdots & k_r \\ \epsilon_1 & \epsilon_2 & \cdots & \epsilon_r \end{array} \right|_t \tag{29}$$

where

$$q_i b + (-1)^{\epsilon_i} a_i = t^{k_i} a_{i+1}, \quad i = 1, 2, \cdots, r \quad (a_{r+1} = a_r) \tag{30}$$

and there are no repeats among the a_i, $i = 1, 2, \cdots, r$; recall that we know that each a_i is prime to b, $t \nmid a_i$, and $a_i < \frac{b}{2}$. Our symbol (29) is, of course, based on the Ψ-algorithm.

Notice that, if we use the Φ-algorithm instead of the Ψ-algorithm, we get the *reverse symbol*, incorporating the same r facts (30); thus, we may replace (29) by

$$b \begin{pmatrix} a_1 & a_r & a_{r-1} & \cdots & a_2 \\ k_r & k_{r-1} & k_{r-2} & \cdots & k_1 \\ \epsilon_r & \epsilon_{r-1} & \epsilon_{r-2} & \cdots & \epsilon_1 \end{pmatrix}_t \tag{31}$$

For neatness, we rewrite (31) as

$$b \begin{pmatrix} c_1 & c_2 & \cdots & c_r \\ \ell_1 & \ell_2 & \cdots & \ell_r \\ \eta_1 & \eta_2 & \cdots & \eta_r \end{pmatrix}_t \tag{32}$$

so that $c_1 = a_1$, $c_j = a_{r+2-j}$, $2 \leq j \leq r$, $\ell_j = k_{r+1-j}$, $\eta_j = \epsilon_{r+1-j}$, and (30) is to be rewritten as

$$q'_j b + (-1)^{\eta_j} c_{j+1} = t^{\ell_j} c_j \tag{33}$$

(where, in fact, $q'_j = q_{r+1-j}$).

We set $L = \sum \ell_j$, $E = \sum \eta_j$ from (32); of course, it is equally true that $L = \sum k_i$, $E = \sum \epsilon_i$ from the symbol (29).

Example 4 Form the symbol (29) with $b = 19$, $a_1 = 6$, $t = 4$.

Solution From the following calculations we obtain the symbol below:

$$\begin{aligned} 2\cdot 19 - 6 &= 4^2\cdot 2\\ 2\cdot 19 - 2 &= 4^1\cdot 9\\ 19+9 &= 4^1\cdot 7\\ 19-7 &= 4^1\cdot 3\\ 19-3 &= 4^2\cdot 1\\ 19+1 &= 4^1\cdot 5\\ 19+5 &= 4^1\cdot 6 \end{aligned}$$

$$19\left|\begin{matrix} 6 & 2 & 9 & 7 & 3 & 1 & 5\\ 2 & 1 & 1 & 1 & 2 & 1 & 1\\ 1 & 1 & 0 & 1 & 1 & 0 & 0 \end{matrix}\right|_4$$

Example 5 Form the symbol (29) with $b = 19$, $a_1 = 6$, $t = 5$.

Solution From the following calculations we obtain the symbol below:

$$\begin{aligned} 19+6 &= 5^2\cdot 1\\ 19+1 &= 5^1\cdot 4\\ 19-4 &= 5^1\cdot 3\\ 2\cdot 19 - 3 &= 5^1\cdot 7\\ 2\cdot 19 + 7 &= 5^1\cdot 9\\ 19-9 &= 5^1\cdot 2\\ 2\cdot 19 + 2 &= 5^1\cdot 8\\ 2\cdot 19 - 8 &= 5^1\cdot 6 \end{aligned}$$

$$19\left|\begin{matrix} 6 & 1 & 4 & 3 & 7 & 9 & 2 & 8\\ 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{matrix}\right|_5$$

Now, let's take a break so you can get some practice!

• • • BREAK 9

(1) Form the symbol with $b = 13$, $a_1 = 6$, $t = 4$, using the Ψ-algorithm. (Hint: In calculating $\Psi(a)$ we must find out which of $b - a$, $b + a$, $2b - a$ is divisible by 4.)

(2) Form the symbol with $b = 17$, $a_1 = 3$, $t = 5$, using the Ψ-algorithm. (Hint: In calculating $\Psi(a)$ we must find out which of $b - a$, $b + a$, $2b - a$, $2b + a$ is divisible by 5.)

We are now ready to prove the general Quasi-Order Theorem. We suppose given a Φ-symbol (32); of course, we might also suppose given the associated Ψ-symbol (29), so that the quantities L, E may be read off from either symbol as the sum of the second row and the sum of the third row, respectively. Our theorem is then the following.

Theorem 6 **(*The General Quasi-Order Theorem*)** *Given a (contracted, reduced)* Φ*-symbol* (32)*, then the quasi-order of* t mod b *is* $L = \sum \ell_j$*. Indeed,*

$$t^L \equiv (-1)^E \bmod b, \quad \textit{where} \quad E = \sum \eta_j.$$

Proof We consider the sequence of $L + 1$ integers

$$c_1, tc_1, \dots, t^{\ell_i - 1}c_1, c_2, tc_2, \dots, t^{\ell_2 - 1}c_2, c_3, \dots, c_r, tc_r, \dots, t^{\ell_r - 1}c_r, c_1 \tag{34}$$

We say that a ***switch*** takes place when we pass from $t^{\ell_i - 1}c_i$ to c_{i+1}, $i = 1, 2, \cdots, r$ $(c_{r+1} = c_1)$. We also write (34) as

$$u_1, u_2, \dots, u_{L+1} \tag{35}$$

We note that, following rule (21) for defining the Φ-algorithm,

$$u_j < \frac{b}{2}, \quad 1 \le j \le L + 1 \tag{36}$$

We also note from (33) that

$$\begin{cases} u_{j+1} = tu_j & \text{if no switch occurs,} \\ u_{j+1} \equiv (-1)^{\eta_i} tu_j \bmod b & \text{if a switch occurs to } c_{i+1}. \end{cases} \tag{37}$$

It follows that

$$u_{L+1} \equiv (-1)^E t^L u_1 \bmod b$$

or

$$c_1 \equiv (-1)^E t^L c_1 \bmod b$$

But c_1, b are coprime, so $t^L \equiv (-1)^E \bmod b$, as claimed.

It remains to show that L is indeed the quasi-order of $t \bmod b$. Suppose not; then there exists an M, where $1 \le M < L$, with $t^M \equiv \pm 1 \bmod b$. Thus, from (37),

$$u_{M+1} \equiv \pm u_1 \bmod b \tag{38}$$

We first show that $u_{M+1} \not\equiv u_1 \bmod b$. For, by (36), if $u_{M+1} \equiv u_1 \bmod b$, then $u_{M+1} = u_1$. If u_{M+1} arose at a switch, then, remembering that $M + 1 < L + 1$, we see that $u_{M+1} = u_1$ contradicts the fact that the symbol (32) is contracted. If u_{M+1} did not arise at a switch, then $t | u_{M+1}$, $t \nmid u_1$, again contradicting $u_{M+1} = u_1$. Hence $u_{M+1} \not\equiv u_1 \bmod b$.

Finally, we show that $u_{M+1} \not\equiv -u_1 \bmod b$. For if

$$u_{M+1} \equiv -u_1 \bmod b,$$

then $b | (u_{M+1} + u_1)$. But, by (36), $u_{M+1} + u_1 < b$, so this is impossible. Thus (38) is false, and L is, as claimed, the quasi-order of $t \bmod b$. □

Corollary 7 ***(Alternative form of the General Quasi-Order Theorem)*** *Given a (contracted, reduced) Ψ-symbol (29), then the quasi-order of $t \bmod b$ is $K = \sum k_i$. Indeed,*

$$t^K \equiv (-1)^E \bmod b$$

where $E = \sum \epsilon_i$.

Remark Recall our claim that the *proof* of Theorem 3 is scarcely more difficult in the general case than the special case $t = 2$; the only (slight) complication arises from the fact that η_j in (32), (33) may be 0 or 1 in the general case, whereas we always have $\eta_j = 1$ if $t = 2$. On the other hand, we do not expect you to doubt that preparing the ground for Theorem 3, that is, defining the Φ- and Ψ-algorithms, was *much* more difficult in the general case!

Example 4 (revisited) From the symbol we created in Example 4 we see that, referring to Corollary 7, $K = 9$, $E = 4$, so that the quasi-order of 4 mod 19 is 9, and indeed

$$4^9 \equiv (-1)^4 \bmod 19 = 1 \bmod 19$$

Example 5 (revisited) From the symbol we created in Example 5 we see that, referring to Corollary 7, $K = 9$, $E = 4$, so that the quasi-order of 5 mod 19 is 9, and indeed

$$5^9 \equiv (-1)^4 \bmod 19 = 1 \bmod 19$$

(Why do you think the values of K are the same in these two examples?)

• • • **BREAK 10**

(1) Refer to the Ψ-symbol you constructed in Break 9(1) and write down the Φ-symbol for $b = 13$, $c_1 = 6$, $t = 4$. From either symbol obtain the quasi-order L of 4 mod 13, and determine whether $4^L \equiv +1$ or -1 mod 13.

(2) Refer to the Ψ-symbol you constructed in Break 9(2) and write down the Φ-symbol for $b = 17$, $c_1 = 3$, $t = 5$. From either symbol obtain the quasi-order L of 5 mod 17, and determine whether $5^L \equiv +1$ or -1 mod 17.

(3) (Harder) Show that the quasi-order of t mod b, where b is an odd prime, is always a factor of $\frac{1}{2}(b-1)$. (Hint: Use Fermat's Little Theorem (Ch. 2 of [2].))

(4) Answer the question following Example 5 using the hint given for solving question (3).

★

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