

# Existence Theorems for the 3D–Navier–Stokes System Having as Initial Conditions Sums of Plane Waves

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We study the 3-dimensional Navier–Stokes system on  $\mathbb{R}^3$  without external forcing and prove local and global existence theorems for initial conditions which are the Fourier transforms of finite linear combinations of  $\delta$ –functions. Bibliography: 8 titles.

**1. Formulation of the result.** We consider the 3D Navier–Stokes system on  $\mathbb{R}^3$  for incompressible fluids. The viscosity is taken to be 1, and no external forcing is assumed. After the Fourier transform the system takes the form

$$v(k, t) = \exp\{-t|k|^2\}v(k, 0) + i \int_0^t \exp\{-(t-s)|k|^2\} ds \int_{\mathbb{R}^3} \langle v(k-k', s), k \rangle P_k v(k', s) d^3 k', \quad (1)$$

where  $v(k, 0)$  is the initial condition,  $P_k$  is the orthogonal projection to the subspace orthogonal to  $k$ , i.e.,  $P_k v = v - \frac{\langle v, k \rangle}{\langle k, k \rangle} k$ . The values  $v(k, s) \in \mathbb{C}^3$ , the incompressibility means that  $\langle v(k, t), k \rangle = 0$ . If  $v(k, t)$  is the Fourier transform of a real-valued function  $u(x, t)$ , then

$$v(-k, t) = \overline{v(k, t)}. \quad (2)$$

However, in this paper, we consider arbitrary  $v(k, t)$  not assuming (2) (see also [5]).

Beginning with the works of Leray [1], Hopf [3], Kato [4], people considered the problem of local and global existence of solutions of (1) with initial conditions having finite energy

$$E(0) = \int_{\mathbb{R}^3} \langle v(k, 0), \overline{v(k, 0)} \rangle d^3 k < \infty.$$

In this paper, we deal with another class of initial conditions having infinite energy. Namely, we consider the initial condition of the form

$$v(k, 0) = \sum_{j=1}^n B_j \delta(k - k_j), \quad (3)$$

where the sum is taken over a finite set  $\{k_j\}$  of points  $k_j \neq 0$ ,  $\langle B_j, k_j \rangle = 0$  for any  $j$ , and  $\delta(k - k_j)$  is the delta-function concentrated at  $k_j$ . Since  $\delta(k - k_j) \notin L^2(\mathbb{R}^3)$ , the initial condition  $v(k, 0)$  has infinite energy.

Denote by  $G(k_1, \dots, k_n)$  the semigroup generated by a finite set  $\{k_j\}$ , i.e.,  $k \in G(k_1, \dots, k_n)$  if and only if  $k = \sum_{j=1}^n p_j k_j$  for some nonnegative integers  $p_j$ . The main results of this paper are the following theorems.

**Theorem 1.** *Let  $v(k, 0) = \sum_{j=1}^n B_j \delta(k - k_j)$ . There exists  $T > 0$  depending on  $\{k_j\}$ ,  $\{B_j\}$  such that there exists a solution  $v(k, t)$  of (1)*

on the interval  $0 \leq t \leq T$  which can be written as a signed measure

$$v(k, t) = \sum_{g \in G(k_1, \dots, k_n), g \neq 0} B_g(t) \delta(k - g). \tag{4}$$

The coefficients  $B_g(t)$  satisfy the inequalities:

$$|B_g(t)| < t^{\frac{p(g)}{2}} C^{p(g)} \left( \sum_{j=1}^n |B_j| \right)^{p(g)} \left( 1 - t^{\frac{1}{2}} C \sum_{j=1}^n |B_j| \right)^{-1}, \tag{5}$$

where  $p(g) = \min_{\sum p_j k_j = g} \sum_{i=1}^n p_i$ ,  $C$  depends only on  $k_j$ .

**Theorem 2.** Let  $k_j$ ,  $1 \leq j \leq n$ , belong to some cone with angle less than  $\pi$ , i.e., the angle between any  $k_{j_1}$  and  $k_{j_2}$  is less than  $\pi$ . Then for a sufficiently small  $B$  and an initial condition  $v(k, 0)$  such that  $\sum_{j=1}^n |B_j| < B$  there exists a global solution of (1) having the form (4) for which

$$\sum_{g \in G(k_1, \dots, k_n), g \neq 0} |B_g(t)| < \infty.$$

Some existence results for a similar class of initial conditions can be found in [2, 8].

**2. Proof of Theorems 1 and 2.** The proofs of both theorems are based on the method of power series introduced in [6, 7]. We consider a one-parameter family of initial conditions  $Av(k, 0) = v_A(k, 0)$  and write down the solution of (1) as a power series with respect to the complex parameter  $A$ :

$$v_A(k, t) = Ah_1(k, t) + \sum_{p>1} A^p \int_0^t \exp\{-(t-s)|k|^2\} h_p(k, s) ds, \tag{6}$$

where  $h_1(k, t) = \exp\{-t|k|^2\}v(k, 0)$ . Substituting (6) into (1), we obtain the following system of recurrent relations between the functions  $h_p(k, t)$ :

$$h_2(k, t) = i \int_{\mathbb{R}^3} \langle v(k - k', 0), k \rangle P_k v(k', 0) \exp\{-t|k - k'|^2 - t|k'|^2\} d^3 k'$$

and for  $p > 2$

$$h_p(k, t) = i \int_0^t ds_2 \int_{\mathbb{R}^3} \langle v(k - k', 0), k \rangle P_k h_{p-1}(k', s_2)$$

$$\begin{aligned}
& \times \exp\{-t|k - k'|^2 - (t - s_2)|k'|^2\} d^3 k' \\
& + i \sum_{\substack{p_1 + p_2 = p \\ p_1, p_2 > 1}} \int_0^t ds_1 \int_0^t ds_2 \int_{\mathbb{R}^3} \langle h_{p_1}(k - k', s_1), k \rangle P_k h_{p_2}(k', s_2) \\
& \times \exp\{-(t - s_1)|k - k'|^2 - (t - s_2)|k'|^2\} d^3 k' \\
& + i \int_0^t ds_1 \int_{\mathbb{R}^3} \langle h_{p-1}(k - k', s_1), k \rangle P_k v(k', 0) \\
& \times \exp\{-(t - s_1)|k - k'|^2 - t|k'|^2\} d^3 k'. \tag{7}
\end{aligned}$$

In our case of solutions of type (4), the convolutions given by the integrals in (7) are well defined, and we can write

$$h_1(k, t) = \sum_{j=1}^n \exp\{-t|k_j|^2\} B_j \delta(k - k_j).$$

Recall that  $B_j$  are 3-dimensional vectors  $\langle B_j, k_j \rangle = 0$ . Further,

$$\begin{aligned}
h_2(k, t) &= i \int_{\mathbb{R}^3} \sum_{j_1, j_2=1}^n \langle B_{j_1}, k \rangle \left( B_{j_2} - \frac{\langle B_{j_2}, k \rangle k}{\langle k, k \rangle} \right) \\
& \times \exp\{-t|k - k'|^2 - t|k'|^2\} \delta(k - k' - k_{j_1}) \delta(k' - k_{j_2}) d^3 k' \\
&= \sum_{j_1, j_2=1}^n B_{j_1, j_2}(t) \delta(k - (k_{j_1} + k_{j_2}))
\end{aligned}$$

with

$$\begin{aligned}
B_{j_1, j_2}(t) &= i \langle B_{j_1}, k_{j_2} \rangle \exp\{-t|k_{j_1}|^2 - t|k_{j_2}|^2\} \\
& \times \left( B_{j_2} - \frac{\langle B_{j_2}, (k_{j_1} + k_{j_2}) \rangle (k_{j_1} + k_{j_2})}{\langle (k_{j_1} + k_{j_2}), (k_{j_1} + k_{j_2}) \rangle} \right).
\end{aligned}$$

If  $k_{j_1} + k_{j_2} = 0$ , then the corresponding term in the last sum is zero.

From the last formula it easily follows that

$$|B_{j_1, j_2}(t)| \leq C_1 |B_{j_1}| |B_{j_2}|, \tag{8}$$

$C_1 = \max_{1 \leq j \leq n} |k_j|$ , and

$$\sum_{j_1, j_2=1}^n |B_{j_1, j_2}(t)| \leq C_1 \left( \sum_{j=1}^n |B_j| \right)^2. \tag{9}$$

The proof of Theorem 1 is based on the following assertion.

**Lemma 1.** *Assume that for any initial condition (3) and  $0 < t < 1$  the functions  $h_q(k, t)$ ,  $2 \leq q < p$ , can be written in the form*

$$h_q(k, t) = \sum_{1 \leq j_1, \dots, j_q \leq n} B_{j_1 \dots j_q}(t) \delta\left(k - \sum_{j=1}^q k_j\right), \quad (10)$$

where  $B_{j_1 \dots j_q}(t)$  are continuous functions of  $t$  and

$$|B_{j_1 \dots j_q}(t)| \leq C_2^{q-1} t^{\frac{q-2}{2}} \prod_{l=1}^q |B_{j_l}|, \quad (11)$$

$C_2$  is another constant which depends only on the vectors  $k_j$ ,  $1 \leq j \leq n$ . Then (8) and (9) are valid for  $q = p$ .

Using Lemma 1, we derive Theorem 1. We have

$$\begin{aligned} v(k, t) &= \sum_{j=1}^n B_j \exp\{-t|k|^2\} \delta(k - k_j) \\ &+ \sum_{p>1} \sum_{j_1, \dots, j_p=1}^n \tilde{B}_{j_1 \dots j_p}(t) \delta\left(k - \sum_{l=1}^p k_{j_l}\right) \end{aligned} \quad (12)$$

and

$$\tilde{B}_{j_1 \dots j_p}(t) = \int_0^t \exp\{-(t-s)|k|^2\} B_{j_1 \dots j_p}(s) ds.$$

If  $B_{j_1 \dots j_p}(t)$  satisfies (8), then from Lemma 1 it follows that

$$|\tilde{B}_{j_1 \dots j_p}(t)| \leq \frac{2}{p} C_2^{p-1} t^{\frac{p}{2}} \prod_{l=1}^p |B_{j_l}|$$

and

$$\sum_{j_1, \dots, j_p=1}^n |\tilde{B}_{j_1 \dots j_p}(t)| \leq \frac{2}{p} C_2^{p-1} t^{\frac{p}{2}} \left(\sum_{j=1}^n |B_j|\right)^p. \quad (13)$$

Therefore, the series  $\sum_{p>1} \sum_{j_1, \dots, j_p=1}^n |\tilde{B}_{j_1 \dots j_p}(t)|$  converges absolutely if

$$t^{\frac{1}{2}} < \min\left(1, \left(C_2 \sum_{j=1}^n |B_j|\right)^{-1}\right).$$

Taking together all terms with the same value of the sum  $k_{j_1} + \dots + k_{j_p}$ , we get the representation of the solution  $v(k, t)$  as

$$\sum_{g \in G(k_{j_1}, \dots, k_{j_p})} B_g(t) \delta(k - g),$$

where for each  $B_g(t)$  we have the estimate (5). Theorem 1 is proved.  $\square$

PROOF OF LEMMA 1. For  $p = 2$  the statement of the lemma is already proved (see (8), (9)). For  $p \geq 3$  we use the recurrent relation (7). Denote by  $h_p^{(p_1, p_2)}(k, t)$  the term in (7) which corresponds to  $p_1, p_2$ . Consider the case  $p_1, p_2 \geq 2$ . We have

$$\begin{aligned} h_p^{(p_1, p_2)}(k, t) &= i \int_0^t ds_1 \int_0^t ds_2 \int_{\mathbb{R}^3} \langle h_{p_1}(k - k', s_1), k \rangle \\ &\quad \times P_k h_{p_2}(k', s_2) \exp\{-(t - s_1)|k - k'|^2 - (t - s_2)|k'|^2\} d^3 k' \\ &= i \int_0^t ds_1 \int_0^t ds_2 \sum_{j_1 \dots j_{p_1}=1}^n \langle B_{j_1 \dots j_{p_1}}(s_1), k \rangle \\ &\quad \times \left( B_{j_{p_1+1} \dots j_p}(s_2) - \frac{\langle B_{j_{p_1+1} \dots j_p}(s_2), k \rangle k}{|k|^2} \right) \\ &\quad \times \left[ \left( \exp\{-(t - s_1)|k|^2\} \delta\left(k - \sum_{l=1}^{p_1} k_{j_l}\right) \right) \right. \\ &\quad \left. \otimes \left( \exp\{-(t - s_2)|k|^2\} \delta\left(k - \sum_{l=p_1+1}^p k_{j_l}\right) \right) \right] \\ &= \sum_{j_1 \dots j_p}^n B_{j_1 \dots j_p}^{(p_1, p_2)}(s) \delta\left(k - \sum_{l=1}^p k_{j_l}\right), \end{aligned}$$

where  $\otimes$  is the convolution and

$$\begin{aligned} B_{j_1 \dots j_p}^{(p_1, p_2)}(t) &= i \int_0^t ds_1 \int_0^t ds_2 \left\langle B_{j_1 \dots j_{p_1}}(s_1), \sum_{l=p_1+1}^p k_{j_l} \right\rangle \\ &\quad \times \exp\left\{ -(t - s_1) \left| \sum_{l=1}^{p_1} k_{j_l} \right|^2 - (t - s_2) \left| \sum_{l=p_1+1}^p k_{j_l} \right|^2 \right\} \end{aligned}$$

$$\times \left( B_{j_{p_1+1} \dots j_p}(s_2) - \frac{\langle B_{j_{p_1+1} \dots j_p}(s_2), \sum_{l=1}^p k_{j_l} \rangle \sum_{l=1}^p k_{j_l}}{\left| \sum_{l=1}^p k_{j_l} \right|^2} \right).$$

Here, we used the incompressibility condition

$$\langle B_{j_1 \dots j_{p_1}}(s_1), \sum_{l=1}^{p_1} k_{j_l} \rangle = 0.$$

It is clear that  $B_p^{(p_1, p_2)}(t)$  is continuous as a function of  $t \in [0, 1]$ . By the assumption of Lemma 1,

$$\begin{aligned} \left| B_{j_1 \dots j_p}^{(p_1, p_2)}(s) \right| &\leq \int_0^s s_1^{\frac{p_1-2}{2}} \exp\left\{-(s-s_1) \left| \sum_{l=1}^{p_1} k_{j_l} \right|^2\right\} ds_1 \\ &\quad \times \int_0^s s_2^{\frac{p_2-2}{2}} \exp\left\{-(s-s_2) \left| \sum_{l=p_1+1}^p k_{j_l} \right|^2\right\} \left| \sum_{l=p_1+1}^p k_{j_l} \right| ds_2 \\ &\quad \times C^{p_1-1} \prod_{l=1}^{p_1} |B_{j_l}| C^{p_2-1} \prod_{l=p_1+1}^p |B_{j_l}|. \end{aligned}$$

In the integral with respect to  $s_1$ , we replace the exponent by 1, and we estimate the integral with respect to  $s_2$  with the help of the Cauchy-Schwarz inequality:

$$\begin{aligned} &\int_0^s s_2^{\frac{p_2-2}{2}} \exp\left\{-(s-s_2) |\bar{k}|^2\right\} |\bar{k}| ds_2 \\ &\leq \left( \int_0^s s_2^{p_2-2} ds_2 \int_0^s \exp\left\{-2(s-s_2) |\bar{k}|^2\right\} |\bar{k}|^2 ds_2 \right)^{\frac{1}{2}}. \end{aligned}$$

Thus, we get

$$\left| B_{j_1 \dots j_p}^{(p_1, p_2)}(s) \right| \leq \frac{s^{\frac{p_1}{2}} s^{\frac{p_2-1}{2}}}{p_1 \sqrt{\frac{p_2-1}{2}}} C^{p-2} \prod_{l=1}^p |B_{j_l}| \leq \frac{s^{\frac{p-2}{2}}}{p_1 \sqrt{\frac{p_2-1}{2}}} C^{p-2} \prod_{l=1}^p |B_{j_l}|.$$

At the last step, we used the fact that  $0 \leq s \leq 1$ . The first and last terms in (7) are estimated in a similar way:

$$|B_{j_1 \dots j_p}^{(1, p-1)}(s)| \leq \frac{s^{\frac{p-2}{2}}}{\sqrt{\frac{p-2}{2}}} C^{p-2} \prod_{l=1}^p |B_{j_l}|$$

and

$$|B_{j_1 \dots j_p}^{(p-1, 1)}(s)| \leq \frac{s^{\frac{p-2}{2}}}{\sqrt{\frac{p-2}{2}}} C^{p-2} \prod_{l=1}^p |B_{j_l}|.$$

For  $h_p(k, s)$  we have the representation:

$$\begin{aligned} h_p(k, s) &= \sum_{p_1=1}^{p-1} h_p^{(p_1, p-p_1)}(s) = \sum_{p_1=1}^{p-1} \sum_{j_1 \dots j_{p-p_1}=1}^n B_{j_1 \dots j_{p-p_1}}^{(p_1, p-p_1)}(s) \delta\left(k - \sum_{l=1}^n k_{j_l}\right) \\ &= \sum_{j_1 \dots j_p=1}^n B_{j_1 \dots j_p}(s) \delta\left(k - \sum_{l=1}^n k_{j_l}\right), \end{aligned}$$

where

$$B_{j_1 \dots j_p}(s) = \sum_{p_1=1}^{p-1} B_{j_1 \dots j_p}^{(p_1, p-p_1)}(s).$$

For  $B_{j_1 \dots j_p}(s)$  we have the estimate

$$\begin{aligned} |B_{j_1 \dots j_p}(s)| &\leq \sum_{p_1=1}^{p-1} |B_{j_1 \dots j_p}^{(p_1, p-p_1)}(s)| \\ &\leq s^{\frac{p-2}{2}} C^{p-2} \prod_{l=1}^p |B_{j_l}| \left( \sum_{p_1=2}^{p-2} \frac{\sqrt{2}}{p_1 \sqrt{p-p_1}} + \frac{2\sqrt{2}}{\sqrt{p-2}} \right). \end{aligned}$$

It is easy to see that for  $p \geq 3$  the sum

$$\sum_{p_1=2}^{p-2} \frac{\sqrt{2}}{p_1 \sqrt{p-p_1}} + \frac{2\sqrt{2}}{\sqrt{p-2}}$$

is bounded by some constant  $C_1$ . Taking  $C_2 = \max(C_1, \max_{1 \leq j \leq n} |k_j|)$ , we get the required inequality for  $q = p$ . The lemma is proved.  $\square$

PROOF OF THEOREM 2. We use the following lemma.

**Lemma 2.** *Let  $\{k_j\}$  be contained in a cone whose angle is less than  $\pi$ . Assume that for  $2 \leq q < p$  the functions  $h_q(k, s)$  have the representation*



(10) and for some constant  $D$  depending on initial conditions

$$|B_{j_1 \dots j_q}(s)| \leq D^{q-1} \prod_{l=1}^q |B_{j_l}|.$$

Then the same representation is valid for  $q = p$ .

PROOF. Using the notation from Lemma 1, we can write

$$\begin{aligned} |B_{j_1 \dots j_p}^{(p_1, p_2)}(s)| &\leq \int_0^s ds_1 \int_0^s ds_2 \left| \sum_{l=p_1}^p k_{j_l} \right| D^{p_1-1} \prod_{l=1}^{p_1} |B_{j_l}| D^{p_2-1} \prod_{l=p_1+1}^p |B_{j_l}| \\ &\quad \times \exp \left\{ - (s - s_1) \left| \sum_{l=1}^{p_1} k_{j_l} \right|^2 - (s - s_2) \left| \sum_{l=p_1+1}^p k_{j_l} \right|^2 \right\} \\ &< D^{p-2} \prod_{l=1}^p |B_{j_l}| \frac{\left| \sum_{l=p_1+1}^p k_{j_l} \right|}{\left| \sum_{l=1}^{p_1} k_{j_l} \right|^2 \left| \sum_{l=p_1+1}^p k_{j_l} \right|^2}. \end{aligned}$$

By the assumption of the lemma,  $\left| \sum_j \alpha_j k_j \right| \geq d \sum \alpha_j$ , where  $\alpha_j \geq 0$  and  $d > 0$  is some constant depending on the initial vectors  $k_j$ . Indeed, under some rotation of  $\mathbb{R}^3$ , the vectors  $k_j$  can be brought inside a cone of angle less than  $\pi$  belonging to the subspace  $k^{(1)} > 0$ . Then

$$\left| \sum_j \alpha_j k_j \right| = \left| \sum_j \alpha_j \bar{k}_j \right| \geq \sum_j \alpha_j \min_j \bar{k}_j^{(1)},$$

where  $\bar{k}_j$  is the image of  $k_j$  under the above-mentioned rotation.

Put  $d = \min_j \bar{k}_j^{(1)}$ . Now, we can write

$$|B_{j_1 \dots j_p}^{(p_1, p_2)}(s)| \leq D^{p-2} \prod_{l=1}^p |B_{j_l}| \frac{p_2 \max_j |k_j|}{p_1^2 p_2^2 d^4}.$$

Similarly, for  $p_1 = 1$

$$|B_{j_1 \dots j_p}^{(1, p-1)}(s)| \leq D^{p-2} \prod_{l=1}^p |B_{j_l}| \frac{(p-1) \max_j |k_j|}{(p-1)^2 d^2}$$

and for  $p_2 = 1$

$$|B_{j_1 \dots j_p}^{(p-1,1)}(s)| \leq D^{p-2} \prod_{l=1}^p |B_{j_l}| \frac{(p-1) \max_j |k_j|}{(p-1)^2 d^2}.$$

As in the proof of Lemma 1, we have

$$\begin{aligned} |B_{j_1 \dots j_p}(s)| &\leq \sum_{p_1=1}^{p-1} |B_{j_1 \dots j_p}^{(p_1, p-p_1)}(s)| \\ &< D^{p-2} \prod_{l=1}^p |B_{j_l}| \max_{1 \leq j \leq n} |k_j| \left[ \frac{2}{(p-1)d^2} + \sum_{p_1=2}^{p-2} \frac{1}{p_1^2(p-p_1)d^4} \right]. \end{aligned}$$

The expression in the square brackets is bounded by some constant  $D_0$  depending on initial conditions. Taking  $D \geq D_0 \max_{1 \leq j \leq n} |k_j|$ , we get the required inequality for  $q = p$ . The lemma is proved.  $\square$

Now, we derive Theorem 2 from Lemma 2. Using the notation from the proof of Theorem 1, we can write

$$|\tilde{B}_{j_1 \dots j_p}(t)| < D^{p-1} \frac{\prod_{l=1}^p |B_{j_l}|}{\left| \sum_{l=1}^p k_{j_l} \right|} < \frac{D^{p-1} \prod_{l=1}^p |B_{j_l}|}{(pd)^2}$$

and

$$\sum_{j_1 \dots j_p=1}^n |\tilde{B}_{j_1 \dots j_p}(t)| < \frac{D^{p-1}}{(pd)^2} \left( \sum_{j=1}^n |B_j| \right)^p.$$

Therefore, if

$$\sum_{j=1}^n |B_j| < D^{-1},$$

then the series

$$\sum_{p>0} \sum_{j_1 \dots j_p=1}^n |B_{j_1 \dots j_p}(t)|$$

converges absolutely for all  $t$ . The end of the proof is the same as in Theorem 1.  $\square$

**3. Example.** We consider  $n = 2$  and the initial condition

$$v(k, 0) = B_1 \delta(k - k_1) + B_2 \delta(k - k_2), \tag{14}$$

where the vectors  $k_1$  and  $k_2$  are linearly independent and  $B_1$  is orthogonal to  $k_1$  and  $k_2$ .

**Theorem 3.** *For the initial condition (14) the system (1) has a global solution.*

PROOF. In this case,

$$\begin{aligned}
 h_1(k, s) &= \sum_{j=1}^2 \exp\{-s|k_j|^2\} B_j \delta(k - k_j), \\
 h_2(k, s) &= i \sum_{j_1, j_2=1}^2 \langle k, B_{j_1} \rangle \left( B_{j_2} - \frac{\langle k, B_{j_2} \rangle k}{|k|^2} \right) \\
 &\quad \times \exp\{-s|k_{j_1}|^2 - s|k_{j_2}|^2\} \delta(k - (k_{j_1} + k_{j_2})) \\
 &= i \sum_{j_1, j_2=1}^2 \langle k_{j_1} + k_{j_2}, B_{j_1} \rangle \left( B_{j_2} - \frac{\langle k_{j_1} + k_{j_2}, B_{j_2} \rangle (k_{j_1} + k_{j_2})}{|k_{j_1} + k_{j_2}|^2} \right) \\
 &\quad \times \exp\{-s|k_{j_1}|^2 - s|k_{j_2}|^2\} \delta(k - (k_{j_1} + k_{j_2})).
 \end{aligned}$$

It is easy to see that only the term with  $j_1 = 2, j_1 = 1$  is different from zero. Therefore,

$$h_2(k, s) = \langle k_1, B_2 \rangle B_2 \exp\{-s|k_1|^2 - s|k_2|^2\} \delta(k - (k_1 + k_2)).$$

From the recurrent relation it easily follows that, in the sum (7), only the term with  $p_1 = p - 1$  is different from zero. If

$$h_q(k, s) = B^{(q)}(s) \delta(k - (k_1 + (q - 1)k_2))$$

and

$$|B^{(q)}(s)| < \frac{C^{q-1}}{\sqrt{(q-2)!}} |B_1| |B_2|^{q-1},$$

then the same inequality is valid for  $q = p$  (see the proof of Lemma 1). Theorem 3 is proved. □

**Remark.** Theorem 3 is equivalent to the estimates of simple diagrams in [7].

**Acknowledgments.** The second author thanks NSF for the financial support (grant no. DMS-0600996).

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Bardos, C.; Fursikov, A.V. (Eds.)

2008, XXXVI, 364 p., Hardcover

ISBN: 978-0-387-75216-7