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Pierce Bohl, 1865–1921
Jacques Hadamard, 1865–1963
Luitzen Egbertus Jan Brouwer, 1881–1966
George David Birkhoff, 1884–1944
Solomon Lefschetz, 1884–1972
James Waddell Alexander, 1888–1971
Stefan Mazurkiewicz, 1888–1945
Jacob Nielsen, 1890–1959
Stefan Banach, 1892–1945
Harold Marston Morse, 1892–1977
Bronislaw Knaster, 1893–1980
Heinz Hopf, 1894–1971
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John von Neumann, 1903–1957
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Fixed Point Theory
Granas, A.; Dugundji, J.
2003, XVI, 690 p., Hardcover