Contents

I  Steady-State Solutions of the Navier–Stokes Equations:
    Statement of the Problem and Open Questions .......... 1
    Introduction ............................................................ 1
    I.1 Flow in Bounded Regions ................................. 4
    I.2 Flow in Exterior Regions ......................... 8
       I.2.1 Three-Dimensional Flow ............................ 10
       I.2.2 Plane Flow ........................................ 14
    I.3 Flow in Regions with Unbounded Boundaries ....... 17

II  Basic Function Spaces and Related Inequalities ....... 25
    Introduction ............................................................ 25
    II.1 Preliminaries ................................................. 25
       II.1.1 Basic Notation ...................................... 26
       II.1.2 Banach Spaces and their Relevant Properties .... 29
       II.1.3 Spaces of Smooth Functions ....................... 35
       II.1.4 Classes of Domains and their Properties ....... 36
    II.2 The Lebesgue Spaces \(L^q\) ............................. 40
    II.3 The Sobolev Spaces \(W^{m,q}\) and Embedding
        Inequalities .............................................. 48
    II.4 Boundary Inequalities and the Trace of Functions
        of \(W^{m,q}\) .................................................. 61
    II.5 Further Inequalities and Compactness Criteria
        in \(W^{m,q}\) ................................................ 69
    II.6 The Homogeneous Sobolev Spaces \(D^{m,q}\) and Embedding
        Inequalities ................................................ 80
    II.7 Approximation of Functions from \(D^{m,q}\) by Smooth
        Functions and Characterization of Space
        \(\tilde{D}_0^{m,q}\) ......................................... 102
    II.8 The Normed Dual of \(D_0^{m,q}(\Omega)\). The Spaces
        \(D_0^{m,q}\) ................................................ 109
    II.9 Pointwise behavior at Large Distances of Functions
        from \(D^{1,q}\) ................................................... 115
    II.10 Boundary Trace of Functions from
         \(D^{m,q}(\mathbb{R}_n^+))\) ................................... 121
    II.11 Some Integral Transforms and Related Inequalities .... 125
    II.12 Notes for the Chapter ................................. 134
III  The Function Spaces of Hydrodynamics .......................... 139
  Introduction ........................................................................ 139
  III.1  The Helmholtz–Weyl Decomposition of the Space $L^q$ ....... 141
  III.2  Relevant Properties of the Spaces $H_q$ and $G_q$ ............... 155
  III.3  The Problem $\nabla \cdot v = f$ .................................... 161
  III.4  The Spaces $H^1_q$ .................................................. 193
    III.4.1  Bounded Domains .......................................... 196
    III.4.2  Exterior Domains .......................................... 197
    III.4.3  Domains with Noncompact Boundary ..................... 198
  III.5  The Spaces $D^{1,q}_0$ ............................................. 214
  III.6  Approximation Problems in Spaces $H^1_q$ and $D^{1,q}_0$ ....  218
  III.7  Notes for the Chapter ............................................ 226

IV  Steady Stokes Flow in Bounded Domains ......................... 231
  Introduction ........................................................................ 231
  IV.1  Generalized Solutions. Existence and Uniqueness .......... 233
  IV.2  Existence, Uniqueness, and $L^q$-Estimates in the Whole Space. The Stokes Fundamental Solution .... 238
  IV.4  Interior $L^q$-Estimates ........................................... 263
  IV.5  $L^q$-Estimates Near the Boundary ............................. 271
  IV.6  Existence, Uniqueness, and $L^q$-Estimates in a Bounded Domain ..................................................... 279
  IV.7  Existence and Uniqueness in Hölder Spaces. Schauder Estimates ....................................................... 287
  IV.8  Green’s Tensor, Green’s Identity and Representation Formulas ......................................................... 288
  IV.9  Notes for the Chapter ................................................. 294

V  Steady Stokes Flow in Exterior Domains ............................ 299
  Introduction ........................................................................ 299
  V.1  Generalized Solutions. Preliminary Considerations and Regularity Properties ...................................... 304
  V.2  Existence and Uniqueness of Generalized Solutions for Three-Dimensional Flow ................................. 306
  V.3  Representation of Solutions. Behavior at Large Distances and Related Results .................................... 310
  V.4  Existence, Uniqueness, and $L^q$-Estimates: Strong Solutions ................................................................. 320
  V.5  Existence, Uniqueness, and $L^q$-Estimates: $q$-generalized Solutions ....................................................... 337
  V.6  Green’s Tensor and Some Related Properties ................ 349
  V.7  A Characterization of Certain Flows with Nonzero Boundary Data. Another Form of the Stokes Paradox .... 351
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>V.8</td>
<td>Further Existence and Uniqueness Results for ( q )-generalized Solutions</td>
<td>354</td>
</tr>
<tr>
<td>V.9</td>
<td>Notes for the Chapter</td>
<td>362</td>
</tr>
<tr>
<td>VI</td>
<td>Steady Stokes Flow in Domains with Unbounded Boundaries</td>
<td>365</td>
</tr>
<tr>
<td>VI.1</td>
<td>Leray’s Problem: Existence, Uniqueness, and Regularity</td>
<td>370</td>
</tr>
<tr>
<td>VI.2</td>
<td>Decay Estimates for Flow in a Semi-infinite Straight Channel</td>
<td>379</td>
</tr>
<tr>
<td>VI.3</td>
<td>Flow in Unbounded Channels with Unbounded Cross Sections. Existence, Uniqueness, and Regularity</td>
<td>387</td>
</tr>
<tr>
<td>VI.4</td>
<td>Pointwise Decay of Flows in Channels with Unbounded Cross Section</td>
<td>393</td>
</tr>
<tr>
<td>VI.5</td>
<td>Existence, Uniqueness, and Asymptotic Behavior of Flow Through an Aperture</td>
<td>407</td>
</tr>
<tr>
<td>VI.6</td>
<td>Notes for the Chapter</td>
<td>415</td>
</tr>
<tr>
<td>VII</td>
<td>Steady Oseen Flow in Exterior Domains</td>
<td>417</td>
</tr>
<tr>
<td>VII.1</td>
<td>Generalized Solutions. Regularity and Uniqueness</td>
<td>420</td>
</tr>
<tr>
<td>VII.2</td>
<td>Existence of Generalized Solutions for Three-Dimensional Flow</td>
<td>424</td>
</tr>
<tr>
<td>VII.3</td>
<td>The Oseen Fundamental Solution and the Associated Volume Potentials</td>
<td>429</td>
</tr>
<tr>
<td>VII.4</td>
<td>Existence, Uniqueness, and ( L^q )-Estimates in the Whole Space</td>
<td>445</td>
</tr>
<tr>
<td>VII.5</td>
<td>Existence of Generalized Solutions for Plane Flows in Exterior Domains</td>
<td>459</td>
</tr>
<tr>
<td>VII.6</td>
<td>Representation of Solutions. Behavior at Large Distances and Related Results</td>
<td>467</td>
</tr>
<tr>
<td>VII.7</td>
<td>Existence, Uniqueness, and ( L^q )-Estimates in Exterior Domains</td>
<td>475</td>
</tr>
<tr>
<td>VII.8</td>
<td>Limit of Vanishing Reynolds Number. Transition to the Stokes Problem</td>
<td>487</td>
</tr>
<tr>
<td>VII.9</td>
<td>Notes for the Chapter</td>
<td>492</td>
</tr>
<tr>
<td>VIII</td>
<td>Steady Generalized Oseen Flow in Exterior Domains</td>
<td>495</td>
</tr>
<tr>
<td>VIII.1</td>
<td>Generalized Solutions. Regularity and Existence</td>
<td>499</td>
</tr>
<tr>
<td>VIII.2</td>
<td>Generalized Solutions. Uniqueness</td>
<td>505</td>
</tr>
<tr>
<td>VIII.3</td>
<td>The Fundamental Solution to the Time-Dependent Oseen Problem and Related Properties</td>
<td>514</td>
</tr>
<tr>
<td>VIII.4</td>
<td>On the Unique Solvability of the Oseen Initial-Value Problem</td>
<td>526</td>
</tr>
</tbody>
</table>
VIII.5 Existence, Uniqueness, and Pointwise Estimates of Solutions in the Whole Space .......... 546
VIII.6 On the Pointwise Asymptotic Behavior of Generalized Solutions .................. 554
VIII.7 Existence, Uniqueness, and $L^q$-Estimates. The case $R = 0$ ........ 560
VIII.8 Existence, Uniqueness, and $L^q$-Estimates. The Case $R \neq 0$ ........ 572
VIII.9 Notes for the Chapter ................................ 578

IX Steady Navier–Stokes Flow in Bounded Domains ........ 583
Introduction ................................................. 583
IX.1 Generalized Solutions. Preliminary Considerations .......... 586
IX.2 On the Uniqueness of Generalized Solutions ............ 591
IX.3 Existence and Uniqueness with Homogeneous Boundary Data ........................................ 596
IX.4 Existence and Uniqueness with Nonhomogeneous Boundary Data .................................. 602
IX.5 Regularity of Generalized Solutions ........................ 621
IX.6 Limit of Infinite Viscosity: Transition to the Stokes Problem .................................... 640
IX.7 Notes for the Chapter ................................ 643

X Steady Navier–Stokes Flow in Three-Dimensional Exterior Domains. Irrotational Case .... 649
Introduction ................................................. 649
X.1 Generalized Solutions. Preliminary Considerations and Regularity Properties ................ 653
X.2 On the Validity of the Energy Equation for Generalized Solutions .................... 659
X.3 Some Uniqueness Results .................................. 668
X.4 Existence of Generalized Solutions ............................ 676
X.5 On the Asymptotic Behavior of Generalized Solutions: Preliminary Results and Representation Formulas .... 688
X.6 Global Summability of Generalized Solutions
when $v_\infty \neq 0$ ........................................ 698
X.7 The Energy Equation and Uniqueness for Generalized Solutions when $v_\infty \neq 0$ ............... 705
X.8 The Asymptotic Structure of Generalized Solutions
when $v_\infty \neq 0$ ........................................ 709
X.9 On the Asymptotic Structure of Generalized Solutions
when $v_\infty = 0$ ........................................ 721
X.10 Limit of Vanishing Reynolds Number: Transition to the Stokes Problem ................ 731
X.11 Notes for the Chapter ................................ 742
XI Steady Navier–Stokes Flow in Three-Dimensional Exterior Domains. Rotational Case ........................................ 745
Introduction .................................................................... 745
XI.1 Generalized Solutions. Existence of the Pressure and Regularity Properties ............................. 749
XI.2 On the Energy Equation and the Uniqueness of Generalized Solutions ..................................................... 755
XI.3 Existence of Generalized Solutions ........................................ 762
XI.4 Global Summability of Generalized Solutions when $v_0 \cdot \omega \neq 0$ ............................................................. 770
XI.5 The Energy Equation and Uniqueness for Generalized Solutions when $v_0 \cdot \omega \neq 0$ ....................... 772
XI.6 On the Asymptotic Structure of Generalized Solutions When $v_0 \cdot \omega \neq 0$ ....................................................... 775
XI.7 On the Asymptotic Structure of Generalized Solutions When $v_0 \cdot \omega = 0$ ....................................................... 790
XI.8 Notes for the Chapter ......................................................... 795

XII Steady Navier–Stokes Flow in Two-Dimensional Exterior Domains ........................................ 797
Introduction .................................................................... 797
XII.1 Generalized Solutions and $D$-Solutions ........................................ 800
XII.2 On the Uniqueness of Generalized Solutions ........................................ 801
XII.3 On the Asymptotic Behavior of $D$-Solutions ........................................ 804
XII.4 Asymptotic Decay of the Vorticity and its Relevant Consequences ......................................................... 824
XII.5 Existence and Uniqueness of Solutions for Small Data and $v_\infty \neq 0$ ........................................ 836
XII.6 A Necessary Condition for Non-Existence with Arbitrary Large Data ........................................ 853
XII.7 Global Summability of Generalized Solutions when $v_\infty \neq 0$ ............................................................. 855
XII.8 The Asymptotic Structure of Generalized Solutions when $v_\infty \neq 0$ ............................................................. 866
XII.9 Limit of Vanishing Reynolds Number: Transition to the Stokes Problem ........................................ 885
XII.10 Notes for the Chapter ......................................................... 891

XIII Steady Navier–Stokes Flow in Domains with Unbounded Boundaries ........................................ 897
Introduction .................................................................... 897
XIII.1 Leray’s Problem: Generalized Solutions and Related Properties ......................................................... 900
XIII.2 On the Uniqueness of generalized Solutions to Leray’s Problem ......................................................... 903
XIII.3 Existence and Uniqueness of Solutions to
Leray’s Problem .................................................... 910
XIII.4 Decay Estimates for Steady Flow in a Semi–Infinite
Straight Channel .................................................... 916
XIII.5 Flow in an Aperture Domain, Generalized Solutions and
Related Properties ................................................. 926
XIII.6 Energy Equation and Uniqueness for Flows in an
Aperture Domain ................................................. 930
XIII.7 Existence and Uniqueness of Flows in an
Aperture Domain ................................................. 935
XIII.8 Global Summability of Generalized Solutions for Flow
in an Aperture Domain ........................................... 949
XIII.9 Asymptotic Structure of Generalized Solutions for Flow
in an Aperture Domain ......................................... 958
XIII.10 Notes for the Chapter .......................................... 970

Bibliography .............................................................. 975

Index ................................................................. 1009